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**FUNCTIONAL CHANGE OF VARIABLES  
IN THE WHEELER–DE WITT EQUATION**

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**ABSTRACT**

I present a new way to solve the Wheeler–de Witt equation using the invariance of the classical lagrangian under reparametrization. This property allows one to introduce an arbitrary function for each degree of freedom of the wave function  $\Psi$ : this arbitrariness can be used to fix the asymptotic behaviour of  $\Psi$  so as to obtain a wave function representing a closed universe or a wormhole. These considerations are applied in detail to the Kantowsky–Sachs spacetime.

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## 1. Introduction.

Different problems arise in writing and solving the Wheeler–de Witt (WDW) equation. An important one is how to associate the quantum operators to the classical quantities. Generally we start with a classical lagrangian which is a function of some number of variables: for example in the minisuperspace formalism the euclidean action for gravity minimally coupled to a scalar field is [1,2]

$$S_E = -\frac{1}{2} \int dt [a\dot{a}^2 + a - a^3\dot{\varphi}^2] \quad (1.1)$$

and the problem is formally reduced to two degrees of freedom. To quantize this system we must associate a representation of the quantum operators representing the conjugate momenta to the variables  $a$  and  $\varphi$ , i.e.  $\Pi_a$  and  $\Pi_\varphi$ . For instance we may put:

$$\Pi_a^2 \rightarrow \frac{1}{a^p} \frac{\partial}{\partial a} \left[ a^p \frac{\partial}{\partial a} \right] \quad (1.2a)$$

$$\Pi_\varphi^2 \rightarrow \frac{1}{\varphi^q} \frac{\partial}{\partial \varphi} \left[ \varphi^q \frac{\partial}{\partial \varphi} \right] \quad (1.2b)$$

where  $p$  and  $q$  are two arbitrary positive integers. Choosing  $p = 1$  and  $q = 0$  the WDW equation assumes the form:

$$\left[ \frac{1}{a} \frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{1}{a^2} \frac{\partial^2}{\partial \varphi^2} - a^2 \right] \Psi(a, \varphi) = 0. \quad (1.3)$$

However, using the procedure described above we have two ambiguities: the factor ordering of the quantum operators (i.e.  $p$  and  $q$ ) and the choice of the variables to quantize: for instance to obtain wave functions describing wormholes [2] a better choice is to quantize the lagrangian written in terms of the variables

$$x = a \sinh \varphi, \quad y = a \cosh \varphi.$$

With these variables the WDW equation becomes ( $p = 0, q = 0$ ):

$$\left[ \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} - y^2 + x^2 \right] \Psi(x, y) = 0. \quad (1.4)$$

In this way we obtain the solutions:

$$\Psi(x, y) = \psi_n(x) \psi_n(y) \quad (1.5)$$

where  $\psi_n$  represents the wave function of the one dimensional harmonic oscillator of order  $n$ . In the case of more degrees of freedom a simple choice for the variables to quantize is generally not possible: for example in the case of a

Kantowsky–Sachs spacetime (see section 4) it is not trivial find the appropriate variable transformations in order to obtain a definite type solution, for instance a wormhole wave function.

In the following I will show that it is possible to quantize the system and write the WDW equation without fixing the transformations. The form of these can be chosen later by specifying the desired asymptotic behaviour of the wave function. We want to emphasize that the difference between the two methods is that in our approach we fix the variable transformations only after quantization of the system. In this way the quantization is independent of the transformations which can be interpreted like “gauge” functions because a particular choice of them does not change the physics.

The structure of the paper is the following: in the next section we shall recall briefly the main aspects of the WDW equation and we will define the notations; in the third section we shall introduce the transformations for the general case; finally in the last section we apply these considerations to the simple case of a Kantowsky–Sachs spacetime.

## 2. WDW equation.

In the following let us introduce the standard notations for the WDW equation. In particular we will write the line element in the form [3]:

$$ds^2 = (N^2 - N_i N^i) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j \quad (2.1)$$

where  $N$  represents the lapse function,  $N^i$  is the shift vector and  $h_{ij}$  is the three-space metric. The euclidean hamiltonian for general relativity ( $M_p$  is the Planck mass) is then [4]

$$H_E = \int (N \mathcal{H}_G + N_i \mathcal{H}^i) d^3x \quad (2.2)$$

where (in the following  ${}^{(3)}R$  is the scalar curvature tensor relative to the space metric)

$$\mathcal{H}_G = 16\pi M_p^{-2} \mathcal{H}_{ijkl} \Pi^{ij} \Pi^{kl} + \sqrt{h} \left[ \left( \frac{M_p^2}{16\pi} \right) {}^{(3)}R + T_{44} \right], \quad (2.3)$$

and (“|” represents the covariant derivative with respect to the space metric)

$$\mathcal{H}^i = -2\Pi^{ij}{}_{|j} - \sqrt{h} T^{4i}. \quad (2.4)$$

(2.3) and (2.4) are written in terms of the superspace metric

$$\mathcal{H}_{ijkl} = \frac{1}{2\sqrt{h}} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}), \quad (2.5)$$

in terms of the conjugate momentum to  $h_{ij}$  ( $\mathbf{K}^{ij}$  is the second fundamental form)

$$\Pi^{ij} = \frac{M_p^2}{16\pi} \sqrt{h} (\mathbf{K}^{ij} - h^{ij} \mathbf{K}), \quad (2.6)$$

and in terms of the stress energy tensor of the matter field  $T_{\mu\nu}$ . We suppose also that the cosmological constant is zero. The lapse function and the shift vector can be considered as Lagrange multipliers; then the classical equations of motion can be written:

$$\mathcal{H}_G = 0, \quad (2.7a)$$

$$\mathcal{H}^i = 0. \quad (2.7b)$$

The quantization of the problem can be achieved identifying the classical quantities  $\Pi^{ij}$  with the operators [3]

$$\Pi^{ij} \rightarrow - \left( \frac{M_p^2}{16\pi} \right) \frac{\delta}{\delta h_{ij}} \quad (2.8)$$

which satisfy the euclidean commutation relations:

$$\begin{aligned} [h_{ij}, h_{kl}] &= 0, \\ [\Pi_{ij}, \Pi_{kl}] &= 0, \\ [h_\alpha, \Pi_\beta] &= \frac{M_p^2}{16\pi} \delta_{\alpha\beta}, \quad \alpha = (ij), \quad \beta = (kl). \end{aligned} \quad (2.9)$$

and analogously for the classical momenta of the matter fields. With this identification we obtain from (2.7a) the WDW equation describing the quantum properties of the gravitational field coupled to matter:

$$\left[ \mathcal{H}_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} + \sqrt{h} \left[ {}^{(3)}R + \frac{16\pi}{M_p^2} T_{44} \right] \right] \Psi(h_{ij}, \phi) = 0 \quad (2.10)$$

where now the stress energy tensor of the matter fields  $\phi$  is a quantum operator and  $\Psi$  represents the wave function of the system. From (2.7b) we obtain the equation:

$$\left[ -2 \left( \frac{M_p^2}{16\pi} \right) \left[ \frac{\delta}{\delta h_{ij}} \right]_{|j} + \sqrt{h} T^{4i} \right] \Psi(h_{ij}, \phi) = 0 \quad (2.11)$$

that represents a constraint identically satisfied by the wave function.

### 3. Transformations for the general WDW equation.

Let us come back to the line element (2.1) and perform the transformation:

$$h_{ij} \rightarrow h'_{ij} = f_{ij}^{-1}(h_{kl}). \quad (3.1)$$

We stress that (3.1) is not a change of coordinates in the metric but a functional transformation of the space metric. To make this consideration explicit we can consider, for example, the line element

$$ds^2 = dt^2 + a^2(t)d\vec{x}^2 \quad (3.2)$$

which corresponds to (2.1) with  $N = 1$ ,  $N_i = 0$  and  $h_{ij} = a^2(t)\delta_{ij}$ . A transformation (3.1) could be

$$a^2(t) = b(t). \quad (3.3)$$

In this case we do not change the coordinates, only the functional form of the scale factor. Let us now substitute (3.1) into the lagrangian: we can calculate the hamiltonian with respect to the new space metric and conjugate momentum. This latter will be related to the previous expression (2.6) by the following relation:

$$(\Pi^{ij})' = \Pi^{kl} \frac{\partial f_{kl}}{\partial h'_{ij}}. \quad (3.4)$$

Now we can repeat all the steps described in the previous paragraph to find the WDW equation. (2.8) becomes

$$(\Pi^{ij})' \rightarrow -\left(\frac{M_p^2}{16\pi}\right) \frac{\delta}{\delta h'_{ij}} \quad (3.5)$$

which implies that the quantum operator that must be substituted for  $\Pi^{ij}$  in (2.3) is

$$\Pi^{ij} \rightarrow -\frac{\partial f_{kl}^{-1}}{\partial h_{ij}} \left(\frac{M_p^2}{16\pi}\right) \frac{\delta}{\delta h'_{kl}}. \quad (3.6)$$

We can easily note that the WDW equation is now formally different from the previous expression (2.10) because of the presence of the factors

$$\frac{\partial f_{kl}^{-1}}{\partial h_{ij}}$$

which represent the arbitrariness in the choice of the variables to be quantized. The functions  $f_{ij}$  can then be interpreted as “gauge” functions and can be fixed to obtain an appropriate form for the WDW equation.

#### 4. Kantowsky–Sachs spacetime.

Now we will apply the considerations of the latter paragraph to a particular ansatz. The simplest non trivial case is obtained when the gravitational lagrangian depends on two degrees of freedom; we assume then the spacetime manifold to be described by the Kantowsky–Sachs line element:

$$ds^2 = N^2(t)dt^2 + a^2(t)d\chi^2 + b^2(t)d\Omega_2^2 \quad (4.1)$$

where  $\chi$  is the coordinate of the 1-sphere,  $0 \leq \chi < 2\pi$ ,  $d\Omega_2^2$  represents the line element of the 2-sphere and  $N(t)$  is the lapse function. In the pure gravity case this problem was first solved by Fishbone [5] using the ‘‘exponential route’’ choice of variables [6]. Because the WDW equation is independent of the lapse function we can simplify the line element choosing  $N = 1$ . The euclidean action of our problem is

$$S_E = \int_{\Omega} d^4x \sqrt{g} \left[ -\frac{M_p^2}{16\pi} R + L(\phi) \right] + \int_{\partial\Omega} d^3x \sqrt{h} \frac{M_p^2}{8\pi} \mathbf{K} \quad (4.2)$$

where  $\Omega$  is the compact four dimensional manifold described by (4.1),  $R$  is the curvature scalar,  $L(\phi)$  is the matter lagrangian,  $\mathbf{K}$  is the trace of the extrinsic curvature of the boundary  $\partial\Omega$  of  $\Omega$  and  $h$  is the determinant of the induced metric over  $\partial\Omega$ .

Let us discuss first the pure gravity case. Substituting (4.1) in (4.2) we obtain:

$$S_E = - \int dt [a\dot{b}^2 + 2\dot{a}b\dot{b} + a] \quad (4.3)$$

where the dot represents differentiation with respect to  $t$  and we have integrated over the coordinates of the 1-sphere and 2-sphere. The problem is then formally reduced to two degrees of freedom:  $a$  and  $b$ . From (4.3) it is easy to derive the classical equations for  $a$  and  $b$  varying with respect to the two scale factors [7]:

$$2\frac{\ddot{b}}{b} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} = 0, \quad (4.4a)$$

$$\frac{\ddot{b}}{b} + \frac{\ddot{a}}{a} + \frac{\dot{a}\dot{b}}{ab} = 0. \quad (4.4b)$$

Now we look for the quantum solutions. To associate the classical quantities  $a$  and  $b$  and their momenta to quantum operators we need a new form for the action (4.3): this is necessary because the lagrangian is not quadratic in the canonical momenta. Following the prescriptions of the previous section we introduce two functions in order to separate the canonical momenta:

$$a = f(x, y), \quad (4.5a)$$

$$b = g(x, y). \quad (4.5b)$$

Substituting into the lagrangian we can require that the term proportional to  $\dot{x}\dot{y}$  vanishes; if  $f$  does not depend on  $y$  this condition implies that the functions  $f$  and  $g$  fulfill

$$f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} = 0. \quad (4.6)$$

Analogous results can be obtained with the conditions  $f(x, y) \equiv f(y)$ , or  $g(x, y) \equiv g(x)$ , or  $g(x, y) \equiv g(y)$ . From (4.6) we obtain the final form for  $a$  and  $b$ :

$$a = f \equiv f(x), \quad (4.7a)$$

$$b = g = \frac{h(y)}{f(x)}, \quad (4.7b)$$

where  $h(y)$  is an arbitrary function of  $y$ . Substituting (4.7) into the lagrangian we can write the classical hamiltonian in terms of the new canonical variables  $x$  and  $y$  and their canonical momenta:

$$H = -\frac{f}{4h^2} \left[ \frac{1}{(h'/h)^2} \Pi_y^2 - \frac{1}{(\dot{f}/f)^2} \Pi_x^2 - 4h^2 \right]. \quad (4.8)$$

Now we can quantize the system substituting for the classical quantities quantum operators. The WDW equation (2.10) becomes:

$$\left[ \frac{1}{(h'/h)^2} \frac{\partial^2}{\partial y^2} - \frac{1}{(\dot{f}/f)^2} \frac{\partial^2}{\partial x^2} - 4h^2 \right] \Psi(x, y) = 0, \quad (4.9)$$

where now the dot represents differentiation with respect to  $x$  and the prime denotes differentiation with respect to  $y$ . We can easily solve (4.9) putting

$$\Psi(x, y) = \chi(x)\sigma(y). \quad (4.10)$$

With this substitution (4.9) separates into the following two equations:

$$\frac{d^2\chi}{dx^2} = K \left( \frac{\dot{f}}{f} \right)^2 \chi, \quad (4.11a)$$

$$\frac{d^2\sigma}{dy^2} = (K + 4h^2) \left( \frac{h'}{h} \right)^2 \sigma, \quad (4.11b)$$

where  $K$  is an arbitrary constant. We want to discuss the asymptotic behaviour of (4.10) in terms of the old variables  $a$  and  $b$ . Using (4.7) we can rewrite (4.11) in the following form:

$$\frac{d^2\chi}{df^2} + \frac{\ddot{f}}{\dot{f}^2} \frac{d\chi}{df} - \frac{K}{f^2} \chi = 0, \quad (4.12a)$$

$$\frac{d^2\sigma}{dh^2} + \frac{h''}{h'^2} \frac{d\sigma}{dh} - \left( 4 + \frac{K}{h^2} \right) \sigma = 0. \quad (4.12b)$$

Now we can fix the asymptotic behaviour of the wave functions in  $a$  and  $b$  simply fixing their behaviour in  $f$  and  $h$ . Moreover the asymptotic behaviours can be

chosen imposing suitable conditions for these two functions. Let us see some examples.

If we want to obtain a solution damped for large values of  $a$  and  $b$  we can choose  $K = \nu^2 > 0$  and fix the “gauge” in the following way:

$$\frac{h''}{h'^2} = \frac{1}{h}. \quad (4.13)$$

With this condition (4.12b) becomes:

$$\frac{d^2\sigma}{dh^2} + \frac{1}{h} \frac{d\sigma}{dh} - \left(4 + \frac{\nu^2}{h^2}\right)\sigma = 0. \quad (4.14)$$

Then the solutions of equation (4.14) are

$$\sigma = Z_\nu(2ih) = Z_\nu(2iab) \quad (4.15)$$

where  $Z_\nu$  is any Bessel function of index  $\nu$ . With  $K = \nu^2$  (4.12a) becomes:

$$\frac{d^2\chi}{df^2} + \frac{\ddot{f}}{f^2} \frac{d\chi}{df} - \frac{\nu^2}{f^2}\chi = 0. \quad (4.16)$$

Choosing  $\ddot{f} = 0$  we obtain:

$$\frac{d^2\chi}{df^2} - \frac{\nu^2}{f^2}\chi = 0 \quad (4.17)$$

which has the non singular solution

$$\chi = \sqrt{f} \cdot f^{\sqrt{1+4\nu^2}/2}. \quad (4.18)$$

Finally we can write the complete form of the wave function

$$\Psi(a, b) = \sqrt{a} \cdot a^{\sqrt{1+4\nu^2}/2} Z_\nu(2iab). \quad (4.19)$$

If we want to obtain the transformations between  $a$ ,  $b$  and  $x$ ,  $y$  that give rise to the solution (4.19) we can solve (4.13) and the corresponding equation for  $f$ . We obtain:

$$\begin{aligned} ab &= C_1 \cdot \exp(C_2 \cdot y), \\ a &= C_3 + C_4 x, \end{aligned} \quad (4.20)$$

where  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  are arbitrary constants. (4.20) represent the transformations (3.1) in our case. Choosing for simplicity  $C_1 = C_2 = C_4 = 1$  and  $C_3 = 0$  we obtain that the line element (4.1) can be written in the following way:

$$ds^2 = dt^2 + x^2(t)d\chi^2 + \frac{e^{2y(t)}}{x^2(t)}d\Omega_2^2. \quad (4.21)$$



As we have mentioned, the choice of the asymptotic behaviour of the wave function (4.19) for large values of the scale factor  $b$  can be:

$$\Psi \approx \exp(-ab). \quad (4.22)$$

This is exactly the asymptotic behaviour of a solution representing a wormhole with spatial symmetry  $S^2 \times S^1$  [8]; then (4.19) can be interpreted as describing a quantum wormhole.

In the same way we can discuss the case with  $L(\phi) \neq 0$ . Let us see the cases of scalar and electromagnetic fields. If we add a scalar field the action (4.3) becomes

$$S_E = - \int dt [a\dot{b}^2 + 2\dot{a}b\dot{b} + a - ab^2\dot{\Phi}^2] \quad (4.23)$$

and the WDW equation assumes the form

$$\left[ \frac{\partial^2}{\partial \Phi^2} - \frac{1}{(h'/h)^2} \frac{\partial^2}{\partial y^2} + \frac{1}{(f/f)^2} \frac{\partial^2}{\partial x^2} + 4h^2 \right] \Psi(x, y, \Phi) = 0. \quad (4.24)$$

The matter degree of freedom can be easily separated by putting  $\Psi = \zeta(\Phi)\xi(x, y)$ . With this substitution (4.24) separates into

$$\frac{d^2 \zeta}{d\Phi^2} + \omega^2 \zeta = 0, \quad (4.25a)$$

$$\left[ -\frac{1}{(h'/h)^2} \frac{\partial^2}{\partial y^2} + \frac{1}{(f/f)^2} \frac{\partial^2}{\partial x^2} + 4h^2 - \omega^2 \right] \xi(x, y) = 0. \quad (4.25b)$$

(4.25b) can be separated to obtain

$$\frac{d^2 \chi}{dx^2} = (\omega^2 + K) \left( \frac{f}{f} \right)^2 \chi, \quad (4.26a)$$

$$\frac{d^2 \sigma}{dy^2} = (K + 4h^2) \left( \frac{h'}{h} \right)^2 \sigma. \quad (4.26b)$$

We can easily solve (4.25a) and (4.26) to get the wave function. In the case  $K = \nu^2$  we find:

$$\Psi(a, b, \Phi) = \sqrt{a} \cdot a^{\sqrt{1+4(\nu^2+\omega^2)}/2} Z_\nu(2iab) e^{\pm i\omega\Phi}. \quad (4.27)$$

Now let us examine the case with an electromagnetic field. If we choose for the vector potential the ansatz [7]

$$A_\mu = A(t)\delta_{\chi\mu}, \quad (4.28)$$

the action (4.3) becomes:

$$S_E = - \int dt \left[ ab\dot{b}^2 + 2\dot{a}b\dot{b} + a - \frac{b^2}{a} \dot{A}^2 \right] \quad (4.29)$$

and the corresponding WDW equation is:

$$\left[ \frac{\partial^2}{\partial A^2} + \frac{1}{f^2} \left[ -\frac{1}{(h'/h)^2} \frac{\partial^2}{\partial y^2} + \frac{1}{(\dot{f}/f)^2} \frac{\partial^2}{\partial x^2} + 4h^2 \right] \right] \Psi(x, y, A) = 0 \quad (4.30)$$

which is easily separated using an ansatz similar to the previous case. In the case  $K = \nu^2$  we obtain the solutions:

$$\Psi(a, b, A) = \sqrt{a} Z_\alpha(i\omega a) Z_\nu(2iab) e^{\pm i\omega A}, \quad (4.31a)$$

$$\Psi(a, b, A) = Z_\nu(i\omega a) Z_\nu(2iab) e^{\pm i\omega A}, \quad (4.31b)$$

where  $\alpha = \pm\sqrt{1 + 4\nu^2}/2$  and we have used (4.13) for  $h$  and respectively  $\ddot{f} = 0$  and  $\ddot{f}/\dot{f}^2 = 1/f$  for  $f$ .

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