Discrete approximation of nonlocal-gradient energies

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Abstract

We study a discrete approximation of functionals depending on nonlocal gradients. The discretized functionals are proved to be coercive in classical Sobolev spaces. The key ingredient in the proof is a formulation in terms of circulant Toeplitz matrices.

Keywords: nonlocal gradients, peridynamics, fractional Sobolev spaces, discrete approximations, discrete-to-continuum convergence.

1 Introduction

Variational problems involving nonlocal gradients $\nabla_\rho u$ defined by

$$\nabla_\rho u(x) = \int_{\mathbb{R}^d} \rho(\xi)\frac{(u(x+\xi)-u(x))\xi}{|\xi|}d\xi, \tag{1}$$

where ρ is a suitable symmetric positive kernel, have been recently considered e.g. in [20, 14]. In particular, power kernels have been used in connection with fractional Sobolev spaces (as in [6, 5, 11, 21]), in which case, and in general in the case of singular kernels, this integral must be considered as a principal value.

Fractional-gradient integral functionals take the form

$$\int_{\mathbb{R}^d} f(\nabla_\rho u)\,dx, \tag{2}$$

and boundary-value problems can be addressed on suitably defined spaces. These energies allow to consider problems stated in a weaker form than in usual Sobolev spaces. On the other hand, by scaling such gradients, an approximation can be provided of classical functionals of the Calculus of Variations [2]. More precisely, after considering scaled kernels ρ_ε defined by $\rho_\varepsilon(\xi) = \frac{1}{\varepsilon^d}\rho(\frac{\xi}{\varepsilon})$, from the weak convergence of u_ε to u,

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upon some boundedness conditions on the L^p-norm of $\nabla_{\rho_\varepsilon} u_\varepsilon$ $(1 < p < \infty)$, we may deduce the weak convergence of $\nabla_{\rho_\varepsilon} u_\varepsilon$ to (a multiple of) the usual weak gradient ∇u. In particular arguing as in [6] (see also [16]) we can deduce the convergence

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} |\nabla_{\rho_\varepsilon} u|^p\,dx = C_\rho^p \int_{\mathbb{R}^d} |\nabla u|^p\,dx, \qquad \text{where} \qquad C_\rho = \frac{1}{d}\int_{\mathbb{R}^d} \rho(\xi)|\xi|\,d\xi, \tag{3}$$

in the spirit of the seminal paper by Bourgain, Brezis and Mironescu [3], as well as the related Γ-convergence analysis (see Ponce [18]). These results can be achieved thanks to the characterization of nonlocal gradients in distributional form [6, 7, 20], which guarantees that weak limits of nonlocal gradients are nonlocal gradients of the weak limit, or even, in the case of the convergence in (3), classical weak gradients.

In this paper we propose a discretized approach to energies depending on nonlocal gradients as in (1). Even though this subject has a clear connection with numerical methods in the treatment of fractional problems (see e.g. [9, 8, 10, 23, 24]), this work is rather meant as part of the exploration of the use of recent techniques in the analysis of discrete systems by variational methods. In order to explain the spirit of such an approach, we can compare the convergence in (3) with the analogous convergence in fractional-type Sobolev spaces shown by Bourgain, Brezis and Mironescu [3], where

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\times\mathbb{R}^d} \rho_\varepsilon(x-y)\frac{|u(x)-u(y)|^p}{|x-y|^p}\,dxdy = \widetilde{C}_\rho \int_{\mathbb{R}^d} |\nabla u|^p\,dx, \text{ and } \widetilde{C}_\rho = \int_{\mathbb{R}^d} \rho(\xi)\,d\xi, \tag{4}$$

For the asymptotic analysis of functionals of type (4) a discretization approach is possible, proving their equivalence with discrete energies depending on differences $u_i - u_j$ parameterized on a cubic lattice (see [4]). Such differences can be interpreted as difference quotients of some interpolation, for which the finiteness of the energy implies boundedness in some classical Sobolev space.

In the case of nonlocal gradients such an equivalence is more delicate by the possible cancellations in (1). We focus on the one-dimensional case, proposing an extension to higher dimension at the end of the paper. In order to define discrete nonlocal gradients in parallel with (1) it is convenient to note that, thanks to the symmetry of ρ, we also have

$$\nabla_\rho u(x) = \int_{\mathbb{R}} \rho(\xi)\frac{u(x+\xi)\,\xi}{|\xi|}d\xi = -\int_0^{+\infty} \rho(\xi)\,u(x-\xi)d\xi + \int_0^{+\infty} \rho(\xi)\,u(x+\xi)d\xi. \tag{5}$$

With this formula in mind, if $u\colon \mathbb{Z} \to \mathbb{R}$ then we define its *discrete nonlocal gradient* as the function $u'_\rho\colon \mathbb{Z} \to \mathbb{R}$, whose value at $k \in \mathbb{Z}$ is

$$(u'_\rho)_k = -\sum_{i=1}^{M} \rho_i\,u_{k+1-i} + \sum_{i=1}^{M} \rho_i\,u_{k+i}, \tag{6}$$

where $u_k = u(k)$ and ρ_i are positive values representing a discretization of the kernel ρ. The trivial (and exceptional) case is $M = 1$ (and $\rho_1 = 1$), for which we just have the finite difference $u_{k+1} - u_k$. Note that in order to avoid considering the value of ρ at 0 we have introduced an asymmetry in this definition, which amounts to a translation of $\frac{1}{2}$ (see Remark 2.1). This is not surprising if we view u_k as an average of a continuum function over the interval $[k, k+1]$, whose center is $k+\frac{1}{2}$. A formally more symmetric definition would be

$$(u'_\rho)_k = -\sum_{i=1}^{M} \rho_i\, u_{k-i} + \sum_{i=1}^{M} \rho_i\, u_{k+i}, \tag{7}$$

but this definition will not lead to coercive energies, as shown below.

Next, we scale this definition. If $\varepsilon > 0$ and $u\colon \varepsilon\mathbb{Z} \to \mathbb{R}$, then the *nonlocal gradient at scale* ε is the function $u'_{\rho_\varepsilon}\colon \varepsilon\mathbb{Z} \to \mathbb{R}$ related to the discrete kernel ρ_ε, which is defined from ρ by scaling in the same way as for continuous kernels where $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho\big(\frac{x}{\varepsilon}\big)$. The value of u'_{ρ_ε} at εk, with $k \in \mathbb{Z}$ is

$$(u'_{\rho_\varepsilon})_k = \frac{1}{\varepsilon}\Big(-\sum_{i=1}^{M} \rho_i u_{k+1-i} + \sum_{i=1}^{M} \rho_i u_{k+i}\Big), \tag{8}$$

where now $u_k = u(\varepsilon k)$.

The main result proved in this paper (Section 4) is that we can improve the weak convergence of the discrete nonlocal gradients to the weak convergence of the gradients of the interpolations. That is, if $u^\varepsilon\colon \varepsilon\mathbb{Z} \to \mathbb{R}$ and $u\colon \mathbb{R} \to \mathbb{R}$ are such that the piecewise-constant interpolations of u^ε weakly converge in $L^1_{\rm loc}$ to u and the piecewise-constant interpolations of $(u^\varepsilon)'_\rho$ weakly converge in L^2, then indeed the piecewise-affine interpolations of u^ε weakly converge in $W^{1,2}_{\rm loc}$ to u.

The improved convergence is not a trivial fact and requires some minimal assumptions on ρ_i, since in (8) we may have cancellations due to the changing sign of the coefficients. Indeed, if we have a constant $\rho_i = \overline{\rho}$ then

$$(u'_{\rho_\varepsilon})_k = \frac{\overline{\rho}}{\varepsilon}\big(-u_{k-M+1} - u_{k-M+2} - \ldots - u_{k-1} - u_k + u_{k+1} + u_{k+2} + \ldots + u_{k+M}\big).$$

If M is even and we take $u^\varepsilon_k = (-1)^k$ then the nonlocal gradient (at scale ε) of u^ε is 0, but the interpolations of u^ε only converge weakly in $L^p_{\rm loc}$ and are not bounded in any Sobolev space. The same counterexample holds with arbitrary ρ_i, also not constant, if the symmetric definition of gradient (7) is used.

As an application, in a discrete-to-continuum setting [1], we can consider functionals of the form

$$\sum_{k\in\mathbb{Z}} \varepsilon f\Big(\frac{1}{\varepsilon}\Big(\sum_{j=1}^{M}(u_{k+j} - u_k)\rho_j - \sum_{j=1}^{M}(u_{k-j+1} - u_k)\rho_j\Big)\Big), \tag{9}$$

and prove their convergence with respect to the weak convergence of the interpolations in $H^1(\mathbb{R})$ to $\int f(Ku')dt$ with $K = \sum\limits_{j=1}^{M} \rho_j(2j-1)$.

These results on discrete functions can be read in the continuum case as statements on average values of sequences with bounded energies (see Section 5). For example, if ρ has support $[-1, 1]$ and $\rho_i = \rho(\frac{i}{M})$ then we are considering a piecewise-constant approximation of ρ. Given a continuum u and defined the value u_k as the average of u on $[\varepsilon\frac{k}{M}, \varepsilon\frac{k+1}{M}]$, the discrete nonlocal gradient of $\{u_k\}_k$ corresponds to the continuum nonlocal gradient of u for the discretized kernel at εk, and the result above can be read as a compactness result in $H^1_{\rm loc}$ for (the piecewise-affine interpolations of) such averages.

The main tool in the proof of the results is the theory of circulant Toeplitz matrices, for which it is possible to bound the lowest eigenvalue (see Section 3). Its use allows to give lower bounds for energies depending on discrete nonlocal gradients in terms of coercive nearest-neighbour energies, thus ensuring compactness properties for piecewise-affine interpolations. Even though natural, this argument seems new in the context of discrete-to-continuum analysis. It must be noted that in principle this analysis holds in any dimension, upon suitably defining discrete nonlocal partial derivatives (see Section 6), but the properties of the kernels ρ_i that ensure a bound on the lowest eigenvalue of the related Toeplitz matrices are more involved and we do not address their study here.

2 Discrete nonlocal gradients

We consider $M \in \mathbb{N}$ and a decreasing array of positive numbers $\rho_1, \dots, \rho_M$. Let $u\colon \mathbb{Z} \to \mathbb{R}$ and let $u_k = u(k)$. The *discrete non-local gradient* related to ρ is the function $u'_\rho\colon \mathbb{Z} \to \mathbb{R}$, whose value at $k \in \mathbb{Z}$ is defined by (6). Note that we can equivalently write this quantity as

$$\begin{aligned}
(u'_\rho)_k &= \rho_M(u_{k+M} - u_{k-M+1}) + \rho_{M-1}(u_{k+M-1} - u_{k-M+2}) + \dots + \rho_1(u_{k+1} - u_k)\\
&= \rho_M \sum_{j=k-M+2}^{k+M} (u_j - u_{j-1}) + \rho_{M-1} \sum_{j=k-M+3}^{k+M-1} (u_j - u_{j-1}) + \dots + \rho_1(u_{k+1} - u_k)\\
&= \rho_M(u_{k-M+2} - u_{k-M+1}) + (\rho_{M-1} + \rho_M)(u_{k-M+3} - u_{k-M+2}) + \dots\\
&\quad + (\rho_2 + \rho_3 + \dots + \rho_M)(u_k - u_{k-1}) + (\rho_1 + \rho_2 + \dots + \rho_M)(u_{k+1} - u_k)\\
&\quad + (\rho_2 + \rho_3 + \dots + \rho_M)(u_{k+2} - u_{k+1}) + \dots\\
&\quad + (\rho_{M-1} + \rho_M)(u_{k+M-1} - u_{k+M-2}) + \rho_M(u_{k+M} - u_{k+M-1}). \qquad (10)
\end{aligned}$$

Remark 2.1. Note that we can rewrite the one-dimensional non-local gradient at a point $k \in \mathbb{Z}$ as

$$(u'_\rho)_k = \sum_{i\in\mathbb{Z},\ i>0} \rho_i u_{k+i} - \sum_{i\in\mathbb{Z},\ i\leqslant 0} \rho_i u_{k+i} = \sum_{i\in\mathbb{Z}} \rho_i u_{k+i} \,\mathrm{sign}\Big(i - \frac{1}{2}\Big), \qquad (11)$$

where

$$\rho_i = \rho\Big(\Big|i - \frac{1}{2}\Big|\Big), \qquad (12)$$

and ρ a symmetric kernel. This shows a slight asymmetry of the discrete definition with respect to the continuum one.

We will consider $\varepsilon > 0$ and the *scaled discrete non-local gradients* defined for functions $u\colon \varepsilon\mathbb{Z} \to \mathbb{R}$ in (8) as the functions $u'_{\rho_\varepsilon} : \varepsilon\mathbb{Z} \to \mathbb{R}$ given at the point εk by

$$(u'_{\rho_\varepsilon})_k = \frac{1}{\varepsilon}(u'_\rho)_k, \qquad (13)$$

where u'_ρ is given by (6) and we have used the notation $u_k = u(\varepsilon k)$.

Note that if we regard the value of u_k as a mean value of a continuous function u over the interval $[k, k+1]$ then we loose some symmetry. In particular, the analogous formula of (1), obtained from (8) subtracting u_k from all terms, reads as

$$(u'_{\rho_\varepsilon})_k = \frac{1}{\varepsilon}\Big(-\rho_M(u_{k-M+1}-u_k) - \rho_{M-1}(u_{k-M+2}-u_k) - \ldots - \rho_2(u_{k-1}-u_k) \\ +\rho_1(u_{k+1}-u_k) + \ldots + \rho_{M-1}(u_{k+M-1}-u_k) + \rho_M(u_{k+M}-u_k)\Big). \tag{14}$$

Even though by (10) $(u'_{\rho_\varepsilon})_k$ can be seen as a combination of the difference quotients

$$\frac{u_{j+1}-u_j}{\varepsilon}$$

for $k-M+1 < j \leqslant k+M-1$, due to the sign changes in (14), in general $(u'_\rho)_k$ cannot be interpreted in terms of difference quotients of some interpolation for which a bound in some Sobolev space can be derived, except for $M = 1$, in which case only one term is present and we have a classical nearest-neighbour interaction problem.

If $u_i = \varphi_i = \varphi(\varepsilon i)$ for some C^1-function, since

$$\frac{u_{k+j}-u_{k+j-1}}{\varepsilon} = \frac{\varphi(\varepsilon(k+j)) - \varphi(\varepsilon(k+j-1))}{\varepsilon} = \varphi'(\varepsilon k) + o(1)$$

as $\varepsilon \to 0$ for all $j \in \{-M+2, \ldots, M\}$, by (10) we have

$$(u'_{\rho_\varepsilon})_k = \varphi'(\varepsilon k)\sum_{j=1}^{M}\rho_j(2j-1) + o(1),$$

so that the piecewise-affine (or, equivalently, the piecewise-constant) interpolations of $\varphi'_{\rho_\varepsilon}$ converge to φ' times the constant $K := \sum_{j=1}^M \rho_j(2j-1)$.

We will examine the asymptotic behaviour of functionals of the form

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}} \varepsilon f\big((u'_{\rho_\varepsilon})_k\big) \tag{15}$$

when $f(z) \geqslant c_1|z|^2$ with respect to the convergence of the interpolations; that is the convergence u^ε is meant as the convergence of the piecewise-constant interpolations u_ε defined by $u_\varepsilon(x) = u^\varepsilon_{\lfloor x/\varepsilon\rfloor}$.

In order to show coerciveness properties of F_ε, let $u^\varepsilon\colon \varepsilon\mathbb{Z} \to \mathbb{R}$. Note that if $\varphi \in C_c^\infty(\mathbb{R})$, from the equality

$$\sum_{k\in\mathbb{Z}} \varepsilon((u^\varepsilon)'_{\rho_\varepsilon})_k\varphi_k = -\sum_{k\in\mathbb{Z}}\varepsilon(\varphi'_{\rho_\varepsilon})_k u^\varepsilon_k,$$

we deduce that, if the interpolations of $\{u^\varepsilon_k\}_k$ weakly converge to some u in $L^2(\mathbb{R})$ as $\varepsilon \to 0$ and the interpolations of $\{((u^\varepsilon)'_{\rho_\varepsilon})_k\}_k$ weakly converge to some v in $L^2(\mathbb{R})$ as $\varepsilon \to 0$, then

$$\int_{\mathbb{R}} v\varphi\,dx = -\int_{\mathbb{R}} K\varphi' u\,dx. \tag{16}$$

Hence, $u \in W^{1,2}(\mathbb{R})$ and the interpolations of $\{((u^\varepsilon)'_{\rho_\varepsilon})_k\}_k$ weakly converge to Ku'. Our aim is to improve this convergence showing that actually the piecewise-affine interpolations of $\{u^\varepsilon_k\}_k$ converge in $W^{1,2}(\mathbb{R})$.

3 Eigenvalues of banded circulant matrices

Using the second equality in (10) we will express the discrete nonlocal gradient as a linear combination of differences of nearest neighbours through a Toeplitz matrix, or, equivalently considering boundary conditions, a circulant matrix. Coerciveness properties can be deduced from bounds on minimal eigenvalues of such a matrix, for which a general result can be proved.

We consider symmetric n-banded circulant matrices; that is, $N \times N$ matrices of the form

$$A = \begin{pmatrix} \sigma_0 & \sigma_1 & \sigma_2 & & \cdots & \sigma_{N-1} \\ \sigma_{N-1} & \sigma_0 & \sigma_1 & \sigma_2 & & \vdots \\ & \sigma_{N-1} & \sigma_0 & \sigma_1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & \sigma_2 \\ & & & & & \sigma_1 \\ \sigma_1 & \cdots & & & \sigma_{N-1} & \sigma_0 \end{pmatrix} \tag{17}$$

with $\sigma_{N-j} = \sigma_j$, $\sigma_n \neq 0$ and $\sigma_j = 0$ if $j \in \{n+1, \ldots, N-n-1\}$.

We assume that the following convexity condition holds

$$\sigma_{j-1} - \sigma_j > \sigma_j - \sigma_{j+1} \quad \text{for all} \quad j \in \{1, \ldots, n+1\}, \tag{18}$$

which in particular implies that $\sigma_j > 0$ for $j \in \{0, \ldots, n\}$. Indeed, since $\sigma_{n+1} = \sigma_{n+2} = 0$, from (18) with $j = n+1$ we deduce that $\sigma_n - \sigma_{n+1} > 0$, and then that $\sigma_j - \sigma_{j-1} > 0$ for all $j \leqslant n$. In particular, we can use a finite backwards-induction argument: we have $\sigma_n = \sigma_n - \sigma_{n+1} > 0$, while if $\sigma_j > 0$ then $\sigma_{j-1} > \sigma_j + (\sigma_j - \sigma_{j+1}) > 0$.

Lemma 3.1. *Let $N > 2n$. Then $\lambda_{\min}$, the minimal eigenvalue of A, is larger than a positive constant independent of N.*

Proof. By [13, Chapter 3] and the symmetry of the matrix, the minimal eigenvalue of A is bounded from below by the minimum of the function

$$\Phi(t) = \sigma_0 + 2 \sum_{j=1}^{n} \sigma_j \cos(jt)$$

for $t \in [0, \pi]$. The positivity of this trigonometric sum is a classical result due to Fejér (see [12] for the orginal source or [17, Chapter 4] for a review and English translation): in summary, using a closed form of Fejér kernels $\sum\limits_{|k|<j} (j-k)\cos(kt) = \frac{1-\cos(jt)}{1-\cos t}$, we can rewrite Φ as

$$\Phi(t) = \sum_{j=1}^{n+1} (\sigma_{j-1} - 2\sigma_j + \sigma_{j+1}) \frac{1-\cos(jt)}{1-\cos t}. \tag{19}$$

By (18) each coefficient $\sigma_{j-1} - 2\sigma_j + \sigma_{j+1}$ is strictly positive, so that Φ is a sum of non-negative functions. In particular

$$\min \Phi \geqslant \sigma_0 - 2\sigma_1 + \sigma_2 > 0$$

and the claim. □

Remark 3.2. If $A = (a_{i,j})$ is a symmetric, n-banded, $N \times N$ Toeplitz matrix; that is, $a_{i,j} = \sigma_{|i-j|}$ for $|i-j| \leqslant n$ with $\sigma_n \neq 0$ and $a_{i,j} = 0$ otherwise, then a general result about Hermitian Toeplitz matrices [13, Lemma 4.1] ensures that the eigenvalues of A belong to the interval $[m_\Phi, M_\Phi]$ whose endpoints are respectively the minimum and the maximum of the Fourier series

$$\Phi(t) = \sum_{k=-\infty}^{+\infty} \sigma_k e^{ikt} = \sigma_0 + 2 \sum_{k=1}^{n} \sigma_k \cos(kt).$$

Henceforth, if the numbers $\{\sigma_k\}_{k=0,\dots,n}$ satisfy convexity condition (18) then Lemma 3.1 also holds for this class of Toeplitz matrices.

4 Coerciveness and discrete-to-continuum convergence

We first examine the coerciveness properties of reference quadratic energies as follows.

Theorem 4.1. *Let $\rho_i > 0$, $i \in \{1, \dots M\}$ be a decreasing array of real numbers. Let (a,b) be a bounded interval in $\mathbb{R}$ and let the energies*

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}} \varepsilon \left| \frac{1}{\varepsilon} \Big(\sum_{j=1}^{M} (u_{k+j} - u_k)\rho_j - \sum_{j=1}^{M} (u_{k-j+1} - u_k)\rho_j \Big) \right|^2 \tag{20}$$

be defined for $u\colon \varepsilon\mathbb{Z} \to \mathbb{R}$ with $u(x) = 0$ if $x \in \mathbb{R}\backslash(a,b)$. Then there exists a constant Λ such that

$$F_\varepsilon(u) \geqslant \Lambda \sum_{k\in\mathbb{Z}} \varepsilon \left| \frac{u_{k+1} - u_k}{\varepsilon} \right|^2 \tag{21}$$

for all u and $\varepsilon < \frac{1}{2M}$.

Proof. We can suppose without loss of generality that $(a,b) = (0,1)$.

Let A be the matrix defined in (17) with $n = M - 1$ and

$$\sigma_j = \sum_{k=j+1}^{M} \rho_k.$$

Note that the monotonicity condition on ρ_i ensures that (18) holds, and that the function in (19) is given by

$$\Phi(t) = \sum_{j=1}^{n+1} (\rho_j - \rho_{j+1}) \frac{1 - \cos(jt)}{1 - \cos t}.$$

By Lemma 3.1 for all z with $\sum_{k=1}^{N} z_k^2 = 1$ we have

$$|Az| \geqslant |\langle Az, z\rangle| \geqslant \lambda_{\min},$$

so that for all z we have $|Az|^2 \geqslant \Lambda|z|^2$, where $\Lambda = (\lambda_{\min})^2$. Hence, (21) follows upon taking $N \geqslant \frac{1}{\varepsilon} + 4M$ and applying the previous estimate to $z_k = \frac{1}{\varepsilon}(u_{k+2M} - u_{k+2M-1})$. Note that $z_k = 0$ for $k \in \{1, \dots, 2M\}$ and $k \in \{N - 2M + 1, \dots, N\}$, so that

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}} \varepsilon\big|(Az)_{k-2M}\big|^2,$$

and the claim follows. □

Remark 4.2. Note that in the definition of discrete nonlocal gradient we may also consider kernels such that $\rho_i > 0$ for all $i \in \mathbb{Z}$, for which we may give a meaning to (10) as a converging series. However, for such kernels it is not clear how to use an equivalence between banded Toeplitz and circulant matrices as in the proof of the previous proposition, for which the size of the matrix must be large with respect to the support of ρ.

The following result proves a discrete-to-continuum convergence for discrete energies using the improved coerciveness.

Theorem 4.3. *Let $\rho_i > 0$, $i \in \{1, \dots M\}$ be a decreasing array of real numbers. Let f be a convex function with $c_1|z|^2 + c_0 \leqslant f(z) \leqslant c_2|z|^2 + c_3$ with $c_1, c_2 > 0$. Let (a, b) be a bounded interval in $\mathbb{R}$ and let the energies*

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}} \varepsilon f\bigg(\frac{1}{\varepsilon}\Big(\sum_{j=1}^{M}(u_{k+j} - u_k)\rho_j - \sum_{j=1}^{M}(u_{k-j+1} - u_k)\rho_j\Big)\bigg) \tag{22}$$

be defined for $u\colon \varepsilon\mathbb{Z} \to \mathbb{R}$ with $u(x) = 0$ if $x \in \mathbb{R}\backslash(a, b)$. Then there exists the Γ-limit of F_ε with respect to the weak L^2-convergence of interpolations as $\varepsilon \to 0$ and

$$\Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon(u) = \int_{(a,b)} f(Ku')dt, \qquad K = \sum_{j=1}^{M}(2j-1)\rho_j. \tag{23}$$

with domain $H^1_0(a, b)$.

Proof. Let u^ε converge weakly to u and let $F_\varepsilon(u^\varepsilon)$ be equibounded. Then by the previous theorem the sequence of the corresponding piecewise-affine interpolations is weakly precompact in $H^1(\mathbb{R})$, so that indeed u^ε weakly converges to u in $H^1(\mathbb{R})$. Since $u^\varepsilon = 0$ outside (a, b) the convergence is actually in $H^1_0(a, b)$. Since for all fixed j all interpolations of difference quotients $\frac{1}{\varepsilon}(u^\varepsilon_{k+j-1} - u^\varepsilon_{k+j})$ weakly converge to u', by (10) the weak limit of $(u^\varepsilon)'_\rho$ is Ku' (this can also be obtained as in (16)). By the weak lower semicontinuity of $z \mapsto \int f(z)\,dt$ we then obtain the liminf inequality.

If $u \in C^\infty_c(a, b)$, extended by 0 outside (a, b), then we have

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}} \varepsilon f(Ku'(\varepsilon k)) + o(1),$$

as $\varepsilon \to 0$, and we obtain the pointwise convergence to $\int_{(a,b)} f(Ku')dt$. The limsup inequality follows by density. □

5 Application to continuum interpolations

We will use discretizations to provide an approximation for Γ-limits of continuum functionals of the form

$$F_\varepsilon(u) = \int_{\mathbb{R}} f(\nabla_{\rho_\varepsilon} u)\,dx$$

in the one-dimensional setting, with respect to the weak convergence in L^2. As above, here $\rho_\varepsilon(\xi) = \frac{1}{\varepsilon}\rho(\frac{\xi}{\varepsilon})$.

Note that the equality

$$\int_{\mathbb{R}} \varphi(x)\nabla_{\rho_\varepsilon} u(x)\,dx = -\int_{\mathbb{R}} u(x)\nabla_{\rho_\varepsilon}\varphi(x)\,dx$$

implies as in (16) that if u_ε weakly converges in $L^2(\mathbb{R})$ to u and the sequence $\nabla_{\rho_\varepsilon} u_\varepsilon$ is bounded in $L^2(\mathbb{R})$, then actually $u \in H^1(\mathbb{R})$ and the sequence $\nabla_{\rho_\varepsilon} u_\varepsilon$ weakly converges in $L^2(\mathbb{R})$ to Ku', where

$$K = \int_{\mathbb{R}} \rho(\xi)|\xi|\,d\xi, \tag{24}$$

and, if a growth condition of the type $f(z) \geqslant c_1|z|^2$ holds, we deduce the Γ-convergence of F_ε to

$$F(u) = \int_{\mathbb{R}} f(Ku')\,dx \tag{25}$$

with respect to the weak convergence in $L^2(\mathbb{R})$. However, we observe that for sequences of functions $u_\varepsilon \in L^2(\mathbb{R})$ with $F_\varepsilon(u_\varepsilon)$ equibounded in general we cannot deduce any stronger coerciveness. Indeed, note that if ρ is a non-negative even continuous integrable kernel then for each $\varepsilon > 0$ $\nabla_{\rho_\varepsilon}$ is a continuous operator in $L^2(\mathbb{R})$, so that for a fixed function $u \in H^1(\mathbb{R})$ we can find u_ε tending to u in the L^2-norm such that $\nabla_{\rho_\varepsilon} u_\varepsilon$ is close to $\nabla_{\rho_\varepsilon} u$ but with ∇u_ε unbounded in $L^2(\mathbb{R})$. More in general, we can give an explicit counterexample valid also for Riesz fractional gradients. Let $\rho(\xi) = |\xi|^{-1-\alpha}$ with $\alpha \in (0,1)$. Let $R > 1$ be fixed and let φ be the cut-off function defined as $\varphi(t) = \min\{1, (R-|t|)^+\}$. We define

$$u_\varepsilon(t) = \varepsilon^2 \sin\Big(\frac{t}{\varepsilon^2}\Big)\varphi(t).$$

We then have

$$F_\varepsilon(u_\varepsilon) \leqslant 2\int_{-R}^{R}\Big(\frac{1}{\varepsilon}\int_{-\infty}^{\infty}\rho_\varepsilon(x-y)(u_\varepsilon(x)-u_\varepsilon(y))\frac{x-y}{|x-y|}\,dx\Big)^2 dy.$$

By using the bounds $|u_\varepsilon(t) - u_\varepsilon(s)| \leqslant 2|t-s|$ and $|u_\varepsilon(t)| \leqslant \varepsilon^2$, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big|\int_{-\infty}^{\infty}\rho_\varepsilon(x-y)(u_\varepsilon(x)-u_\varepsilon(y))\frac{x-y}{|x-y|}\,dx\Big| \\
&\quad\leqslant \frac{1}{\varepsilon^2}\int_{\{|\xi|<\varepsilon^{3/2}\}}\frac{\varepsilon^{1+\alpha}}{|\xi|^{1+\alpha}}|u_\varepsilon(y+\xi)-u_\varepsilon(y)|\,d\xi + \frac{1}{\varepsilon^2}\int_{\{|\xi|>\varepsilon^{3/2}\}}\frac{\varepsilon^{1+\alpha}}{|\xi|^{1+\alpha}}|u_\varepsilon(y+\xi)|\,d\xi \\
&\quad\leqslant \frac{4}{\varepsilon^{1-\alpha}}\int_0^{\varepsilon^{3/2}}\frac{1}{\xi^\alpha}\,d\xi + 2\varepsilon^{1+\alpha}\int_{\varepsilon^{3/2}}^{+\infty}\frac{1}{\xi^{1+\alpha}}\,d\xi = \frac{4}{1-\alpha}\varepsilon^{\frac{1-\alpha}{2}} + \frac{2}{\alpha}\varepsilon^{1-\frac{\alpha}{2}} \leqslant c(\alpha)\varepsilon^{\frac{1-\alpha}{2}}.
\end{aligned}$$

Hence,
$$F_\varepsilon(u_\varepsilon) \leqslant 4R\, c(\alpha)^2\varepsilon^{1-\alpha}$$
which is infinitesimal as $\varepsilon \to 0$, so that there is no constant c such that $F_\varepsilon(u_\varepsilon) \geqslant c\|u'_\varepsilon\|^2_{L^2}$.

The following theorem shows a family of discrete nonlocal-gradient energies $F^{M,\varepsilon}$ that can be interpreted as an approximation of the family F_ε in the sense that the limit as $M \to +\infty$ of the Γ-limit of F^M_ε coincides with that of F_ε. Moreover, for fixed M, the family $\{F^{M,\varepsilon}\}_\varepsilon$ is equicoercive in $H^1(\mathbb{R})$ in the sense specified in the first part of the paper.

Theorem 5.1. *Let $\rho : \mathbb{R} \to [0,+\infty)$ be a non-negative even continuous kernel with support $[-1,1]$ and decreasing on $[0,1]$. For $M \in \mathbb{N}$ we let $\rho^M_i = \rho(\frac{i}{M})$ for $i \in \{1,\dots,M\}$ and define the even piecewise-constant function ρ^M by*
$$\rho^M(\xi) = \rho^M_i \quad \text{in} \quad \Big(\frac{i-1}{M}, \frac{i}{M}\Big). \tag{26}$$
We also set $\rho^M_\varepsilon(\xi) = \frac{1}{\varepsilon}\rho^M(\frac{\xi}{\varepsilon})$. Let $F^{M,\varepsilon}$ be the discrete energies defined on functions $u\colon \frac{\varepsilon}{M}\mathbb{Z} \to \mathbb{R}$ by
$$F^{M,\varepsilon}(u) = \sum_{k\in\mathbb{Z}} \frac{\varepsilon}{M} f\big((u'_{\rho^M_\varepsilon})_k\big). \tag{27}$$
Then we have that
$$\lim_{M\to+\infty}\Big(\Gamma\text{-}\lim_{\varepsilon\to 0} F^{M,\varepsilon}\Big) = \Gamma\text{-}\lim_{\varepsilon\to 0} F_\varepsilon = F,$$
with F given by (25) and (24).

Proof. The sequence $F^{M,\varepsilon}$ can be related to the sequence of continuum energies
$$F^M_\varepsilon(u) = \int_{\mathbb{R}} f(\nabla_{\rho^M_\varepsilon} u)\, dx.$$
Given a sequence $\{u_\varepsilon\}$ with $F^M_\varepsilon(u_\varepsilon)$ equibounded, we can suppose, up to a small translation, that
$$F^M_\varepsilon(u_\varepsilon) \geqslant \sum_{k\in\mathbb{Z}} \frac{\varepsilon}{M} f\bigg(\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(\xi)\, u\Big(\frac{\varepsilon k}{M} + \xi\Big)\frac{\xi}{|\xi|}\, d\xi\bigg).$$
We define the sequence of discrete functions $u^{\varepsilon,M}\colon \frac{\varepsilon}{M}\mathbb{Z} \to \mathbb{R}$ as
$$u^{\varepsilon,M}\Big(\frac{\varepsilon}{M}j\Big) = u^{\varepsilon,M}_j = \frac{\varepsilon}{M}\int_{\frac{\varepsilon(j-1)}{M}}^{\frac{\varepsilon j}{M}} u_\varepsilon(t)\, dt.$$
Moreover, we define the functions $z^{\varepsilon,M}\colon \frac{\varepsilon}{M}\mathbb{Z} \to \mathbb{R}$ by
$$z^{\varepsilon,M}\Big(\frac{\varepsilon}{M}j\Big) = z^{\varepsilon,M}_j = \frac{M}{\varepsilon}(u^{\varepsilon,M}_j - u^{\varepsilon,M}_{j-1}).$$
These functions are the difference quotients of $u^{\varepsilon,M}$ on $\frac{\varepsilon}{M}\mathbb{Z}$.

We have

$$\begin{aligned}
\nabla_{\rho_\varepsilon^M} u\Big(\frac{\varepsilon k}{M}\Big) &= \frac{1}{\varepsilon}\int_{\mathbb{R}} \rho_\varepsilon^M(\xi) u\Big(\frac{\varepsilon k}{M}+\xi\Big)\frac{\xi}{|\xi|}\,d\xi \\
&= \frac{1}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon} \rho^M\Big(\frac{\xi}{\varepsilon}\Big) u\Big(\frac{\varepsilon k}{M}+\xi\Big)\frac{\xi}{|\xi|}\,d\xi \\
&= \frac{1}{\varepsilon M}\Big(\sum_{i=1}^{M}\rho_i^M u_{i+k}^{\varepsilon,M} - \sum_{i=-M+1}^{0}\rho_{1-i}^M u_{i+k}^{\varepsilon,M}\Big) \\
&= \frac{1}{\varepsilon M}\Big(\rho_1^M(u_{1+k}^{\varepsilon,M}-u_k^{\varepsilon,M}) + \rho_2^M(u_{2+k}^{\varepsilon,M}-u_{-1+k}^{\varepsilon,M}) + \cdots + \rho_M^M(u_{M+k}^{\varepsilon,M}-u_{-M+1+k}^{\varepsilon,M})\Big) \\
&= \frac{1}{M^2}\Big((\sum_{j=1}^{M}\rho_j^M) z_{1+k}^{\varepsilon,M} + (\sum_{j=2}^{M}\rho_j^M)(z_k^{\varepsilon,M}+z_{2+k}^{\varepsilon,M}) + \cdots + \rho_M^M(z_{-M+1+k}^{\varepsilon,M}+z_{M+k}^{\varepsilon,M})\Big),
\end{aligned}$$

so that $\nabla_{\rho_\varepsilon^M} u$ at $\frac{\varepsilon}{M}k$ coincides with the discrete gradient of $u^{\varepsilon,M}$ at k.

For $i \in \{0, \dots, M-1\}$ we consider

$$\sigma_i = \sigma_i^M = \sum_{j=i+1}^{M} \rho_j^M .$$

Let u^ε be compactly supported, so that u_k^ε is not zero for $k \in [-N_\varepsilon, N_\varepsilon]$, and we consider the circulating matrix as defined above of dimension $2N_\varepsilon + 2M + 1$, which is denoted by A^M.

If we take $z = z^{\varepsilon,M}$ as the vector with components $z_j^{\varepsilon,M}$, we obtain

$$\begin{aligned}
&\sum_{k=-N_\varepsilon-M}^{N_\varepsilon+M} \frac{\varepsilon}{M}\Big(\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\rho_\varepsilon(\xi)u\Big(\frac{\varepsilon k}{M}+\xi\Big)\frac{\xi}{|\xi|}\,d\xi\Big)^2 \\
&\quad = \sum_{k=-N_\varepsilon-M}^{N_\varepsilon+M} \frac{\varepsilon}{M}\Big(\frac{1}{M^2}\langle A^M z, e_{k+1}\rangle\Big)^2 = \frac{\varepsilon}{M}\Big(\frac{1}{M^2}|A^M z|\Big)^2 \\
&\quad \geqslant \frac{\varepsilon}{M}\Big(\frac{\lambda_{\min}}{M^2}\Big)^2|z|^2 = \Big(\frac{\lambda_{\min}}{M^2}\Big)^2\sum_{k=-N_\varepsilon-M}^{N_\varepsilon+M}\frac{\varepsilon}{M}\Big(\frac{u_k^{\varepsilon,M}-u_{k-1}^{\varepsilon,M}}{\frac{\varepsilon}{M}}\Big)^2,
\end{aligned}$$

which proves the equicoerciveness of $\{F_\varepsilon^M\}_\varepsilon$. Moreover, if u_ε weakly converges to u in $L^2(\mathbb{R})$ then $\{u^{\varepsilon,M}\}_\varepsilon$ weakly converge to u in $H^1(\mathbb{R})$ and we have the lower bound

$$\liminf_{\varepsilon\to0^+} F_\varepsilon^M(u_\varepsilon) \geqslant \liminf_{\varepsilon\to0^+} F^{M,\varepsilon}(u^{\varepsilon,M}).$$

A direct computation for $u \in C^1(\mathbb{R})$ gives also the upper bound and shows the equality of the Γ-limits. By letting M tend to $+\infty$ we obtain an approximation of the Γ-limit of F_ε. □

Remark 5.2. The assumption that ρ be of compact support can be removed, upon requiring that the constant K in (24) be finite, and suitably defining the functionals $F^{\varepsilon,M}$. For example, we can define $F^{\varepsilon,M}$ as in (27) upon requiring that ρ_i^M be defined as in (26) for $|i| \leqslant M^2$. The same proof as above also works in this case.

6 Generalization to higher dimensions

We can follow the arguments used above to provide a discrete approximation in dimension higher than one. The definition of discrete nonlocal gradient needs some care in order to avoid excessive cancellations of terms. This can be done using some slight asymmetries as for the similar problems already encountered in dimension one.

In order to generalize the one-dimensional definition, we take into account the asymmetric definition in Remark 2.1 starting from a continuum kernel. This definition could be transposed to functions defined in $\mathbb{Z}^d$, for which the discrete nonlocal gradient is a vector in $\mathbb{R}^d$. The discrete nonlocal derivative in the n-th direction could be defined as

$$\Big(\frac{\partial_\rho u}{\partial x_n}\Big)_k = \sum_{i\in\mathbb{Z}^d} \rho_i u_{k+i} \frac{i_n - \frac{1}{2}}{\Big|i - \Big(\frac{1}{2}, \dots \frac{1}{2}\Big)\Big|} \tag{28}$$

for $k \in \mathbb{Z}^d$, where

$$\rho_i = \rho\Big(\Big|i - \Big(\frac{1}{2}, \dots \frac{1}{2}\Big)\Big|\Big). \tag{29}$$

However, this definition would not allow to describe properties of the interpolation of functions u, since, for example, the oscillating function u with value $u_k = (-1)^{|k_1|+\dots+|k_d|}$ would have zero discrete non-local gradient.

We slightly modify the definition in (28) introducing an additional asymmetry between the n-th direction and the others, which forbids oscillations with zero gradient.

Definition 6.1. *Let $\rho : [0, +\infty) \to [0, +\infty)$ be a positive kernel with support $[0, M]$, decreasing in $[0, M]$. We define the* discrete nonlocal partial derivative *in the n-th direction of a function $u : \mathbb{Z}^d \to \mathbb{R}$ as the function defined by*

$$\Big(\frac{\partial_\rho u}{\partial x_n}\Big)_k = \sum_{i\in\mathbb{Z}^d} \rho^n_i u_{k+i} \tag{30}$$

for $k \in \mathbb{Z}^d$, where

$$\rho^n_i = \rho\Big(\Big|i - \frac{1}{2}e_n\Big|\Big) \frac{i_n - \frac{1}{2}}{|i - \frac{1}{2}e_n|}. \tag{31}$$

Note that ρ^n_i is non-negative for $i_n > 0$ and non-positive for $i_n \leqslant 0$, in analogy with the one-dimensional definition. The discrete nonlocal gradient *is the vector $\nabla_\rho u$ in $\mathbb{R}^d$ whose n-th component is the discrete nonlocal derivative in the n-th direction.*

If $\varepsilon > 0$ the scaled discrete nonlocal partial derivatives $\frac{\partial_{\rho_\varepsilon} u}{\partial x_n}$ and related gradient are defined by scaling as in the one-dimensional case.

We can state now a higher-dimensional analogue of the improved coerciveness and Γ-convergence result showed in one-dimension. We state it for functions defined on a bounded domain Ω with zero boundary values. The assumption on the kernel is given in a more abstract form than in the one-dimensional case, by directly requiring that the related Toeplitz matrices have strictly positive eigenvalues. While this condition could be directly translated into properties of ρ in dimension one, in this case we do not pursue further the very interesting issue of the optimization of the conditions on ρ to ensure coerciveness. A slight difference with the one-dimensional case is that the piecewise-affine interpolations are not uniquely defined. We can use for example the ones with an underlying Kuhn's simplicial decomposition (see [15] and a recent application to interpolations of discrete functions in [22]).

Theorem 6.2. *Suppose that for N large enough there exist banded Toeplitz matrices $A_1^N, \dots, A_j^N$ with entries independent of N whose minimal eigenvalues are larger than a strictly positive constant independently of N, such that*

$$\Big(\frac{\partial_\rho u}{\partial x_j}\Big)_k = (A_j^N z)_k, \tag{32}$$

for all $u : [-N, N]^d \to \mathbb{R}$ such that $u = 0$ on a M-neighbourhood of $\partial[-N, N]^d$, where $z_k = u_{k+e_1} - u_k$.

Let Ω be a Lipschitz open set compactly contained in $[-1, 1]^d$, and let $f : \mathbb{R}^d \to [0, +\infty)$ be a convex function with $f(z) \geqslant c_2|z|^2 - c_1$ for some positive constants c_1 and c_2, and let

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}^d} \varepsilon^d f\big((\nabla_{\rho_\varepsilon} u)_k\big)$$

be defined on functions $u\colon \varepsilon Z^d \to \mathbb{R}$ with $u = 0$ outside Ω. Then

(1) *if $F_\varepsilon(u^\varepsilon) \leqslant C < +\infty$ then the piecewise-affine interpolations of u^ε in the related Kuhn's simplexes are weakly precompact in $W_0^{1,2}(\Omega)$;*

(2) *the Γ-limit of F_ε with respect to the weak L^2-convergence of the piecewise-affine interpolations (which by* (1) *is equivalent to the weak convergence in $W_0^{1,2}(\Omega)$ of the piecewise-affine interpolations) is given by*

$$\int_\Omega f(K_\rho \nabla u)\,dx, \qquad K_\rho = \sum_{k\in\mathbb{Z}^d} \rho\Big(\Big|k - \frac{1}{2}e_1\Big|\Big)\frac{k_1(2k_1 - 1)}{|2k - e_1|}$$

for $u \in W_0^{1,2}(\Omega)$.

Proof. (1) The proof is achieved following the one-dimensional argument of Theorem 4.1, since hypothesis (32) ensures that we can bound energies F_ε with a nearest-neighbour coercive energy.

(2) The proof is the same as in the one-dimensional case, noting the weak convergence of discrete nonlocal partial derivatives to the corresponding $K_\rho \frac{\partial u}{\partial x_j}$, the difference quotients in directions different than j in the definition of discrete nonlocal partial derivatives cancelling out. □

We give an example in dimension two, where we can compute conditions on the kernel ρ so that condition (32) is satisfied.

Example 6.3. In dimension $d = 2$ we consider the simplest non-trivial case with $M = 2$. In this case we have

$$\begin{aligned}\Big(\frac{\partial_\rho u}{\partial x_1}\Big)_k &= \rho_1(u_{k+e_1} - u_k) + \rho_2(u_{k+2e_1} - u_{k-e_1}) \\ &\quad + \varrho(u_{k+e_1+e_2} - u_{k+e_2}) + \varrho(u_{k+e_1-e_2} - u_{k-e_2}),\end{aligned} \tag{33}$$

where

$$\rho_1 = \rho\Big(\frac{1}{2}\Big), \qquad \rho_2 = \rho\Big(\frac{3}{2}\Big), \qquad \varrho = \rho\Big(\frac{\sqrt{5}}{2}\Big)\frac{1}{\sqrt{5}}.$$

Following the one-dimensional argument, we rewrite this sum, in terms of the differences $z_k = u_{k+e_1} - u_k$, as

$$\Big(\frac{\partial_\rho u}{\partial x_1}\Big)_k = (\rho_1+\rho_2)z_k + \rho_2 z_{k+e_1} + \rho_2 z_{k-e_1} + \varrho z_{k+e_2} + \varrho z_{k-e_2}.$$

Suppose now that the support of u be contained in $[-N+2, N-2]^2$. Then we can write

$$\Big(\frac{\partial_\rho u}{\partial x_1}\Big)_k = (A^N z)_k,$$

where A^N is a $N^2 \times N^2$ symmetric Toeplitz circulant matrix with $\rho_1 + \rho_2$ on the diagonal, ρ_1 on the two next off-diagonal terms, and ϱ on the N-th neighbours. We can then use [13, Chapter 3] and the symmetry of the matrix, to bound the minimal eigenvalue of A^N i by the minimum of the function

$$\Phi^N(t) = \rho_1 + \rho_2 + 2\rho_2\cos(t) + 2\varrho\cos(Nt).$$

A sufficient condition independent of N that ensures that the minimal eigenvalue of A^N is strictly positive is

$$\rho_1 > \rho_2 + 2\varrho. \tag{34}$$

We can argue in the same way for the partial derivative in the x_2-direction. If condition (34) is satisfied then we can argue as in the proof of Theorem 4.1. Namely, if we define

$$F_\varepsilon(u) = \sum_{k\in\mathbb{Z}^2} \varepsilon^2 \big|(\nabla_{\rho_\varepsilon} u)_k\big|^2, \tag{35}$$

then there exists Λ such that

$$F_\varepsilon(u) \geqslant \Lambda \sum_{k,\ell\in\mathbb{Z}^2, |k-\ell|=1} \varepsilon^2 \Big|\frac{u_k - u_\ell}{\varepsilon}\Big|^2, \tag{36}$$

where $u_k = u(\varepsilon k)$ for $u : \varepsilon\mathbb{Z}^2 \to \mathbb{R}$.

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