Almost Global Stochastic Feedback Stabilization of Conditional Quantum Dynamics *

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Abstract

We propose several parametrization-free solutions to the problem of quantum state reduction control by means of continuous measurement and smooth quantum feedback. In particular, we design a feedback law for which almost global stochastic feedback stabilization can be proved analytically by means of Lyapunov techinques. This synthesis arises very naturally from the physics of the problem, as it relies on the variance associated with the quantum filtering process.

Keywords: Quantum Feedback, Stochastic Stabilization, Nonlinear Stochastic Matrix Differential Equations, Quantum Filtering.

1 Introduction

Experimental techniques in quantum optics permit nowadays to continuously monitor and modify the dynamics of a cloud of cold atoms confined in an optical cavity [15]. The knowledge of the system state can be described by a conditional dynamical equation, the so-called Stochastic Master Equation (SME), obtained from a suitable quantum system-field interaction model by means of non-commutatitive filtering theory [31, 7], and based on monitoring the outgoing field from the cavity. The stochastic perturbation induced by the indirect measurement process produces an effective *dynamical reduction model* [1]. In other words, it makes the system state converge to one of the maximal information, pure states for the system physical observable interacting with the field.

If we have a second controllable field, acting as a time dependent Hamiltonian perturbation, then we can use the real-time estimate of the system state to

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modify the state reduction process. One possible use of such a feedback control strategy can be *choosing to which pure state of the monitoring observable the system will converge.* Beside possible interest for quantum measurement theory, the overall strategy can then be seen as a technique for engineering quantum *state preparation.*

From a control theoretic viewpoint, the problem is doubtless challenging. The SME is a nonlinear affine in control matrix Stochastic Differential Equation (SDE) living in the convex cone of positive semidefinite N-dimensional Hermitian matrices. In the particular case of perfect measurement efficiency and maximal information on the initial condition, a SME turns out to be equivalent to a Stochastic Schrödinger Equation (SSE), i.e. a vector-valued, norm-preserving diffusion in $\mathbb{C}^N$ [1, 8]. Influencing the open loop state reduction by means of the measurement is therefore a stochastic nonlinear feedback stabilization problem. Partial solutions to this problem were presented for example in [29] for 2 level SSE and, in more general terms, in [28] based on a (convex) numerical Lyapunov design. This last solution, however, suffers from scalability problems as the dimension of the system grows, since it is based on explicit parametrizations which grow with N^2 if N is the dimension of the system.

For nonlinear (vector) SDE, most of the results on state feedback stabilization are due to Florchinger [12, 11, 13] (see also e.g. [9, 10, 5] for other possible approaches) and consists of extensions of Lyapunov-based techniques, like the Jurdjevic-Quinn condition, control Lyapunov function constructions, passivity-based methods and so on, to the stochastic case.

For our SME, these systematic construction methods have a limited success, and allow only to attain local stabilization in the particular case of SSE. The feedback that achieves it is the simple linear feedback already used for deterministic unitary feedback stabilization of quantum ensembles [3]. However, owing to the nature of the problem, local convergence results are of no practical interest. It is the structure of the problem itself that suggests how to improve the design: the state reduction encoded in the SME is "certified" by the variance of the continuous measurement process, a multiequilibrium Lyapunov-like function (with N equilibria corresponding to the N eigenvectors of the observable being measured) which is also a Morse function and can be used to enlarge the region of attraction of the controller. We shall in particular discuss two nonlinear feedback laws induced by the use of the variance, both more effective than the linear controller. Both allow for simple explicit proofs of convergence: one corresponds to a closed loop stochastic generator which is a sum of squares, the other to the square of a sum. The former achieves almost global stabilization for the perfect efficiency case, but cannot cancel all invariant sets of the dynamics in the more general SME and thus suffers from similar problems as the linear controller. The latter instead corresponds to a feedback stabilization design for the SME which is almost global, up to the $N-1$ isolated repulsive critical points (the remaining $N-1$ eigenvectors of the measured observable). This feedback strategy extends the idea of "symmetry breaking" enunciated in [28]. Indeed it works by using the uncontrollable part of the drift term to evade from the zero-control locus. We also show how the rate of convergence can be modified by tuning opportunely a pair of gains corresponding to the relative weights given to the controlled and uncontrolled parts of the stochastic generator.

2 Model formulation and background material

We need to recall some basics of quantum mechanics, quantum filtering and stochastic stability theory we will use later on. For an excellent introductory exposition to the statistical description of quantum systems see e.g. [21]. More details can be found in e.g. [22, 25] and references therein. For the theory of stochastic stability, main references are [4, 19], while for stochastic feedback stabilization we shall make use of the works of Florchinger [12, 11, 13]. Needless to say, the paper [28] presents similar control-theoretic perspective on most of the material mentioned below.

2.1 Quantum finite dimensional systems

In the standard statistical formulation of quantum mechanics [17, 26], to each quantum system is associated an Hilbert space $\mathcal{H}$, whose dimension depends essentially on the observable quantities we want to describe. In fact, physical observables are modelled as self-adjoint operators in the Hilbert space, the set of possible outcomes they can assume being their spectrum. In what follows, we will consider only *observables* with finite spectrum, thus represented as Hermitian matrices $C \in \mathrm{Herm}$ acting on $\mathbb{C}^N$.

Our knowledge of the system will be represented by a *density matrix* ρ belonging to the convex set

$$\mathcal{M} = \{\rho = \rho^\dagger \geq 0 \text{ s. t. } \mathrm{tr}(\rho) = 1\}.$$

The extremals of $\mathcal{M}$ are the one dimensional orthogonal projections. These are called *pure states*, and are equivalent to unit vectors in $\mathcal{H}_S$ up to an overall phase factor, by setting $\rho = |\psi\rangle\langle\psi| = \langle\psi, \cdot\rangle\psi$. We will use Dirac's notation for vectors: $|\psi\rangle \in \mathcal{H}$, $\langle\psi| \in \mathcal{H}^\dagger$. Unit vectors in $\mathcal{H}$ will be thus called *state vectors*. The usual notations $\langle\psi, \phi\rangle$ and $\langle\psi, C\phi\rangle$ will be replaced by $\langle\psi|\phi\rangle$, $\langle\psi|C|\phi\rangle$.

Consider an observable C and its spectral decomposition $C = \sum_i c_i P_i$, where $\{P_i\}$ is a *spectral family* of orthogonal projectors summing to the identity. Assume one can perform an ideally instantaneous measurement of C. The probability to obtain c_i as an outcome is then given by

$$p(c_i) = \mathrm{tr}(P_i \rho).$$

Thus, a density matrix determines also the expectation value of an observable

$$\langle C\rangle = \mathrm{tr}(\rho C).$$

If c_i is the observed outcome, the conditioned density matrix is given by the Lüders-von Neumann postulate as

$$\rho_i = \frac{P_i \rho P_i}{\mathrm{tr}(P_i \rho)}. \tag{1}$$

We will assume throughout the paper to work in measurement units such that $\hbar = 1$, where $\hbar$ is the Plank constant divided by 2π. In absence of measurements, the time evolution of an isolated quantum system is driven by the Hamiltonian H, i.e., the energy observable, as specified by:

$$\frac{d}{dt}\rho = -i[H, \rho].$$

Notice that the evolution of the unobserved system is deterministic, and, if $\rho = |\psi\rangle\langle\psi|$ is pure, it is equivalent to the Scrödinger equation

$$\frac{d}{dt}|\psi\rangle = -iH|\psi\rangle.$$

Beside of these basic postulates and definitions, to tackle our main problem we will need more sophisticated tools to deal with continuous-time measurement and subsequent state conditioning.

2.2 Continuous Measurement and Filtering Equation

For explicit derivations and more detailed discussions of the following topics, we refer to e.g. [7, 6, 8, 28].

In the description of a classical uncertain system, observable quantities are represented by real *random variables* defined on a suitable probability space $(\Omega, \Sigma, \mathbb{P})$. The state of the system, i.e., our knowledge about it, is subsumed in the probability measure $\mathbb{P}$. The observables form a commutative algebra, $L^\infty(\Omega, \Sigma, \mathbb{P}; \mathbb{R})$.

The quantum setting presented in the previous section can be interpreted as a non commutative generalization of a classical, discrete probability space [21]. The need for non-commutativity emerges experimentally, motivating the standard axiomatic formulation of the theory and being essentially captured by the canonical commutation relations [26]. Briefly, quantum *observables*, or non commutative random variables, form a (generally non commutative) Von Neumann algebra $\mathcal{A}$ and *events* are represented by orthogonal projections $\mathcal{E} \subset \mathcal{A}$ in the algebra. A *generalized probability measure* on $\mathcal{E}$ is needed to compute probabilities of events.

The finite dimensional setting we are working in leads to a concrete representation of the above abstract notions. We can identify $\mathcal{A}$ with Herm, and the set of generalized probability densities with $\mathcal{M}$, determining probabilities through $p(E) = \mathrm{tr}(\rho E)$, $E \in \mathcal{E}$.

One can then apply *quantum filtering theory* to obtain a conditional equation on $\mathcal{A}$ for the dynamics [7]. It essentially plays the role of the classical Kushner-Stratonovich equation. Thus assume we are continuously monitoring the observable C for the system of interest. In quantum optics, this can be accomplished e.g. for an angular momentum observable C by an homodyne detection experimental setup [31][1]. Since the observed C is time-invariant, we are

[1]In general, with homodyne detection one makes continuous-time measurements of gener-

conditioning the dynamics on the observation of a *commuting* quantum stochastic process, that leads to a dynamical equation driven by classical white noise (see e.g. [6]).

Let $(\Omega, \mathcal{E}, P)$ a (classical) probability space and $\{W_t, t \in \mathbb{R}^+\}$ a standard $\mathbb{R}$-valued Wiener process defined on this space. The homodyne detection measurement record can be written as the output of a stochastic dynamical system of the form:

$$dY_t = \sqrt{\eta}\,\mathrm{tr}(\rho_t C)dt + dW_t, \tag{2}$$

where $0 \leq \eta \leq 1$ represents the efficiency of the measurement. Denote with $\mathcal{E}_t$ the filtration associated to $\{W_t, t \in \mathbb{R}^+\}$.

Then one can derive the filtering equation determining the conditional evolution of the state for the measurement record (2), the Quantum Filtering or Stochastic Master Equation (SME) à la Itô:

$$\begin{aligned} d\rho_t &= (\mathcal{F}(H, \rho_t) + \mathcal{D}(C, \rho_t))dt + \mathcal{G}(C, \rho_t)dW_t \\ &= \left(-i[H, \rho_t] + \mu C\rho_t C - \frac{\mu}{2}(C^2\rho_t + \rho_t C^2)\right) dt + \sqrt{\mu\eta}(C\rho_t + \rho_t C - 2\mathrm{tr}(C\rho_t)\rho_t)dW_t, \end{aligned} \tag{3}$$

where $\mathcal{F}$ represent the Hamiltonian part, with H given by a drift and a (bilinear) control part $H = H_a + uH_b$, $\mathcal{D}$ and $\mathcal{G}$ are the drift and diffusion parts of the weak measurement performed along the observable $C = C^\dagger$. The parameter $\mu > 0$ represents the strength of the measurement.

Here ρ_t, the $\mathcal{M}$-valued solution of (3) given a constant initial condition ρ_0, that can be written explicitly as

$$\begin{aligned} \rho_t &= \Phi(\rho_0, t, 0), \quad \rho_0 \in \mathcal{M} \\ &= \rho_0 + \int_0^t (\mathcal{F}(H, \rho_s) + \mathcal{D}(C, \rho_s))\, ds + \int_0^t \mathcal{G}(C, \rho_s)dW_s, \end{aligned}$$

exists, is unique, adapted to the filtration $\mathcal{E}_t$ and $\mathcal{M}$-invariant by construction, see [7, 28].

Considering (2) and (3) together, one can recognize the basic structure of a *Kalman-Bucy* filter. Since $\langle C\rangle_t = \mathrm{tr}(C\rho_t)$ is the expectation of Y_t at time t, in (2) dW_t plays the role of *innovation process* in a filtering model. Other correspondences with the classical setting have been discussed and highlighted in e.g. [27].

We denote by $\mathcal{L}$ the infinitesimal generator à la Itô associated with the SME

alized operators of the form:

$$\mathcal{R}[\rho] = L\rho + \rho L^\dagger,$$

measuring the outgoing field from an optical cavity where we confine the system. The operator L depends on the system-field interaction occurring in the cavity. We will specialize to the case of Hermitian operators $L = L^\dagger = C$.

(3), written in a "symmetrized" fashion

$$\begin{aligned} \mathcal{L}\cdot = \frac{1}{2}\Bigg(& (\mathcal{F}(H,\rho_t)+\mathcal{D}(C,\rho_t))\frac{\partial\,\cdot}{\partial\rho} + \frac{\partial\,\cdot}{\partial\rho}(\mathcal{F}(H,\rho_t)+\mathcal{D}(C,\rho_t)) \\ & + \mathcal{G}^2(C,\rho_t))\frac{\partial^2\,\cdot}{\partial\rho^2} + \frac{\partial^2\,\cdot}{\partial\rho^2}\mathcal{G}^2(C,\rho_t))\Bigg). \end{aligned} \tag{4}$$

Consider now the case of perfect detection efficiency $\eta = 1$. In this case, a pure ρ_0 remains pure throughout the evolution. In fact, recalling that $\rho_t = |\psi_t\rangle\langle\psi_t|$ if and only if $\mathrm{tr}(\rho_t^2) = 1$, it suffices to prove the following.

Lemma 1 *Consider* (3) *with* ρ_t *a pure state and* $\eta = 1$*. Then* $d\mathrm{tr}(\rho_t^2) = 0$.

Proof. Using Ito's rule, we have:

$$\begin{aligned} d\mathrm{tr}(\rho_t^2) &= \mathrm{tr}(2\rho_t d\rho_t + (d\rho_t)^2) \\ &= \mathrm{tr}\left(2\rho_t\left(\mathcal{F}+\mathcal{D}\right)dt + 2\rho_t\mathcal{G}dW\right) + \mathrm{tr}\left(\mathcal{G}^2 dt\right) \\ &= \mu\mathrm{tr}\left(2\rho_t\left(C\rho_t C - \frac{1}{2}C^2\rho_t - \frac{1}{2}\rho_t C^2\right)\right)dt + \mu\eta\mathrm{tr}\left(((C\rho_t + \rho_t C - 2\langle C\rangle_t)\rho_t)^2\right)dt \\ &+ 4\sqrt{\mu\eta}\mathrm{tr}\left((C - \langle C\rangle_t)\rho_t^2\right)dW_t \\ &= \mu\mathrm{tr}\left(2(1+\eta)\rho_t C\rho_t C - 2(1-\eta)C^2\rho_t^2 + \eta\langle C\rangle_t^2\rho_t^2 - 8\eta\langle C\rangle_t\rho_t^2\right)dt \\ &+ 4\sqrt{\mu\eta}\mathrm{tr}\left((C - \langle C\rangle_t)\rho_t^2\right)dW_t. \end{aligned}$$

If $\eta = 1$ the term $C^2\rho_t^2$ disappears and

$$\begin{aligned} \mathrm{tr}\left(2\rho_t(\mathcal{F}+\mathcal{D}) + \mathcal{G}^2\right) &= \mu\mathrm{tr}(4C\rho_t C\rho_t + 4\langle C\rangle_t^2\rho_t^2 - 8\langle C\rangle_t\rho_t^2) \\ &= 4\mu\mathrm{tr}\left(((C - \langle C\rangle_t)\rho_t)^2\right). \end{aligned} \tag{5}$$

The assumption of starting with a pure state $\rho_t = |\psi_t\rangle\langle\psi_t|$ implies for example $\rho_t^2 = \rho_t$, $\mathrm{tr}(\rho_t^2) = 1$ and $\mathrm{tr}(\rho_t C\rho_t C) = \langle\psi_t|C|\psi_t\rangle\langle\psi_t|C|\psi_t\rangle = \langle C\rangle_t^2$. Hence

$$\begin{aligned} 4\mu\mathrm{tr}\left(((C - \langle C\rangle_t)\rho_t)^2\right) &= 4\mu\mathrm{tr}(C\rho_t C\rho_t - 2\langle C\rangle_t C\rho_t^2 + \langle C\rangle_t^2\rho_t^2) \\ &= 4\mu\left(\langle C\rangle_t^2 - 2\langle C\rangle_t\mathrm{tr}(C\rho_t) + \langle C\rangle_t^2\right) \\ &= 0 \end{aligned} \tag{6}$$

and, likewise,

$$\mathrm{tr}\left((C - \langle C\rangle_t)\rho_t^2\right) = 0 \tag{7}$$

□

Thus, the SME (3) becomes equivalent to a Stochastic Schrödinger Equation (SSE) of the form [8]:

$$d|\psi_t\rangle = \left(-iH - \frac{\mu}{2}(C - \langle C\rangle_t)^2\right)|\psi_t\rangle dt + \sqrt{\mu}(C - \langle C\rangle_t)|\psi_t\rangle dW_t. \tag{8}$$

In particular (see [3] for details), the state space in this case, call it $\mathcal{S}$, reduces to a homogeneous space of the Lie group $U(N)$:

$$\mathcal{S} = U(N)/(U(N-1)\times U(1)) \subset \mathcal{M},$$

of $\dim(\mathcal{S}) = N^2 - N$.

Equations of the form (8) have been proposed as extensions to standard quantum mechanics in order to give a dynamical model for the after measurement "state collapse", i.e. postulate (1) (see e.g. [1] and references therein).

2.3 Elements of stochastic stability

Consider ρ_d an equilibrium solution of (3), i.e. $\rho_d \in \mathcal{M}$: $\mathcal{F}(H,\rho_d)+\mathcal{D}(C,\rho_d) = \mathcal{G}(C,\rho_d) = 0$.

Definition 1 *The equilibrium ρ_d of the SME* (3) *is said to be*

1. stable in probability *if for any $s \geqslant 0$ and $\epsilon \geqslant 0$*

$$\lim_{\rho_0\to\rho_d} P\left(\sup|\Phi(\rho_0,t,s)-\rho_d| > \epsilon\right) = 0; \tag{9}$$

2. locally asymptotically stable in probability *if* (9) *holds and*

$$\lim_{\rho_0\to\rho_d} P\left(\lim_{t\to\infty}|\Phi(\rho_0,t,s)-\rho_d| = 0\right) = 1; \tag{10}$$

3. almost globally asymptotically stable in probability *if* (9) *holds and* (10) *is true $\forall\ \rho_0 \in \mathcal{M}$ except for at most a finite number of isolated points of $\mathcal{M}$;*

4. globally asymptotically stable in probability *if* (9) *holds and*

$$P\left(\lim_{t\to\infty}|\Phi(\rho_0,t,s)-\rho_d| = 0\right) = 1. \tag{11}$$

We shall make use of the following Lyapunov conditions.

Theorem 1 *Denote by $\mathcal{B}_{\mathcal{M}}$ the intersection of an open neighborhood $\mathcal{B} \in \mathrm{Herm}$ with the set of density operators: $\mathcal{B}_{\mathcal{M}} = \mathcal{B}\cap\mathcal{M}$. Assume $\exists$ a $\mathbb{R}$-valued $V \in C^2(\mathcal{B}_{\mathcal{M}},\mathbb{R})$ with $V(\rho_d) = 0$, $V(\mathcal{B}_{\mathcal{M}}\smallsetminus\{\rho_d\}) > 0$ and such that $\mathcal{L}V_t = \mathcal{L}V(\rho_t) \leqslant 0$ (resp. $\mathcal{L}V_t < 0$) $\forall\ \rho_t \in \mathcal{B}_{\mathcal{M}}\smallsetminus\{\rho_d\}$. Then ρ_d is locally stable (resp. locally asymptotically stable) in probability.*

Since (3) is invariant in $\mathcal{M}$, the restriction of a full neighborhood to $\mathcal{B}_{\mathcal{M}}$ is not altering the standard proof of this result (reported for example in [19]).

Just like in the deterministic case, a well-established version of the LaSalle's invariance principle provides the ω-limit set of a stable stochastic process.

Theorem 2 *([20]) If $\exists$ a Lyapunov function $V \in C^2(\mathcal{M}, \mathbb{R})$ such that $\mathcal{L}V_t \leqslant 0$ $\forall\ \rho_t \in \mathcal{M}$, then the solution ρ_t of* (3) *tends with probability 1 to the largest invariant set whose support is contained in $\mathcal{N} = \{\rho_t \in \mathcal{M}$ s. t. $\mathcal{L}V_t = 0\ \ \forall\ t \geqslant 0\}$.*

Since we have the semiclassical approximation $\mathcal{F}(H, \rho_t) = -i[H_a + uH_b, \rho_t]$ with u a control function, the SME (3) belongs to the class of stochastic affine in control nonlinear differential systems, for which a number of stabilizability conditions have been developed [11, 12, 13]. Call $\mathcal{L}_0$ the infinitesimal generator of the uncontrolled part of the dynamics

$$\mathcal{L}_0 = \mathcal{L} - \mathcal{L}_b u,$$

where

$$\mathcal{L}_b \cdot = -\frac{i}{2}\left([H_b, \rho_t]\frac{\partial \cdot}{\partial \rho} + \frac{\partial \cdot}{\partial \rho}[H_b, \rho_t]\right).$$

Definition 2 *The SME* (3) *satisfies a* stochastic Lyapunov condition *at ρ_d if $\exists\ \mathcal{B}_\mathcal{M} \subset \mathcal{M}$ and a Lyapunov function $V \in C^2(\mathcal{B}_\mathcal{M}, \mathbb{R})$ such that for all $\rho_t \in \mathcal{B}_\mathcal{M} \setminus \{\rho_d\}$ for which the Lie derivative $\mathcal{L}_b V_t = 0$ it is $\mathcal{L}_0 V_t < 0$. The stochastic Lyapunov condition is almost global if $\mathcal{B}_\mathcal{M}$ is all of $\mathcal{M}$ except for at most a finite number of isolated points.*

When this condition is fulfilled, V is said to be a stochastic control Lyapunov function for (3). Our feedback synthesis relies on this condition, but does not follow any of the standard constructions for control Lyapunov functions [12].

We shall instead make use of the following Jurjevic-Quinn type of stochastic stabilizability condition (see [12] Def. 3.5 and [11] Def. 3.1).

Theorem 3 *Assume $\exists\ \mathcal{B}_\mathcal{M} \subset \mathcal{M}$, $\rho_d \in \mathcal{B}_\mathcal{M}$ and $V \in C^2(\mathcal{B}_\mathcal{M}, \mathbb{R})$, $V(\rho_d) = 0$, $V(\mathcal{B}_\mathcal{M} \setminus \{\rho_d\}) > 0$, such that*

1. *$\mathcal{L}V_t \leqslant 0\ \forall\ \rho_t \in \mathcal{B}_\mathcal{M}$;*
2. *the set $\left\{\rho_t \in \mathcal{B}_\mathcal{M}\ \text{s. t.}\ \mathcal{L}^{r+1}V_t = \mathcal{L}^r\mathcal{L}_b V_t = 0,\ r \in \mathbb{N}\right\} = \{\rho_d\}$.*

Then the feedback $u_t = -\mathcal{L}_b V_t$ renders the equilibrium solution ρ_d locally asymptotically stable in probability.

All definitions and theorems carry on unchanged when $\mathcal{M}$ and (3) are replaced by $\mathcal{S}$ and (8).

3 Continuous state reduction: the feedback stabilization problem

The problem we will discuss and solve can be stated as follows.

Problem 1 *Find a smooth control law $u(t)$ that (almost) globally stabilizes in probability the pure state $\rho_d = |\psi_d\rangle\langle\psi_d|$ of an N-dimensional quantum system, whose dynamic is described by the filtering equation* (3) *conditioned by the continuous observation of an observable C.*

We shall propose several choices of $u(t)$ as linear and nonlinear feedback laws based on the conditional estimate for the state ρ_t at time t. In the physics literature, this approach has been baptized bayesian feedback [30] and of course requires the real time integration of (3). Since $u(t) = u_t$ is smooth and adapted to the filtration $\mathcal{E}_t$, the closed loop solution exists and is unique in a global sense.

The first feedback law proposed (§ 3.1) is linear and allows to achieve only local stabilizzability for the SSE (§ 3.2). If we choose a Lyapunov function that includes the variance of the measurement (§ 3.3), then two modifications of the linear law are easily identifiable and are presented in § 3.4 and § 3.5. The first one yields almost global asymptotic stability but only for the SSE, while with the second one we achieve almost global asymptotic stability in $\mathcal{M}$ for the SME. The relation between rate of convergence and gain tuning for the latter feedback is discussed in § 3.6.

Let us first make suitable assumptions on $\mathcal{F}$ and C. In order for $|\psi_d\rangle$ to be an equilibrium, assume $|\psi_d\rangle$ is an eigenstate of H_a and of C. To avoid unnecessary complications, assume further that the spectrum of C is non-degenerate and that $[H_a, C] = 0$. With this choice, it is always possible for example to fix a basis such that $\rho_d = |\psi_d\rangle\langle\psi_d|$ is diagonal and so are the free Hamiltonian H_a and C. We want to choose u so that ρ_d is rendered an attractor for the SME. Since the spectrum of C is non-degenerate, $\exists\ N-1$ state vectors other than $|\psi_d\rangle$, $|\psi_j\rangle$, $j = 1, \ldots, N-1$, that are eigenstates of C. When C is diagonal, they correspond to diagonal density matrices $\rho_j = |\psi_j\rangle\langle\psi_j|$ with diagonal elements $\{1, 0, \ldots, 0\}$. Following the terminology of [3], we shall call these *antipodal states* of ρ_d. Denote with $\mathcal{J}$ the union of such antipodal points: $\mathcal{J} = \bigcup_{j=1}^{N-1}\{|\psi_j\rangle\langle\psi_j|\}$. Finally, to avoid trivial cases, assume that Graph(H_b) is connected, i.e., that all transitions between energy levels are enabled by the control field.

3.1 A linear feedback controller

A natural choice for a Lyapunov function is the distance between density operators induced by the Hilbert-Schmidt norm [24]:

$$V_1 = \mathrm{tr}(\rho_d^2) - \mathrm{tr}(\rho_d \rho). \tag{12}$$

One clearly sees that in the stochastic differential (4) the quadratic part can be neglected since V_1 is linear in ρ_t:

$$\begin{aligned} \mathcal{L}V_{1,t} &= -\mathrm{tr}((-i[H, \rho_t] + \mathcal{D}(C, \rho_t))\rho_d) \\ &= -\mathrm{tr}(-i[H_b, \rho_t]\rho_d)u = \mathcal{L}_b V_{1,t} u. \end{aligned} \tag{13}$$

The non-Hamiltonian part vanishes because C and ρ_d commute and the cyclic property of trace holds (see the proof of Proposition 1): $\mathrm{tr}(-i[H_a, \rho_t]\rho_d) = 0$

and

$$\operatorname{tr}\left(\left(C\rho_t C - \frac{1}{2}(C^2\rho_t + \rho_t C^2)\right)\rho_d\right) = \operatorname{tr}\left((C^2\rho_t - C^2\rho_t)\rho_d\right) = 0. \tag{14}$$

Hence in the SME (3) this stabilization design is concerned only with the unitary part of the evolution and has the natural solution

$$u_t = k\,\operatorname{tr}(-i[H_b, \rho_t]\rho_d), \quad k > 0. \tag{15}$$

Since the closed loop system has

$$\mathcal{L}V_{1,t} = -k\,\operatorname{tr}^2(-i[H_b, \rho_t]\rho_d) \leqslant 0,$$

one needs to study the ω-limit set of (3) with the feedback (15). This is the difficult part of the linear feedback design (15). We certainly have the following for the set of (pure or mixed) diagonal density operators, call it $\mathcal{Q}$ (often call eigenensemble [32]).

Proposition 1 *Consider the SME* (3)*. For $\rho_t \in \mathcal{Q}$, the state dynamics are not influenced by the feedback* (15)*. Moreover, $\mathcal{Q}$ is invariant.*

Proof. It suffices to notice that:

$$\operatorname{tr}(-i[H_b, \rho_t]\rho_d) = 0 \quad \forall \rho_t \text{ such that } [\rho_t, \rho_d] = 0, \tag{16}$$

since

$$\operatorname{tr}(-i[H_b, \rho_t]\rho_d) = \operatorname{tr}(-i[H_b, \rho_t\rho_d]) = 0,$$

as any commutator under the trace operation. Similarly, $[\rho_t, C] = [\rho_t, C^2] = 0$. Thus, the only term affecting the dynamics is the diffusion term, which is also diagonal. Hence the diagonal set is invariant. □

Since $\mathcal{Q}$ is a convex set, we have the following.

Corollary 1 *For the system* (3) *with the feedback* (15) $\nexists$ *open neighborhoods* $\mathcal{B} \in \mathrm{Herm}$ *such that ρ_d is locally asymptotically stable in probability in $\mathcal{B}_\mathcal{M} = \mathcal{B} \cap \mathcal{M}$.*

In fact, the dynamics confined to $\mathcal{Q}$ is only a fluctuation and since the probability of collapse to the eigenstate $\rho_a = |\psi_j\rangle\langle\psi_j| \in \mathcal{J}$ is equal to $P_{\rho_a} = \operatorname{tr}(\rho_a\rho)$, it is never 1 if $\rho \in \mathcal{Q} \setminus \mathcal{J}$.

3.2 Local stabilization of a class of Stochastic Schrödinger equations

Consider the case $\eta = 1$. As discussed in Section 2.2, the SME (3) is equivalent to the SSE (8) and the state space is $\mathcal{S}$. From the transversality of $\mathcal{S}$ with respect to the set of diagonal Hermitian matrices (see Theorem E.2 of [14]), the intersection of $\mathcal{S}$ with $\mathcal{Q}$ is just $\mathcal{J} \cup \{\rho_d\}$.

For this relevant particular case, one can show the following.

Theorem 4 *Assume $\eta = 1$ and that the following Kalman-like rank condition is satisfied:*

$$\mathrm{rank}(-i[H_b,\, \rho_d],\, [A,\, -i[H_b,\, \rho_d]], \ldots,\, \underbrace{[\,A\,,\, \ldots\,,\, [\,A\,}_{N^2-N-1\ times},\, -i[H_b,\, \rho_d]]\ldots]) = N^2 - N \tag{17}$$

where A is either $-iH_a$ or C. Then the feedback law (15) *renders the equilibrium solution of* (3) *locally asymptotically stable in probability.*

Proof. In order to prove Theorem 4, we need a related deterministic result. Consider the deterministic unitary bilinear control system obtained from (3) in correspondence of $C = 0$

$$\dot\rho = -i[H_a,\, \rho] - i\, u\, [H_b,\, \rho], \qquad \rho \in \mathcal{S}, \tag{18}$$

and its tangent linear system at ρ_d

$$\dot\rho = -i[H_a,\, \rho] - i\, u\, [H_b,\, \rho_d]. \tag{19}$$

Lemma 2 *Assume H_a strongly regular. If* (19) *satisfies the Kalman rank condition* (17) *with $A = -iH_a$, then ρ_d is locally asymptotically stabilizable by means of the feedback* (15)*.*

Recall that $A = A^\dagger$ strongly regular means A nondegenerate and with all transition frequencies (i.e., all differences of eigenvalues) that are different. The proof of this Lemma is available in [3] (see also [23]). It essentially relies on the Jurdjevic-Quinn condition [18]: starting from the identity

$$\begin{aligned} u &= \mathrm{tr}(-i[H_b,\, \rho]\rho_d) = 0 \\ &= -\mathrm{tr}(-i[H_b,\, \rho_d]\rho) = 0, \end{aligned}$$

compute sufficiently many derivatives

$$(-1)^2\mathrm{tr}(-i[H_a,\, -i[H_b,\, \rho_d]]\rho) = 0 \tag{20a}$$

$$(-1)^3\mathrm{tr}(-i[H_a,\, -i[H_a,\, -i[H_b,\, \rho_d]]]\rho) = 0 \tag{20b}$$

$$\vdots$$

$$(-1)^{r+1}\mathrm{tr}(\underbrace{-i[H_a, \ldots,\, -i[H_a}_{r\ \text{times}},\, -i[H_b,\, \rho_d]]\ldots]\rho) = 0. \tag{20c}$$

The strong regularity condition of H_a guarantees that all the commutators in (20) are linearly independent up to a number equal to $\dim(\mathcal{S})$, i.e., (17) holds with $A = -iH_a$, implying the controllability of the linearization (19) and thus the local stabilizability of the original system (18). Since $\mathcal{S}$ is a manifold (and not an Euclidean space like in [18]), the condition is only local.

Turning to the stochastic system (3) and the stochastic Jurdjevic-Quinn condition of Theorem 3, since we have a linear Lyapunov function, when computing $\mathcal{L}_0\mathcal{L}_b V_{1,t}$ in $u_t = 0$ the quadratic part of $\mathcal{L}_0$ does not appear:

$$\begin{aligned}\mathcal{L}_0\mathcal{L}_b V_{1,t} &= \operatorname{tr}\left(-i[H_b,\, \mathcal{F}(H_a,\rho_t) + \mathcal{D}(C,\rho_t)]\rho_d\right)\\ &= \operatorname{tr}\left(-i[H_b,\, -i[H_a,\, \rho_t]]\rho_d\right) + \operatorname{tr}\left(-i[H_b,\, -\frac{\mu}{2}[C,\, [C,\, \rho_t]]]\rho_d\right)\\ &= (-1)^2\operatorname{tr}\left(-i[H_a,\, -i[H_b,\, \rho_d]]\rho_t\right) + (-1)^3\operatorname{tr}\left(-\frac{\mu}{2}[C,\, [C,\, -i[H_b,\, \rho_d]]]\rho_t\right).\end{aligned} \tag{21}$$

Similarly,

$$\begin{aligned}\mathcal{L}_0^2\mathcal{L}_b V_{1,t} =& (-1)^3\operatorname{tr}\left(-i[H_a,\, -i[H_a,\, -i[H_b,\, \rho_d]]]\rho_t\right)\\ &+ (-1)^4\operatorname{tr}\left(-i[H_a,\, -\frac{\mu}{2}[C,\, [C,\, -i[H_b,\, \rho_d]]]]\rho_t\right)\\ &+ (-1)^4\operatorname{tr}\left(-\frac{\mu}{2}[C,\, [C,\, -i[H_a,\, -i[H_b,\, \rho_d]]]]\rho_t\right)\\ &+ (-1)^5\operatorname{tr}\left(-\frac{\mu}{2}[C,\, [C,\, -\frac{\mu}{2}[C,\, [C,\, -i[H_b,\, \rho_d]]]]]\rho_t\right)\end{aligned} \tag{22}$$

and so on for $\mathcal{L}_0^r\mathcal{L}_b V_{1,t}$, $r > 2$. Hence, in the case of H_a strongly regular the stochastic Jurdjevic-Quinn condition holds whenever Lemma 2 holds, as the Lie algebra spanned by the commutators in $\mathcal{L}_0^r\mathcal{L}_b V_1 = 0$ is at least as large as the one spanned by the commutators appearing in the deterministic conditions $\frac{d^r u}{dt^r} = 0$ [2]. However, even when H_a not strongly regular but C is, the stochastic Jurdjevic-Quinn condition still holds as the terms

$$-\frac{\mu}{2}[C,\, [C,\, \ldots, -\frac{\mu}{2}[C,\, [C,\, -i[H_b,\, \rho_d]]]\ldots]]$$

still provide the needed linearly independent commutators (see [2] for explicit recursive computations of the commutators involved). Since this is implied by (17), the proof is completed. The condition is local just like its deterministic counterpart. □

[2]Notice that since V_1 is linear, in Theorem 4 we are only concerned with the linear part of the infinitesimal generators and this allows to infer the stochastic Jurdjevic-Quinn condition directly in terms of the Lie algebra, just like in its deterministic counterpart. Redoing the computations above for the corresponding Stratonovich equation (for which Lie algebraic conditions can be made fully rigorous for any V_1)

$$d\rho_t = (\mathcal{F}(H,\rho_t) + \mathcal{D}(C,\rho_t) - \frac{1}{2}\mathcal{G}_s(C,\rho_t))dt + \mathcal{G}(C,\rho_t)\circ dW_t,$$

where the quadratic term in the drift, $\mathcal{G}_s(C,\rho_t)$, is (in the case C is traceless)

$$\begin{aligned}\mathcal{G}_s(C,\rho_t) &= \frac{1}{2}\left(\mathcal{G}(C,\rho_t)\frac{\partial\mathcal{G}(C,\rho_t)}{\partial\rho} + \frac{\partial\mathcal{G}(C,\rho_t)}{\partial\rho}\mathcal{G}(C,\rho_t)\right)\\ &= \mu\eta\left(2C\rho_t C + C^2\rho_t + \rho_t C^2 - 4\langle C\rangle_t(C\rho_t + \rho_t C) + 4\langle C\rangle_t^2\rho_t\right),\end{aligned}$$

one arrives at the same conclusion.

Notice that the feedback law (15) rewritten for the SSE (8) is

$$\begin{aligned} u_t &= k\mathrm{tr}(-i[H_b,\, |\psi_t\rangle\langle\psi_t|]|\psi_d\rangle\langle\psi_d|) \\ &= -ik(\langle\psi_t|\psi_d\rangle\langle\psi_d|H_b|\psi_t\rangle - \langle\psi_t|\psi_d\rangle^*\langle\psi_d|H_b|\psi_t\rangle^*) \\ &= -2k\mathrm{Im}(\langle\psi_t|\psi_d\rangle\langle\psi_d|H_b|\psi_t\rangle). \end{aligned} \tag{23}$$

Remark: Assume, without loss of generality, that $\rho_d = \mathrm{diag}\{1, 0, \ldots, 0\}$. Then in order for (17) to hold it must be $(H_b)_{1j} = (H_b)_{j1}^* \neq 0$, i.e., the control Hamiltonian H_b must "enable" all transitions from $|\psi_d\rangle$ to all other eigenstates $|\psi_j\rangle$, $j = 1, \ldots, N-1$.

When $\eta < 1$, the structure of the state space is larger than $\mathcal{S}$ and in particular the transversality of the state space with respect to $\mathcal{Q}$ no longer holds, hence Theorem 4 does not apply.

3.3 A variance-based Lyapunov condition

The feedback (15) is the same linear controller used in [3] to study the deterministic stabilization problem with state feedback (corresponding to $C = 0$). In that setting, its region of attraction does not correspond to the entire state space. In our stochastic problem, there is the additional requirement that the ω-limit set has to be invariant also to the flow of the diffusion part. We will exploit this feature, considering, instead of V_1, the following candidate Lyapunov function:

$$V = V_1 + V_2 = \mathrm{tr}(\rho_d^2) - \mathrm{tr}(\rho_d\rho) + \mathrm{tr}(C^2\rho) - \mathrm{tr}^2(C\rho). \tag{24}$$

Clearly $V \geqslant 0$, $V = 0$ only in $\rho = \rho_d$. The function V_2 in (24) is the variance of the filtering process along C:

$$V_2 = \langle C^2\rangle - \langle C\rangle^2. \tag{25}$$

V_2 has the property of being a positive semidefinite Morse function on $\mathcal{M}$, i.e., a function whose critical points are nondegenerate [16] and can be used to attain a stochastic Lyapunov condition.

Theorem 5 *The system* (3) *satisfies an almost global stochastic Lyapunov condition with respect to the Lyapunov function V given in* (24)*. The only points for which the stochastic Lyapunov condition is not satisfied are the $N-1$ antipodal states of ρ_d.*

Proof. To discuss the asymptotic properties of V_2, it is useful to notice first that, using $[H, C] = 0$, the cyclic property of the trace and Ito's rule:

$$\begin{aligned} d\langle C\rangle_t &= \mathrm{tr}(Cd\rho_t) \\ &= \mathrm{tr}(-i[H,\, \rho_t]C)dt + 2\sqrt{\mu\eta}\mathrm{tr}(C(C - \langle C\rangle_t)\rho_t)dW_t \\ &= u\mathrm{tr}(-i[H_b,\, \rho_t]C)dt + 2\sqrt{\mu\eta}(\mathrm{tr}(C^2\rho_t) - \langle C\rangle_t^2))dW_t \\ &= u\mathrm{tr}(-i[H_b,\, \rho_t]C)dt + 2\sqrt{\mu\eta}V_{2,t}dW_t. \end{aligned} \tag{26}$$

Thus, for the system (3) we have:

$$\begin{aligned}dV_{2,t} &= d\langle C^2\rangle_t - 2\langle C\rangle_t d\langle C\rangle_t - (d\langle C\rangle_t)^2\\ &= (\mathrm{tr}(-i[H,\,\rho_t](C^2 - 2\langle C\rangle_t C)) - 4\mu\eta V_{2,t}^2)dt\\ &+ 2\sqrt{\mu\eta}(\mathrm{tr}(C^3\rho_t - \langle C\rangle_t C^2\rho_t) - 2\mathrm{tr}(\langle C\rangle_t C^2\rho_t) + 2\langle C\rangle_t^3)dW_t\\ &= (u\ \mathrm{tr}(-i[H_b,\,\rho_t](C^2 - 2\langle C\rangle_t C)) - 4\mu\eta V_{2,t}^2)dt + 2\sqrt{\mu\eta}\sigma(C,\rho_t)dW_t,\end{aligned} \tag{27}$$

with $\sigma(C,\rho_t) = \langle C^3\rangle_t - 3\langle C\rangle_t\langle C^2\rangle_t + 2\langle C\rangle_t^3$ the 3rd central moment.

From (24), (27) and (13), the stochastic differential à la Itô for V is

$$\mathcal{L}V_t = -tr(-i[H_b,\rho_t](\rho_d + 2\langle C\rangle_t C - C^2))u - 4\mu\eta\left(\langle C^2\rangle_t - \langle C\rangle_t^2)\right)^2. \tag{28}$$

Considering the zero-control behavior, notice that if $V_{2,t} = 0$ the system state must be in an eigenstate of C and hence $\sigma(C,\rho_t) = 0$. Thus $V_{2,t} = 0$ is stationary for (27). The convergence to the eigenstates follows applying Theorem 1 to any bounded right interval of zero containing $V_{2,0} \neq 0$. In fact,

$$\mathcal{L}V_{2,t}|_{u=0} = -4\mu\eta V_{2,t}^2 < 0$$

proves the convergence in probability of the variance to zero. Hence, for the closed loop system, looking at (28), $\mathcal{L}_0 V_t = \mathcal{L}V_{2,t}|_{u=0} < 0$ everywhere, except at the $N-1$ other eigenvalues of C and Definition 2 applies almost globally. □

3.4 Almost global stabilization of the SSE by nonlinear feedback

Theorem 6 *Assume $\eta = 1$. The feedback law*

$$u_t = k\,tr(-i[H_b,\rho_t](\rho_d + 2\langle C\rangle_t C - C^2)), \quad k > 0, \tag{29}$$

renders the equilibrium solution ρ_d of the SSE (3) *almost globally asymptotically stable in probability, with region of attraction given by $\mathcal{S} \smallsetminus \mathcal{J}$.*

Proof. The feedback (29) makes (28) into a negative semidefinite sum of squares:

$$\mathcal{L}V_t = -ktr^2(-i[H_b,\rho_t](\rho_d + 2\langle C\rangle_t C - C^2)) - 4\mu\eta\left(\langle C^2\rangle_t - \langle C\rangle_t^2)\right)^2 \leqslant 0. \tag{30}$$

Calling $\mathcal{N}_\mathcal{S}$ the set of critical points of V: $\mathcal{N}_\mathcal{S} = \{\rho_t \in \mathcal{S}$ s. t. $\mathcal{L}V_t = 0\}$, from Theorem 2 we need to compute the ω-limit set of the closed loop inside $\mathcal{N}_\mathcal{S}$. Since (30) is a sum of squares, $\mathcal{N}_\mathcal{S}$ must be a subset of $\{\rho_t \in \mathcal{S}$ s. t. $\mathcal{L}V_{2,t}|_{u=0} = 0\} = \mathcal{J}$. Since in $\rho_a \in \mathcal{J}$, $\rho_a \neq \rho_d$, $V(\rho_a) > 0$, ρ_a cannot be asymptotically stable in probability. □

Remark: Notice that unlike Theorem 4, Theorem 6 does not require any special structure for H_b (compare Remark following Theorem 4). In loose terms, one could say that while the design of Theorem 4 relies on a controllable linearization, in Theorem 6 uncontrollable, asymptotically stable modes are allowed in the linearization.

Of course, it can be easily shown that the feedback (29) can be used also in place of (15) in Theorem 4.

3.5 Almost global stabilization of the SME by nonlinear feedback

When $\eta \leqslant 1$, Proposition 1 and Corollary 1 still hold also with the feedback (29). Hence $\rho_t \in \mathcal{Q}$ is not attracted to ρ_d with probability 1. Although simulation results seem to suggest that with both the feedback laws (15) and (29) all non-diagonal density operators are attracted with probability 1 to ρ_d, we do not see any clear way to prove it. The problem can however be solved in full generality by a different choice of feedback.

Theorem 7 *The system* (3) *with feedback law*

$$u_t = tr(-i[H_b, \rho_t](\rho_d + 2\langle C\rangle_t C - C^2)) - 4\sqrt{\mu\eta}\left(\langle C^2\rangle_t - \langle C\rangle_t^2\right) \tag{31}$$

admits ρ_d as equilibrium solution which is almost globally asymptotically stable. The only states in $\mathcal{M}$ which are not attracted in probability to ρ_d are its $N-1$ antipodal states.

Proof.

Consider still the Lyapunov function V given in (24). It is easy to see that the nonlinear feedback (31) completes $\mathcal{L}V_t$ in (28) to a square:

$$\mathcal{L}V_t = -\left(\mathrm{tr}(-i[H_b, \rho_t](\rho_d + 2\langle C\rangle_t C - C^2)) - 2\sqrt{\mu\eta}\left(\langle C^2\rangle_t - \langle C\rangle_t^2\right)\right)^2 \leqslant 0.$$

Notice first that $\rho_a \in \mathcal{J}$ is a stationary point of both open and closed loop systems: $\mathcal{L}_0 V(\rho_a) = \mathcal{L}V(\rho_a) = 0$. We need to show that $\mathcal{N}$ cannot belong to the ω-limit set of the closed loop (with the exclusion of $\mathcal{J}$) and that there is no subset of $\mathcal{M} \smallsetminus \mathcal{J}$ which can remain undriven for $t \to \infty$. The crucial difference with respect to (29) is that (31) implies $\mathcal{L}_0 V_t \neq \mathcal{L}V_t|_{u=0}$. From Theorem 5, the stochastic Lyapunov condition holds true and guarantees $\mathcal{L}_0 V_t < 0$ almost globally. In particular, notice that $\mathcal{L}_0 V_t < 0$ everywhere in $\mathcal{N} \smallsetminus \mathcal{J}$, hence $\mathcal{N} \smallsetminus \mathcal{J}$ cannot belong to the ω-limit set. In addition $u_t|_{\mathcal{N}} = -2\sqrt{\mu\eta}V_{2,t} < 0$, i.e., the set of spurious critical points is evaded also by means of the control action. The other claim follow by the similar observation that the zero-feedback locus

$$U = \{\rho_t \in \mathcal{M} \text{ s. t. } \mathrm{tr}(-i[H_b, \rho_t](\rho_d + 2\langle C\rangle_t C - C^2)) = 4\sqrt{\mu\eta}\left(\langle C^2\rangle_t - \langle C\rangle_t^2\right)\}$$

is never invariant to $\mathcal{L}_0 V_t$ outside $\mathcal{J}$. □

Remark: If the domain of attraction of (31) is $\mathcal{M} \smallsetminus \mathcal{J}$, the critical points of $\mathcal{J}$ are automatically repulsive equilibria of the closed loop system. If for some $s \geqslant 0$, $\rho_s \in \mathcal{J}$, then an arbitrarily small unitary open loop perturbation is enough to evade from $\mathcal{J}$.

3.6 Gain tuning and rate of convergence

The performances of the feedback design (31) can be improved by adding, and tuning appropriately, two different gains. Instead of (31), consider the following

$$u_t = k^2 \operatorname{tr}(-i[H_b, \rho_t](\rho_d + (2\langle C\rangle_t C - C^2)/\ell^2)) - 4\frac{k\sqrt{\mu\eta}}{\ell}\left(\langle C^2\rangle_t - \langle C\rangle_t^2\right), \quad (32)$$

with $k > 0$, $\ell > 0$. While k corresponds to the usual feedback gain, ℓ can be thought of as a rescaling of the variance in the Lyapunov function (24):

$$\tilde{V} = V_1 + \frac{1}{\ell^2}V_2.$$

The corresponding closed loop stochastic generator is then:

$$\mathcal{L}\tilde{V}_t = -\left(k \operatorname{tr}(-i[H_b, \rho_t](\rho_d + (2\langle C\rangle_t C - C^2))/\ell^2) - 2\frac{\sqrt{\mu\eta}}{\ell}\left(\langle C^2\rangle_t - \langle C\rangle_t^2\right)\right)^2. \quad (33)$$

Proposition 2 *The equilibrium solution ρ_d of the system* (3) *with the feedback law* (32) *is almost globally asymptotically stable for all $k > 0$ and $\ell > 0$. When $\ell \to \infty$ one recovers the linear feedback* (15).

Proof. From (33), for the closed loop system $\mathcal{L}\tilde{V}_t < 0$ in $\mathcal{N} \setminus \mathcal{J}$, $\forall\, k > 0$ and $\forall$ finite $\ell > 0$, as the proof of Theorem 7 still applies. The limit behavior follows by inspection of (32). □

Notice that tuning ℓ does not correspond to modulating the strength of the weak measurement μ.

The effect of ℓ is to change the influence of V_2 on the closed loop dynamics. In terms of the Hilbert-Schmidt norm V_1, from (13), its close loop differential is given by

$$\begin{aligned}&- k^2 \operatorname{tr}^2(-i[H_b, \rho_t]\rho_d) - \frac{k^2}{\ell^2}\operatorname{tr}(-i[H_b, \rho_t](2\langle C\rangle_t C - C^2))\operatorname{tr}(-i[H_b, \rho_t]\rho_d)\\ &+ 4\frac{k\sqrt{\mu\eta}}{\ell}\operatorname{tr}(-i[H_b, \rho_t]\rho_d)V_{2,t}.\end{aligned}$$

Since only the first term is sign definite, convergence in probability to the target state $\forall\, \rho_0 \in \mathcal{M} \setminus \mathcal{J}$ is guaranteed to be faster if we raise the gain ℓ (recall that V_2 was introduced only to perturb the "symmetry" of the problem).

4 Example: 2 level case

For $N = 2$, it is possible to give a simple pictorial description of the trajectories of the system, provided one chooses a real parametrization, like the triple (x, y, z) representing the Bloch vector: $\rho = \frac{1}{2}(I_2 + x\sigma_x + y\sigma_y + z\sigma_z)$, with

σ_x, σ_y, σ_z the Pauli matrices. Consider the observable $C = \sigma_z$, its eigenstate $\rho_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and the Hamiltonian $H_a = h_a \sigma_z$, $H_b = \sigma_x$. Since

$$\begin{aligned}
-i[H_a, \rho_t] &= h_a(-y_t \sigma_x + x_t \sigma_y), \\
-i[H_b, \rho_t] &= -z_t \sigma_y + y_t \sigma_z, \\
\mathcal{D}(C, \rho_t) &= -\mu(x_t \sigma_x + y_t \sigma_y), \\
\mathcal{G}(C, \rho_t) &= \sqrt{\mu\eta}(-x_t z_t \sigma_x - y_t z_t \sigma_y + (1 - z_t^2)\sigma_z),
\end{aligned}$$

the SME (3) in terms of the Bloch vector is

$$\begin{aligned}
dx_t &= (-h_a y_t - \mu x_t)dt - \sqrt{\mu\eta} x_t z_t dW_t \\
dy_t &= (h_a x_t - u z_t - \mu y_t)dt - \sqrt{\mu\eta} y_t z_t dW_t \\
dz_t &= u y_t dt - \sqrt{\mu\eta}(1 - z_t^2) dW_t.
\end{aligned} \tag{34}$$

The Lyapunov functions are

$$V_1 = \frac{1}{2}(1 - z), \qquad V_2 = 1 - z^2,$$

hence

$$\mathcal{L}\tilde{V}_t = -y_t \left(1 - \frac{4z_t}{\ell^2}\right) u - \frac{4\mu\eta}{\ell^2}(1 - z_t^2)^2.$$

The feedback laws (15), (29) and (31) are, respectively,

$$u_t = k\, y_t, \tag{35}$$

$$u_t = k\, y_t \left(1 - \frac{4z_t}{\ell^2}\right), \tag{36}$$

$$u_t = k^2 y_t \left(1 - \frac{4z_t}{\ell^2}\right) - \frac{4k\sqrt{\mu\eta}}{\ell}(1 - z_t^2), \tag{37}$$

for some $k > 0$, $\ell > 0$. In correspondence of $\tilde{V}$ and (36), the closed loop infinitesimal generator is the sum of squares

$$\mathcal{L}\tilde{V}_t = -k\left(y_t \left(1 - \frac{4z_t}{\ell^2}\right)\right)^2 - \frac{4\mu\eta}{\ell^2}(1 - z_t^2)^2,$$

while for (37) it is the square of a sum

$$\mathcal{L}\tilde{V}_t = -\left(k y_t \left(1 - \frac{4z_t}{\ell^2}\right) - \frac{2\sqrt{\mu\eta}}{\ell}(1 - z_t^2)\right)^2. \tag{38}$$

Looking at the closed loop dynamics, one sees that on the "line" of diagonal densities $x = y = 0$, for (35) and (36) the feedback is 0, the line itself is invariant and the dynamics driven only by the filtering term. This is no longer true for (37), as expected.

The level curves of (38) are visualized in Fig 1 for different choices of the parameters k, ℓ. The figures shows that for ℓ that grows the locus $\mathcal{L}\tilde{V}_t = 0$ tends to become aligned with the axis $y = 0$. The effect of raising k instead is to increase the rate of convergence.

For the feedback (37), in Fig. 2 a few sample trajectories are plotted for different initial conditions (the boldface trajectory corresponds to $\rho_0 = I_2/2$, i.e., to the maximally mixed state). They are reproduced as x_t, y_t, z_t versus time in Fig. 3. The corresponding time courses of u_t, $\mathrm{tr}(\rho_t^2)$, $\tilde{V}_t$ and $\mathcal{L}\tilde{V}_t$ are shown in Fig. 4.

5 Conclusion

It is well known in Control Theory that finding a Lyapunov function for nonlinear systems is more an art than a systematic science, and that the knowledge about the physical process can provide the intuition necessary for this scope. The present work is nothing but a confirm of both these rules of thumb. We consider (some of) the standard design procedures available for the class of stochastic differential equations we deal with and show how they provide only a partial solution to our stabilization problem. Once we integrate this design with some physical insight on the structure of the SDE, however, the feedback synthesis becomes much more efficient and allows for a simple analytic demonstration regardless of the dimension of the system. In addition, since the Lyapunov function is a Morse function on the space of density operators, the feedback stabilization design guarantees global convergence up to a finite number of isolated and repulsive critical points.
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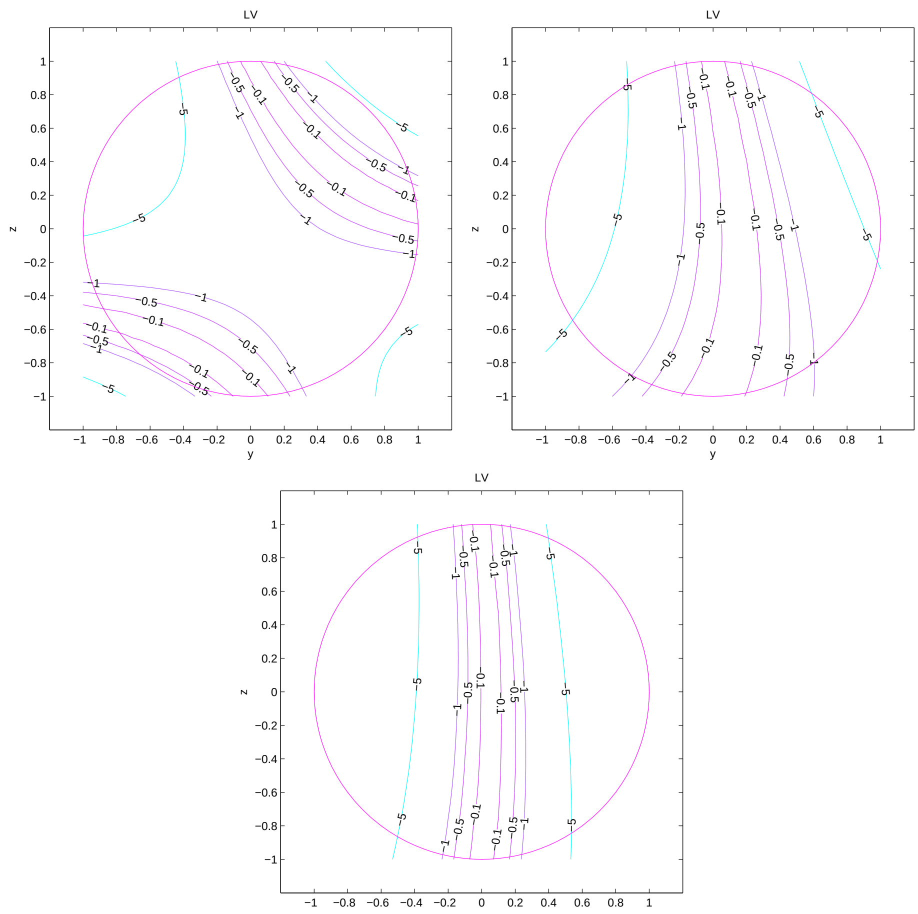

Figure 1: Level surfaces $\mathcal{L}\tilde{V}$ = const (eq.(38)) in the (y, z) plane for different values of k, ℓ, in correspondence of $\mu = 1$, $\eta = 1/2$. Top left: $(k, \ell) = (1, 1)$; Top right: $(k, \ell) = (3, 3)$; bottom: $(k, \ell) = (5, 5)$.

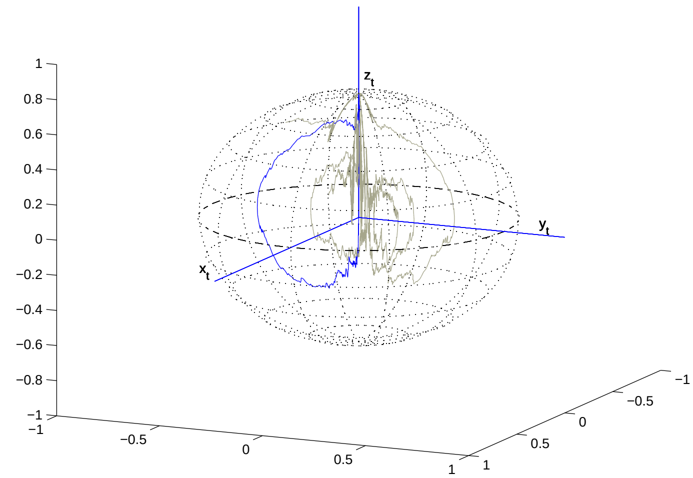

Figure 2: A few trajectories of the closed loop system (34)-(37) for $\mu = 1$, $\eta = 1/2$. The boldface trajectory corresponds to $\rho_0 = I_2/2$.

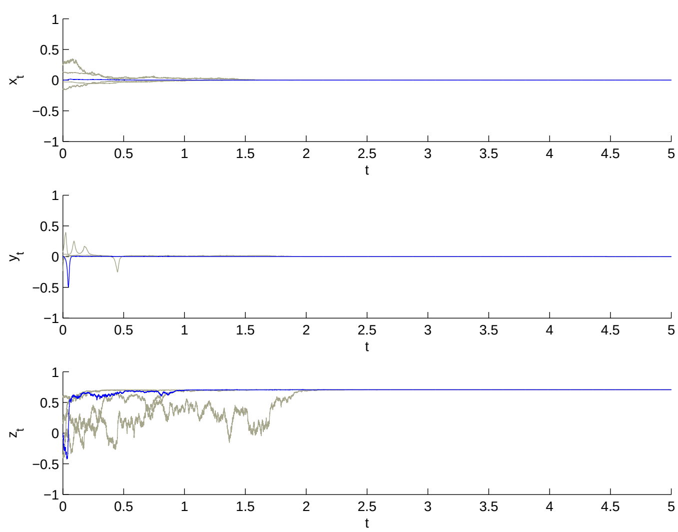

Figure 3: The same trajectories of Fig. 2 versus time.

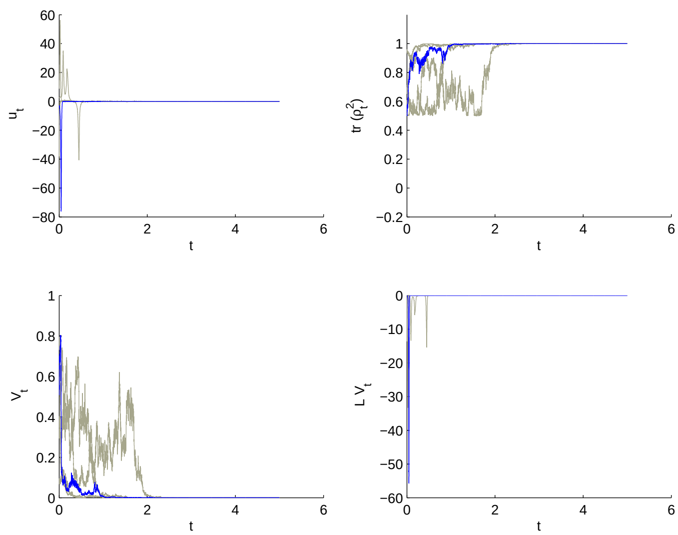

Figure 4: Top left: control signal u_t; top right: $\mathrm{tr}(\rho_t^2)$; bottom left $\tilde{V}_t$; bottom right: $\mathcal{L}\tilde{V}_t$. The trajectories are the same as in Figg. 2-3