SISSA 09/2005/FM

Form factors of descendant operators in the massive Lee-Yang model

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Abstract

The form factors of the descendant operators in the massive Lee-Yang model are determined up to level 7. This is first done by exploiting the conserved quantities of the integrable theory to generate the solutions for the descendants starting from the lowest non-trivial solutions in each operator family. We then show that the operator space generated in this way, which is isomorphic to the conformal one, coincides, level by level, with that implied by the S-matrix through the form factor bootstrap. The solutions we determine satisfy asymptotic conditions carrying the information about the level that we conjecture to hold for all the operators of the model.

1 Introduction

A massive quantum field theory can be seen as the perturbation of a fixed point (conformal) theory by relevant or marginally relevant operators. The perturbation spoils some simple features of the conformal point, as the power law behavior of two-point functions, but preserves some structural properties. In particular, the operator space of the massive theory is expected to be isomorphic to that of the conformal theory. In principle, this expectation can be checked in $1+1$ dimensions even for non-trivial fixed points, due to the existence of integrable quantum field theories. What makes the check far from obvious is that the operator content at and away from criticality is determined in two completely different ways.

On one hand, the operator content at the conformal point is dictated by the representation theory of the Virasoro algebra underlying the infinite dimensional conformal symmetry in two dimensions [1]. This reveals a structure of operator families, each consisting of a primary operator and an infinite tower of descendants organized into multiplets (levels) labeled by integer spaced scaling dimensions. On the massive side, instead, integrable quantum field theories are solved through the exact determination of the S-matrix [2]. The operators are then constructed determining their matrix elements (form factors) on the particle states as solutions of a set of functional equations [3, 4].

It was first shown in [5] for the thermal Ising model, and later for more complicated models [6, 7, 8], that the global counting of independent solutions of the form factor equations matches that expected from conformal field theory. This counting procedure for the form factor solutions, however, disentangles neither the operator families sharing the same symmetry properties nor the levels of the descendants.

In this paper we tackle the problem of the detailed isomorphism between the critical and off-critical operator spaces for the simplest interacting massive theory, the Lee-Yang model. The previously available results for the operators of this model concern the non-trivial primary operator [9] and the first non-trivial scalar descendant $T\bar{T}$ of the identity family [10]. We exploit the first few quantum integrals of motion of the theory to determine, for the two operator families of the model and for each level up to 7, a number of independent form factor solutions coinciding with the number predicted by conformal field theory. We then show that the operators constructed in this way do form a basis for the space of solutions of the form factor equations up to level seven. This is achieved by supplementing the form factor equations with asymptotic conditions carrying the information about the level. We also show that all the new solutions determined in this paper automatically satisfy the asymptotic factorization property for the descendant operators conjectured in [10].

The paper is organized as follows. In the next section we recall the structure of the operator space at criticality as well as the S-matrix and form factor language used for the integrable massive deformations. In section 3 we specialize the discussion to the Lee-Yang model before turning to the construction of the off-critical descendant operators through the use of the conserved quantities in section 4. In section 5 we show the isomorphism with the operator space spanned by the solutions of the form factor equations, while section 6 contains few final remarks. Three appendices conclude the paper.

2 Critical and off-critical operators

The operators at a critical point undergo the general conformal field theory classification [1]. A scaling operator $\Phi(x)$ is first of all labeled by a pair $(\Delta_\Phi, \bar{\Delta}_\Phi)$ of conformal dimensions which determine the scaling dimension X_Φ and the euclidean spin s_Φ as

$$X_\Phi = \Delta_\Phi + \bar{\Delta}_\Phi \tag{2.1}$$

$$s_\Phi = \Delta_\Phi - \bar{\Delta}_\Phi. \tag{2.2}$$

Locality requires that s_Φ is an integer or half-integer number. There exist operator families associated to the highest weight representations of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\,. \tag{2.3}$$

The L_n's generate the conformal transformations associated to the complex variable $z = x_1 + ix_2$, with the central charge c labeling the conformal theory. The same algebra, with the same value of c, holds for the generators $\bar{L}_n$ of the conformal transformations in the variable $\bar{z} = x_1 - ix_2$. The L_n's commute with the $\bar{L}_m$'s. Each operator family consists of a primary operator Φ_0 (which is annihilated by all the generators L_n and $\bar{L}_n$ with $n > 0$) and infinitely many descendant operators obtained through the repeated action on the primary of the Virasoro generators. A basis in the space of descendants is given by the operators

$$L_{-i_1} \ldots L_{-i_I} \bar{L}_{-j_1} \ldots \bar{L}_{-j_J}\, \Phi_0 \tag{2.4}$$

with

$$0 < i_1 \le i_2 \le \ldots \le i_I \tag{2.5}$$

$$0 < j_1 \le j_2 \le \ldots \le j_J\,. \tag{2.6}$$

The levels

$$(l, \bar{l}) = \left(\sum_{n=1}^{I} i_n\,, \sum_{n=1}^{J} j_n \right) \tag{2.7}$$

determine the conformal dimensions of the descendants (2.4) in the form

$$(\Delta, \bar{\Delta}) = (\Delta_{\Phi_0} + l, \bar{\Delta}_{\Phi_0} + \bar{l})\,. \tag{2.8}$$

In general the number of independent operators at level $(l, \bar{l})$ is $p(l)p(\bar{l})$, $p(l)$ being the number of partitions of l into positive integers. This number is reduced in presence of degenerate representations associated to primary operators $\phi_{r,s}$ which possess a vanishing linear combination of descendant operators (null vector) when l or $\bar{l}$ equals rs.

Any conformal theory possesses a primary operator corresponding to the identity I with conformal dimensions $\Delta_I = \bar{\Delta}_I = 0$. All the chiral descendants T_s and $\bar{T}_s$ of the identity at level $(s,0)$ and $(0,s)$, respectively, are local operators satisfying the conservation equations ($\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$)

$$\bar{\partial} T_s = 0 \tag{2.9}$$

$$\partial \bar{T}_s = 0 \tag{2.10}$$

associated to the infinite dimensional conformal symmetry. Since $L_{-1} = \partial$ and $\bar{L}_{-1} = \bar{\partial}$, the first non-vanishing chiral descendants in the identity family are $T = L_{-2}I$ and $\bar{T} = \bar{L}_{-2}I$. They coincide with the non-vanishing components of the energy-momentum tensor at criticality.

We can see a massive theory as the perturbation of a conformal theory. Considering for the sake of simplicity a single relevant perturbing operator $\varphi(x)$, we have the action

$$\mathcal{A} = \mathcal{A}_{CFT} + g \int d^2x \, \varphi(x) \,. \tag{2.11}$$

From the point of view of perturbation theory in g, a well defined renormalization procedure can be implemented to continue the conformal operators away from criticality [9, 11], with the result that the operators of the perturbed theory (2.11) are in one to one correspondence with those of the conformal field theory corresponding to $g = 0$ [2]. A source of ambiguity in the definition of the off-critical operators appears in presence of "resonances" [2] (see also [12]). An operator Ψ is said to have a resonance with the operator Φ if the two operators have the same spin, $s_\Psi = s_\Phi$, and their scaling dimensions satisfy the condition $X_\Psi = X_\Phi + n(2 - X_\varphi)$ for some positive integer n. In such a case the ambiguity $\Psi \to \Psi + \textit{constant } g^n \Phi$ mixes two operators which at criticality are distinguished by a different scaling dimension.

The theory (2.11) is integrable if operators $\Theta_s(x)$ with spin s exist such that the conservation equations (2.9) can be deformed into the off-critical form

$$\bar{\partial} T_s = \partial \Theta_{s-2} \tag{2.12}$$

for an infinite set of integers s which is specific of the model (similarly, (2.10) acquires a total derivative with respect to $\bar{z}$ on the r.h.s.) [2]. Then the quantities

$$Q_s = \int_{-\infty}^{+\infty} dx_1 \, [T_{s+1} + \Theta_{s-1}] \tag{2.13}$$

with spin $s > 0$, together with their counterparts $\bar{Q}_s$ with spin $-s$, are conserved in the massive quantum field theory. In particular, $\Theta_0 \equiv \Theta \sim g\varphi$ is the trace of the energy-momentum tensor, and $Q_1 = P^0 + P^1$ and $\bar{Q}_1 = P^0 - P^1$ are the light-cone components of energy-momentum.

The existence of an infinite number of conserved quantities induces the complete elasticity and factorization of the scattering processes, and allows the exact determination of the S-matrix [13]. In such a framework, the operators are constructed determining their matrix elements on the asymptotic particle states. In order to avoid inessential complications of the notation we consider here the case in which the spectrum of the theory (2.11) possesses a single particle of mass $m \sim g^{1/(2-X_\varphi)}$. The matrix elements (form factors)

$$F_n^\Phi(\theta_1,\ldots,\theta_n) = \langle 0|\Phi(0)|\theta_1\ldots\theta_n\rangle \qquad (2.14)$$

of the local operator $\Phi(x)$ between the vacuum and the n-particle states[1] in the integrable theory satisfy the functional equations [3, 4]

$$F_n^\Phi(\theta_1+\alpha,\ldots,\theta_n+\alpha) = e^{s_\Phi\alpha}F_n^\Phi(\theta_1,\ldots,\theta_n) \qquad (2.15)$$

$$F_n^\Phi(\theta_1,\ldots,\theta_i,\theta_{i+1},\ldots,\theta_n) = S(\theta_i-\theta_{i+1})\,F_n^\Phi(\theta_1,\ldots,\theta_{i+1},\theta_i,\ldots,\theta_n) \qquad (2.16)$$

$$F_n^\Phi(\theta_1+2i\pi,\theta_2,\ldots,\theta_n) = F_n^\Phi(\theta_2,\ldots,\theta_n,\theta_1) \qquad (2.17)$$

$$\text{Res}_{\theta'=\theta}\,F_{n+2}^\Phi(\theta'+\frac{i\pi}{3},\theta-\frac{i\pi}{3},\theta_1,\ldots,\theta_n) = i\Gamma\,F_{n+1}^\Phi(\theta,\theta_1,\ldots,\theta_n) \qquad (2.18)$$

$$\text{Res}_{\theta'=\theta+i\pi}\,F_{n+2}^\Phi(\theta',\theta,\theta_1,\ldots,\theta_n) = i\left[1-\prod_{j=1}^{n}S(\theta-\theta_j)\right]F_n^\Phi(\theta_1,\ldots,\theta_n)\,, \qquad (2.19)$$

where $S(\theta)$ is the two-particle scattering amplitude, and the three-particle coupling Γ is obtained from

$$\text{Res}_{\theta=2i\pi/3}S(\theta) = i\Gamma^2\,. \qquad (2.20)$$

Of course the resonance angle in the last equation as well as in (2.18) will take more general values when spectra with particles of different masses are considered.

The equations (2.15)–(2.19) do not distinguish among operators with the same spin s_Φ. It was shown in [14] for reflection-positive theories that the conformal dimensions determine a quite restrictive upper bound on the asymptotic behavior of form factors at high energies. In general, it is natural to expect a relation between the asymptotic behavior and the level an operator of the massive theory belongs to in the conformal limit. Indeed, the equations (2.15)–(2.19) admit "minimal solutions" characterized by the mildest asymptotic behavior, which are associated to the primary operators. The asymptotic behavior of the non-minimal solutions, which should correspond to the descendants, differs by integer powers of the momenta from that of the minimal ones (see [10]), a pattern that obviously recalls the integer spacing of levels in the conformal limit. Needless to say, the high energy limit is in fact a massless limit toward the ultraviolet conformal point.

Given a solution of the form factor equations corresponding to an operator Φ, the conserved quantities Q_s (as well as the $\bar{Q}_s$) can be used to generate infinitely many new independent solutions. Indeed, the commutator $[Q_s,\Phi]$ is the operator with conformal dimensions $(\Delta_\Phi + s,\bar{\Delta}_\Phi)$ corresponding to the variation of Φ under the transformation generated by Q_s. On the

[1] The rapidity variables θ_k parameterize the on-shell momenta of the particles as $(p_k^0,p_k^1) = (m\cosh\theta_k, m\sinh\theta_k)$. The generic matrix elements with particles also on the left can be obtained by analytic continuation of (2.14).

other hand, the conserved quantities annihilate the vacuum and act diagonally on the asymptotic particle states as[2]

$$Q_s|\theta_1,\ldots,\theta_n\rangle = \Lambda_n^{(s)}(\theta_1,..,\theta_n)\,|\theta_1,\ldots,\theta_n\rangle \tag{2.21}$$

$$\bar{Q}_s|\theta_1,\ldots,\theta_n\rangle = \Lambda_n^{(-s)}(\theta_1,..,\theta_n)\,|\theta_1,\ldots,\theta_n\rangle\,, \tag{2.22}$$

where

$$\Lambda_n^{(s)}(\theta_1,..,\theta_n) \equiv m^{|s|}\sum_{i=1}^{n} e^{s\theta_i}\,. \tag{2.23}$$

Then, writing for simplicity from now on $Q_s\Phi$ instead of $[Q_s,\Phi]$ and $\bar{Q}_s\Phi$ instead of $[\bar{Q}_s,\Phi]$, we have

$$F_n^{Q_s\Phi}(\theta_1,\ldots,\theta_n) = -\Lambda_n^{(s)}(\theta_1,..,\theta_n)\,F_n^{\Phi}(\theta_1,\ldots,\theta_n) \tag{2.24}$$

$$F_n^{\bar{Q}_s\Phi}(\theta_1,\ldots,\theta_n) = -\Lambda_n^{(-s)}(\theta_1,..,\theta_n)\,F_n^{\Phi}(\theta_1,\ldots,\theta_n)\,. \tag{2.25}$$

It is straightforward to check that these expressions satisfy the general form factor equations (2.15)–(2.17) for operators with spin $s_\Phi + s$ and $s_\Phi - s$, respectively. Equation (2.19) leads to the condition

$$e^{s\theta} + e^{s(\theta+i\pi)} = 0\,, \tag{2.26}$$

showing that the values of the spin s of the conserved quantities need to be odd. Finally, if $\Gamma \neq 0$, equation (2.18) gives the condition

$$e^{is\pi/3} + e^{-is\pi/3} = 1\,, \tag{2.27}$$

showing that, in a theory with a single particle A in the spectrum and admitting the fusion $AA \to A$, the allowed values for the spin of the conserved quantities are

$$s = 6n \pm 1\,, \qquad n = 0, \pm 1, \pm 2\ldots\ . \tag{2.28}$$

3 The Lee-Yang model

3.1 Critical point

The Lee-Yang model is the quantum field theory associated to the edge singularity of the zeros of the partition function of the Ising model in an imaginary magnetic field [15, 16, 17]. The critical point is described by the simplest conformal field theory [18], i.e. the minimal model $\mathcal{M}_{2,5}$, with central charge $c = -22/5$, possessing only two local primary operators: the identity $I = \phi_{1,1} = \phi_{1,4}$ with conformal dimensions $(0,0)$, and the operator $\varphi = \phi_{1,2} = \phi_{1,3}$ with conformal dimensions $(-1/5, -1/5)$. The negative values of the central charge and of X_φ show that the model does not satisfy reflection-positivity.

[2]Equations (2.21) and (2.22) also fix our normalization for the conserved quantities.

Level of the descendant	I	φ
$(1,0)$	0	$\partial\varphi$
$(2,0)$	T	$\partial^2\varphi$
$(3,0)$	∂T	$\partial^3\varphi$
$(4,0)$	$\partial^2 T$	$\partial^4\varphi\ ;\ L_{-4}\varphi$
$(5,0)$	$\partial^3 T$	$\partial^5\varphi\ ;\ \partial L_{-4}\varphi$
$(6,0)$	$\partial^4 T\ ;\ L_{-4}T$	$\partial^6\varphi\ ;\ \partial^2 L_{-4}\varphi\ ;\ L_{-6}\varphi$
$(7,0)$	$\partial^5 T\ ;\ \partial L_{-4}T$	$\partial^7\varphi\ ;\ \partial^3 L_{-4}\varphi\ ;\ \partial L_{-6}\varphi$

Table 1: Basis for the left chiral descendants of the two primary fields in the Lee-Yang model up to level 7 in terms of Virasoro generators. The corresponding basis for the right descendants is obtained changing ∂, L_{-n} and T into $\bar\partial$, $\bar L_{-n}$ and $\bar T$, respectively.

The degenerate nature of the primary fields reduces the dimensionality of the space of the level $(l,\bar l)$ descendants of the primary ϕ to $d_\phi(l)d_\phi(\bar l)$, with $d_\phi(n)$ generated by the rescaled character

$$\chi_\phi(q) = \sum_{n=0}^{\infty} d_\phi(n) q^n \,. \tag{3.1}$$

The two characters for the Lee-Yang model are [19, 20, 21]

$$\chi_I(q) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{2+5n})\,(1-q^{3+5n})}$$

$$\chi_\varphi(q) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{1+5n})\,(1-q^{4+5n})}\,.$$

The dimensionalities of the first few levels can then be read from the expansions

$$\chi_I(q) = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + O(q^9)$$

$$\chi_\varphi(q) = 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + O(q^9)\,.$$

A basis for the chiral descendants up to level 7 is given in table 1.

At the conformal point the conserved quantities Q_s are combinations of the Virasoro generators L_n with dimensions $(s,0)$, while the $\bar Q_s$ are combinations of the $\bar L_n$ with dimensions $(0,s)$. However, since Q_s and $\bar Q_s$ are conserved in the massive theory and act diagonally on the particle states, they are much more convenient than the Virasoro generators for labeling the descendant operators in view of the off-critical continuation. This is why we exploit the existence of conserved quantities with spin 5 and 7 in the Lee-Yang model to switch from the basis of table 1 to that given in table 2. Here the descendant operators S_4T, $R_4\varphi$ and $R_6\varphi$ are

Level of the descendant	I	φ
$(1,0)$	0	$\partial\varphi$
$(2,0)$	T	$\partial^2\varphi$
$(3,0)$	∂T	$\partial^3\varphi$
$(4,0)$	$\partial^2 T$	$\partial^4\varphi\ ;\ R_4\varphi$
$(5,0)$	$\partial^3 T$	$\partial^5\varphi\ ;\ Q_5\varphi$
$(6,0)$	$\partial^4 T\ ;\ S_4 T$	$\partial^6\varphi\ ;\ \partial Q_5\varphi\ ;\ R_6\varphi$
$(7,0)$	$\partial^5 T\ ;\ Q_5 T$	$\partial^7\varphi\ ;\ \partial^2 Q_5\varphi\ ;\ Q_7\varphi$

Table 2: Basis for the left chiral descendants of the two primary fields in the Lee-Yang model up to level 7 exploiting the conserved quantities. An analogous basis exists for the right chiral descendants.

defined through the relations

$$Q_5 T = \partial S_4 T \tag{3.2}$$

$$Q_5\varphi = \partial R_4\varphi \tag{3.3}$$

$$Q_7\varphi = \partial R_6\varphi\ . \tag{3.4}$$

The descendant operators $\bar{S}_4\bar{T}$, $\bar{R}_4\varphi$ and $\bar{R}_6\varphi$ defined through the relations

$$\bar{Q}_5\bar{T} = \bar{\partial}\bar{S}_4\bar{T} \tag{3.5}$$

$$\bar{Q}_5\varphi = \bar{\partial}\bar{R}_4\varphi \tag{3.6}$$

$$\bar{Q}_7\varphi = \bar{\partial}\bar{R}_6\varphi \tag{3.7}$$

appear in the right chiral basis.

The linear independence between $\partial^5 T$ and $Q_5 T$ ensures that we can write $Q_5 T \sim \partial^5 T + a\partial L_{-4}T$, with $a \neq 0$. Then (3.2) ensures that $S_4 T$ contains a non-vanishing component of $L_{-4}T$. Similar considerations apply to the other operators we have defined through (3.2)–(3.7). Then we write

$$\begin{aligned} S_4 &\sim \partial^4 + aL_{-4}\,, & \bar{S}_4 &\sim \bar{\partial}^4 + a\bar{L}_{-4} \\ R_4 &\sim \partial^4 + bL_{-4}\,, & \bar{R}_4 &\sim \bar{\partial}^4 + b\bar{L}_{-4} \\ R_6 &\sim \partial^6 + c\partial^2 L_{-4} + dL_{-6}\,, & \bar{R}_6 &\sim \bar{\partial}^6 + c\bar{\partial}^2 L_{-4} + d\bar{L}_{-6}\,, \end{aligned}$$

with a, b and d different from zero.

3.2 Scaling limit

The off-critical Lee-Yang model is obtained perturbing the conformal field theory $\mathcal{M}_{2,5}$ with the primary operator φ, which is the only non-trivial relevant operator in the theory. The massive model is integrable as a consequence of the general results of [2] about perturbed conformal field theories. The mass spectrum contains only one particle A and the exact on-shell solution is provided by the S-matrix [22]

$$S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2i\pi}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2i\pi}{3}\right)}, \tag{3.8}$$

where θ is the rapidity difference of the two colliding particles. The bound state pole located at $\theta = 2i\pi/3$ implies the fusion property $AA \to A$, so that the model possesses conserved quantities with the spins given in (2.28).

The trace of the energy-momentum tensor $\Theta = T^{\mu}_{\mu}/4$ is proportional to the perturbing operator φ and from now on we refer to Θ instead of φ in the off-critical theory. The form factors of Θ were determined in [23, 9] (see also [6]) and can be written as

$$F_n^{\Theta}(\theta_1, \ldots, \theta_n) = U_n^{\Theta}(\theta_1, \ldots, \theta_n) \prod_{i<j} \frac{F_{min}(\theta_i - \theta_j)}{\cosh \frac{\theta_i - \theta_j}{2} \left[\cosh(\theta_i - \theta_j) + \frac{1}{2}\right]}. \tag{3.9}$$

Here the factors in the denominator introduce the bound state and annihilation poles prescribed by (2.18) and (2.19), while

$$F_{min}(\theta) = -i \sinh \frac{\theta}{2} \exp\left\{2 \int_0^{\infty} \frac{dt}{t} \frac{\cosh \frac{t}{6}}{\cosh \frac{t}{2} \sinh t} \sin^2 \frac{(i\pi - \theta)t}{2\pi}\right\} \tag{3.10}$$

is a solution of the equations

$$F(\theta) = S(\theta) F(-\theta) \tag{3.11}$$

$$F(\theta + 2i\pi) = F(-\theta) \tag{3.12}$$

free of zeros and poles for $\text{Im}\theta \in (0, 2\pi)$ and behaving as

$$F_{min}(\theta) \sim e^{|\theta|} \tag{3.13}$$

when $|\theta| \to \infty$. Finally, $U_n^{\Theta}(\theta_1, \ldots, \theta_n)$ are the following entire functions of the rapidities, symmetric and (up to a factor $(-1)^{n-1}$) $2\pi i$-periodic in all θ_j's:

$$U_n^{\Theta}(\theta_1, .., \theta_n) = H_n \left(\frac{1}{\sigma_n^{(n)}}\right)^{(n-1)/2} Q_n^{\Theta}(\theta_1, .., \theta_n) \,. \tag{3.14}$$

Here $\sigma_i^{(n)}$ are the elementary symmetric polynomials generated by

$$\prod_{i=1}^{n} (x + x_i) = \sum_{k=0}^{n} x^{n-k} \sigma_k^{(n)}(x_1, \ldots, x_n), \tag{3.15}$$

with $x_i \equiv e^{\theta_i}$, $H_n = i^{n^2} (3/4)^{n/4} \gamma^{n(n-2)}$,

$$\gamma = \exp\left\{2 \int_0^{\infty} \frac{dt}{t} \frac{\sinh \frac{t}{2} \sinh \frac{t}{3} \sinh \frac{t}{6}}{\sinh^2 t}\right\}, \tag{3.16}$$

and $Q_n^\Theta(\theta_1,\dots,\theta_n)$ is the determinant

$$Q_n^\Theta(\theta_1,\dots,\theta_n) = -\frac{\pi m^2}{4\sqrt{3}}\, det \left\| M_{i,j}^{(n)} \right\| \tag{3.17}$$

of the $(n-1)\times(n-1)$ matrix with entries

$$M_{i,j}^{(n)} = \frac{\sin(2(i-j+1)\frac{\pi}{3})}{\sin\frac{2\pi}{3}}\, \sigma_{2i-j}^{(n)} \,. \tag{3.18}$$

The conservation equations

$$\bar\partial T = \partial\Theta \,, \qquad \partial\bar T = \bar\partial\Theta \tag{3.19}$$

allow the determination of the form factors of the other components of the energy-momentum tensor in the form

$$F_n^T(\theta_1,\dots,\theta_n) = -\frac{\sigma_n^{(n)}\sigma_1^{(n)}}{\sigma_{n-1}^{(n)}}\, F_n^\Theta(\theta_1,\dots,\theta_n) \tag{3.20}$$

$$F_n^{\bar T}(\theta_1,\dots,\theta_n) = -\frac{\sigma_{n-1}^{(n)}}{\sigma_1^{(n)}\sigma_n^{(n)}} F_n^\Theta(\theta_1,\dots,\theta_n) \tag{3.21}$$

for $n>0$. Of course $\langle T\rangle = \langle\bar T\rangle = 0$ as for any operator with non-zero spin. The factors in the denominator of the last two equations do not introduce unwanted singularities since the functions Q_n^Θ have the property of factorizing $\sigma_1^{(n)}$ and $\sigma_{n-1}^{(n)}$, respectively the eigenvalues[3] of ∂ and $\bar\partial$ on the n-particle asymptotic states.

4 Off-critical descendants from conservation laws

In this section we show how the form factors of all the descendant operators in the massive Lee-Yang model up to level 7 can be obtained starting from the known solutions for Θ, T, $\bar T$ and $T\bar T$ exploiting the conserved quantities of the integrable model.

All the solutions we find in this way automatically satisfy the following asymptotic properties that we conjecture to hold for all the operators of the theory[4]:

i) the form factors of a level $(l,\bar l)$ non-chiral descendant $\Phi_{(l,\bar l)}$ of the primary Φ_0 are homogeneous functions, with respect to the variables $x_i \equiv e^{\theta_i}$, of degree $l-\bar l$ (eq. (2.15)), furthermore behaving as

$$F_n^{\Phi_{(l,\bar l)}}(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},...,\theta_n) \sim e^{l\alpha} \tag{4.1}$$

[3] More precisely

$$\partial|\theta_1,\dots,\theta_n\rangle = -im\sigma_1^{(n)}|\theta_1,\dots,\theta_n\rangle \quad ; \quad \bar\partial|\theta_1,\dots,\theta_n\rangle = im\bar\sigma_1^{(n)}|\theta_1,\dots,\theta_n\rangle$$

where

$$\bar\sigma_i^{(n)} \equiv \sigma_{n-i}^{(n)}/\sigma_n^{(n)}.$$

[4] We show in appendix B that these properties hold for all the descendant operators obtained acting on a primary with conserved quantities.

for $\alpha \to +\infty$, $n > 1$ and $1 \leq k \leq n-1$. This equation holds also for the chiral descendants of Θ, while the chiral descendants of I of level $(l,0)$ and $(0,\bar{l})$ have form factors with homogeneous degree l and $-\bar{l}$, respectively, and behave as

$$F_n^{I_{(l,0)}}(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},...,\theta_n) \sim e^{(l-1)\alpha}\,, \quad F_n^{I_{(0,\bar{l})}}(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{l+1},...,\theta_n) \sim e^{-\alpha} \tag{4.2}$$

for $\alpha \to +\infty$, $n > 1$ and $1 \leq k \leq n-1$;

ii) let us denote by $\mathcal{L}_l \bar{\mathcal{L}}_{\bar{l}} \Phi_0$ a level $(l,\bar{l})$ descendant which in the critical limit assumes the form (2.4) with $\mathcal{L}_l$ ($\bar{\mathcal{L}}_{\bar{l}}$) accounting for the product of L_{-i} ($\bar{L}_{-j}$). Then the factorization property

$$\lim_{\alpha\to+\infty} e^{-l\alpha} F_n^{\mathcal{L}_l\bar{\mathcal{L}}_{\bar{l}}\Phi_0}(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},..,\theta_n) = \frac{1}{\langle\Phi_0\rangle} F_k^{\mathcal{L}_l\Phi_0}(\theta_1,..,\theta_k)\, F_{n-k}^{\bar{\mathcal{L}}_{\bar{l}}\Phi_0}(\theta_{k+1},..,\theta_n) \tag{4.3}$$

holds for $k = 0, 1, \ldots, n$.

4.1 Descendants of Θ

The form factors of the descendants of Θ obtained acting on the primary with the conserved quantities are defined without ambiguity by (2.24) and (2.25). Then we are left with the problem of fixing the form factors of the operators obtained acting on Θ with R_4 , R_6 and their negative spin counterparts. We know from (3.3), (3.4), (3.6) and (3.7) that at criticality these operators differ by derivatives from operators obtained through the action of conserved quantities. Since the form factors of Θ factorize the eigenvalues of $\partial\bar{\partial}$, this property is automatically satisfied also in the massive theory. Indeed, stripping off in each case the appropriate derivative eigenvalue we can write

$$F_n^{R_4\Theta} = \left(im\sigma_1^{(n)}\right)^{-1} F_n^{Q_5\Theta}, \qquad F_n^{\bar{R}_4\Theta} = \left(-im\bar{\sigma}_1^{(n)}\right)^{-1} F_n^{\bar{Q}_5\Theta}$$

$$F_n^{R_6\Theta} = \left(im\sigma_1^{(n)}\right)^{-1} F_n^{Q_7\Theta}, \qquad F_n^{\bar{R}_6\Theta} = \left(-im\bar{\sigma}_1^{(n)}\right)^{-1} F_n^{\bar{Q}_7\Theta}$$

$$F_n^{R_4\bar{R}_4\Theta} = \left(m^2\sigma_1^{(n)}\bar{\sigma}_1^{(n)}\right)^{-1} F_n^{Q_5\bar{Q}_5\Theta}, \qquad F_n^{R_4\bar{R}_6\Theta} = \left(m^2\sigma_1^{(n)}\bar{\sigma}_1^{(n)}\right)^{-1} F_n^{Q_5\bar{Q}_7\Theta}$$

$$F_n^{R_6\bar{R}_6\Theta} = \left(m^2\sigma_1^{(n)}\bar{\sigma}_1^{(n)}\right)^{-1} F_n^{Q_7\bar{Q}_7\Theta}, \qquad F_n^{R_6\bar{R}_4\Theta} = \left(m^2\sigma_1^{(n)}\bar{\sigma}_1^{(n)}\right)^{-1} F_n^{Q_7\bar{Q}_5\Theta}.$$

The form factors of Θ given in the previous section satisfy the asymptotic factorization property ($k = 0, 1, \ldots, n$)

$$\lim_{\alpha\to+\infty} F_n^{\Theta}(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},..,\theta_n) = \frac{1}{\langle\Theta\rangle} F_k^{\Theta}(\theta_1,..,\theta_k) F_{n-k}^{\Theta}(\theta_{k+1},..,\theta_n) \tag{4.4}$$

discussed in [24], with

$$\langle\Theta\rangle = F_0^\Theta = -\frac{\pi m^2}{4\sqrt{3}}\,. \tag{4.5}$$

This value of the vacuum expectation value was originally obtained in [25] using the thermodynamic Bethe ansatz.

The factorization properties of the eigenvalues of the conserved quantities given in appendix B then ensure that the solutions determined above satisfy (4.3) with $\Phi_0 = \Theta$. This in turn implies the asymptotic behavior (4.1).

4.2 Descendants of the identity

The matrix elements of the chiral descendants of I obtained through conserved charges are again fixed by (2.24) and (2.25) together with the solution for T and $\bar{T}$ given in the previous section. Concerning the remaining chiral operators S_4T and $\bar{S}_4\bar{T}$, we observe that the form factors of T and $\bar{T}$ factorize the eigenvalues of ∂^2 and $\bar{\partial}^2$, respectively. We can then exploit (3.2) and (3.5) to write

$$F_n^{S_4T} = \left(im\sigma_1^{(n)}\right)^{-1} F_n^{Q_5T} \quad , \quad F_n^{\bar{S}_4\bar{T}} = \left(-im\bar{\sigma}_1^{(n)}\right)^{-1} F_n^{\bar{Q}_5\bar{T}}\,. \tag{4.6}$$

We can now turn to the problem of identifying the off-critical continuation of the non-chiral descendants of I. The first of such operators is $T\bar{T}$. It was shown in [26] that this operator can be conveniently defined away from criticality as

$$T\bar{T}(x) = \lim_{x'\to x}\left[T(x)\bar{T}(x') - \mathcal{R}_{T\bar{T}}(x,x')\right], \tag{4.7}$$

where

$$\mathcal{R}_{T\bar{T}}(x,x') = \Theta(x)\Theta(x') + \text{derivative terms} \tag{4.8}$$

regularizes the divergences arising in the limit away from criticality. The matrix elements of $T\bar{T}$ in the Lee-Yang model have been identified in [10] and read

$$F_n^{T\bar{T}} = \frac{\langle\Theta\rangle}{m^4}\,F_n^{\partial^2\bar{\partial}^2\Theta} + \langle\Theta\rangle^2\,F_n^{K_3} - 2\,\langle\Theta\rangle F_n^\Theta + \langle\Theta\rangle^2\;\delta_{n,0} + c\,F_n^{\partial\bar{\partial}\Theta}\,. \tag{4.9}$$

The kernel solution (see next section) $F_n^{K_3}$ is given explicitly in [10] up to $n = 9$. The operator $\partial\bar{\partial}\Theta$ is resonant with $T\bar{T}$, so that the coefficient c in (4.9) is intrinsically ambiguous.

Going to higher non-chiral descendants of the identity, consider those operators which are obtained as regularized products of conserved quantities acting on T and $\bar{T}$. It is shown in appendix A that the subtraction which regularizes the operator (4.7) regularizes also these composite operators. We exploit, however, the possibility of adding resonant terms ρ_k, and write

$$Q_s\bar{Q}_rT\bar{T}(x) = Q_s\bar{Q}_r\left(T\bar{T}\right)(x) + \sum_k c_k\rho_k(x)\,, \tag{4.10}$$

or in terms of form factors,

$$F_n^{Q_s\bar{Q}_rT\bar{T}} = \Lambda_n^{(s)}\Lambda_n^{(-r)}F_n^{T\bar{T}} + \sum_k c_kF_n^{\rho_k}\,. \tag{4.11}$$

The role of the resonances is the following. We know from table 2 and equations (3.2)-(3.7) that at criticality some of the operators we are considering are derivatives of operators living one level below. We expect this property to continue to hold in the massive theory and have seen how this is automatically achieved for the descendants of Θ and the chiral descendants of I. Concerning the non-chiral descendants of I, the form factor solution for $T\bar{T}$ does not factorize any derivative eigenvalue (see [10]), and this property is not guaranteed a priori. Remarkably, however, it can be recovered by fine tuning the coefficients of some of the resonant terms in the last equation.

To see this, observe first that, since $T\bar{T}$ and Θ have conformal dimensions $(2,2)$ and $(1,1)$, respectively, the level (p,q) descendant of I is resonant with the level $(p-1,q-1)$ descendants of Θ for p and q larger than 1. Letting aside for the time being the operators containing S_4 and $\bar{S}_4$, for p and q less than 7 the above requirement only forces the coefficients of resonances like $R_4\bar{\partial}\Theta$, $\partial\bar{R}_4\Theta$ and $R_4\bar{R}_4\Theta$ (which do not contain the required derivative terms) to vanish.

A more interesting situation arises when considering the descendants of I containing Q_5 or $\bar{Q}_5$ acting on $T\bar{T}$. At level $(7,2)$ we have two operators, $Q_5T\bar{T}$ and $\partial^5T\bar{T}$, which away from criticality are resonant with the space of level $(6,1)$ descendants of Θ spanned by $Q_5\partial\bar{\partial}\Theta$, $\partial^6\bar{\partial}\Theta$ and $R_6\bar{\partial}\Theta$. The latter operator, with form factors

$$F_n^{R_6\bar{\partial}\Theta}=\frac{\bar{\sigma}_1^{(n)}}{\sigma_1^{(n)}}\Lambda_n^{(7)}F_n^{\Theta}\,, \tag{4.12}$$

is the only one among the resonant operators which does not contain a derivative with respect to z. The requirement that also away from criticality the operators $Q_5T\bar{T}$ and $\partial^5T\bar{T}$ are derivatives with respect to z of the two level $(6,2)$ descendants of I is satisfied with the following unique choice of the coefficients of $R_6\bar{\partial}\Theta$

$$\partial^5T\bar{T}(x)=\partial^5\left(T\bar{T}\right)(x) \tag{4.13}$$

$$Q_5T\bar{T}(x)=Q_5\left(T\bar{T}\right)(x)-\frac{5}{7}R_6\bar{\partial}\Theta(x)\,. \tag{4.14}$$

Notice that, since the composite operator $T\bar{T}$ already includes the resonant term $\partial\bar{\partial}\Theta$ in its definition (see (4.9)), the resonances $Q_5\partial\bar{\partial}\Theta$ and $\partial^6\bar{\partial}\Theta$ are automatically included in the space spanned by (4.13) and (4.14). The same analysis gives

$$\bar{\partial}^5T\bar{T}(x)=\bar{\partial}^5\left(T\bar{T}\right)(x) \tag{4.15}$$

$$\bar{Q}_5T\bar{T}(x)=\bar{Q}_5\left(T\bar{T}\right)(x)-\frac{5}{7}\partial\bar{R}_6\Theta(x) \tag{4.16}$$

at level $(2,7)$, and

$$\partial^5\bar{\partial}^5T\bar{T}(x)=\partial^5\bar{\partial}^5\left(T\bar{T}\right)(x) \tag{4.17}$$

$$Q_5\bar{Q}_5T\bar{T}(x)=Q_5\bar{Q}_5\left(T\bar{T}\right)(x)-\frac{5}{7}R_6\bar{\partial}\bar{Q}_5\Theta(x)-\frac{5}{7}\partial Q_5\bar{R}_6\Theta(x) \tag{4.18}$$

$$\partial^5\bar{Q}_5T\bar{T}(x)=\partial^5\bar{Q}_5\left(T\bar{T}\right)(x)-\frac{5}{7}\partial^6\bar{R}_6\Theta(x) \tag{4.19}$$

$$\bar{\partial}^5Q_5T\bar{T}(x)=\bar{\partial}^5Q_5\left(T\bar{T}\right)(x)-\frac{5}{7}R_6\bar{\partial}^6\Theta(x) \tag{4.20}$$

at level $(7,7)$.

We are now in the position of dealing straightforwardly with the operators $S_4T\bar{T}$, $\bar{S}_4T\bar{T}$ and $S_4\bar{S}_4T\bar{T}$. Indeed, the construction above ensures that the equations

$$Q_5T\bar{T} = \partial S_4T\bar{T} \quad ; \quad \bar{Q}_5T\bar{T} = \bar{\partial}\bar{S}_4T\bar{T} \quad ; \quad Q_5\bar{Q}_5T\bar{T} = \partial\bar{\partial}S_4\bar{S}_4T\bar{T} \tag{4.21}$$

hold also away from criticality, so that we can write the form factors

$$F_n^{S_4T\bar{T}} = \left(-im\sigma_1^{(n)}\right)^{-1}\left(\Lambda_n^{(5)}F_n^{T\bar{T}} + \frac{5}{7}\frac{\bar{\sigma}_1^{(n)}}{\sigma_1^{(n)}}\Lambda_n^{(7)}F_n^{\Theta}\right) \tag{4.22}$$

$$F_n^{\bar{S}_4T\bar{T}} = \left(im\bar{\sigma}_1^{(n)}\right)^{-1}\left(\Lambda_n^{(-5)}F_n^{T\bar{T}} + \frac{5}{7}\frac{\sigma_1^{(n)}}{\bar{\sigma}_1^{(n)}}\Lambda_n^{(-7)}F_n^{\Theta}\right) \tag{4.23}$$

$$F_n^{S_4\bar{S}_4T\bar{T}} = \left(m^2\sigma_1^{(n)}\bar{\sigma}_1^{(n)}\right)^{-1}\left(\Lambda_n^{(5)}\Lambda_n^{(-5)}F_n^{T\bar{T}} + \frac{5}{7}\frac{\bar{\sigma}_1^{(n)}}{\sigma_1^{(n)}}\Lambda_n^{(7)}\Lambda_n^{(-5)}F_n^{\Theta} + \frac{5}{7}\frac{\sigma_1^{(n)}}{\bar{\sigma}_1^{(n)}}\Lambda_n^{(-7)}\Lambda_n^{(5)}F_n^{\Theta}\right). \tag{4.24}$$

Since all the non-chiral descendants of the identity up to level 7 can be obtained taking derivatives of the operators $T\bar{T}$, $S_4T\bar{T}$, $\bar{S}_4T\bar{T}$ and $S_4\bar{S}_4T\bar{T}$, our discussion is now complete.

The factorization properties of the eigenvalues of the conserved quantities together with (3.20), (3.21), (4.4) and

$$\lim_{\alpha\to+\infty} e^{-2\alpha}F_n^{T\bar{T}}(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},..,\theta_n) = F_k^{T}(\theta_1,..,\theta_k)\,F_{n-k}^{\bar{T}}(\theta_{k+1},..,\theta_n) \tag{4.25}$$

$(k=0,1,\ldots,n)$ imply that the form factors of the descendants of the identity we determined satisfy (4.3) with $\Phi_0 = I$. In particular, the chiral and non-chiral descendant satisfy the asymptotic behavior (4.2) and (4.1), respectively.

5 Comparison with the form factor bootstrap

In the previous section we used the conserved quantities to construct, starting from the form factor solutions for the energy-momentum tensor and $T\bar{T}$, all the operators in the massive Lee-Yang model as expected from the conformal structure of the operator space, level by level up to level 7. In this section we want to show that exactly the same operator space is generated by the solutions of the form factor bootstrap equations (2.15)-(2.19) supplemented by the level-dependent specification (4.1) on the asymptotic behavior. We do this explicitly for the scalar operators of level (l,l) up to $l=7$, thus generalizing what had been done in [10] for $l \leq 2$.

An essential role in this analysis is played by the so-called kernel solutions. We call minimal scalar N-particle kernel the solution

$$F_N^{K_N}(\theta_1,...,\theta_N) = 2^{N(N-1)}H_N\prod_{1\leq i<j\leq N}F_{min}(\theta_i-\theta_j) \tag{5.1}$$

of the form factor equations (2.15)–(2.19) with $s_\Phi = 0$, $n = N$ and zero on the r.h.s. of (2.18) and (2.19). This kernel is minimal in the sense that it has the mildest asymptotic behavior for large rapidities. The other scalar N-particle kernels are given by

$$U_N\left(\theta_1,..,\theta_N\right) F_N^{K_N}(\theta_1,...,\theta_N)\,, \tag{5.2}$$

where $U_N\left(\theta_1,..,\theta_N\right)$ is a scalar entire function of the rapidities, symmetric and (up to a factor $(-1)^{N-1}$) $2\pi i$-periodic in all θ_j's. We call N-particle kernel solution the solution $F_n^{U_N F_N}$ of the form factor equations having (5.2) as initial condition ($F_n^{U_N F_N} = 0$ for $n < N$ and the M-particle kernel solutions arising for $M > N$ are ignored).

It follows from (3.13) that

$$F_N^{K_N}\left(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},...,\theta_n\right) \sim e^{k(N-k)\alpha}, \tag{5.3}$$

for $\alpha \to +\infty$, $N > 1$ and $1 \le k \le N-1$. So, the maximal asymptotic behavior of $F_N^{K_N}$ is $e^{d_{K_N}\alpha}$ with $d_{K_{2M}} = M^2$ and $d_{K_{2M+1}} = (M+1)M$. Similarly, if $e^{d_{U_N K_N}}$ is the maximal asymptotic behavior of (5.2), we know that $d_{U_N K_N} \ge d_{K_N}$. On the other hand, it is implied by (4.1) that $U_N F_N^{K_N}$ can contribute to the form factor solutions of operators of level (l,l) if and only if $d_{U_N K_N} = l$. Since $d_{K_N} > 7$ for $N > 5$, our analysis for $l \le 7$ can involve only N-particle kernel solutions with $N \le 5$.

In table 3 we list, for each level up to $(7,7)$, all the independent scalar solutions of the form factor equations supplemented by the asymptotic condition (4.1). They are given in terms of the form factors of Θ and of the kernel solutions labeled as specified in table 4.

The asymptotic properties

$$\lim_{\alpha\to+\infty} e^{-k\alpha}\sigma_p^{(n)}\left(x_1e^\alpha,..,x_ke^\alpha,x_{k+1},..,x_n\right) = \sigma_k^{(k)}(x_1,\ldots,x_k)\sigma_{p-k}^{(n-k)}(x_{k+1},\ldots,x_n), \quad k \le p$$

$$\lim_{\alpha\to+\infty} e^{-p\alpha}\sigma_p^{(n)}\left(x_1e^\alpha,..,x_ke^\alpha,x_{k+1},..,x_n\right) = \sigma_p^{(k)}(x_1,\ldots,x_k), \qquad k \ge p$$

of the elementary symmetric polynomials are useful to check that each kernel solution corresponds to the level indicated in table 3.

Tables 5 and 6 list the scalar descendants of Θ and I, respectively, up to level $(7,7)$, as constructed in the previous section by continuing away from criticality the conformal basis. Notice that, for each level, the total number of independent operators obtained putting together the two operator families perfectly matches the number of solutions of the form factor boostrap for that level given in table 3. Hence, the last thing to check in order to show that the solutions of table 3 span the same space of those of tables 5 and 6 is that the kernel solutions of table 4 can be rewritten as linear combinations of the solutions of tables 5 and 6. This change of basis is given explicitly in Appendix C.

(0,0)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	(6,6)	(7,7)
F_n^{Θ}	$D_n F_n^{\Theta}$	$(D_n)^2 F_n^{\Theta}$	$(D_n)^3 F_n^{\Theta}$	$(D_n)^4 F_n^{\Theta}$	$(D_n)^5 F_n^{\Theta}$	$(D_n)^6 F_n^{\Theta}$	$(D_n)^7 F_n^{\Theta}$
$F_n^I = \delta_{n,0}$		$F_n^{K_3}$	$D_n F_n^{K_3}$	$(D_n)^2 F_n^{K_3}$	$(D_n)^3 F_n^{K_3}$	$(D_n)^4 F_n^{K_3}$	$(D_n)^5 F_n^{K_3}$
				F_n^{A,K_3}	$D_n F_n^{A,K_3}$	$(D_n)^2 F_n^{A,K_3}$	$(D_n)^3 F_n^{A,K_3}$
				F_n^{B,K_3}	$D_n F_n^{B,K_3}$	$(D_n)^2 F_n^{B,K_3}$	$(D_n)^3 F_n^{B,K_3}$
						F_n^{C,K_3}	$D_n F_n^{C,K_3}$
						F_n^{D,K_3}	$D_n F_n^{D,K_3}$
				$F_n^{K_4}$	$D_n F_n^{K_4}$	$(D_n)^2 F_n^{K_4}$	$(D_n)^3 F_n^{K_4}$
						F_n^{A,K_4}	$D_n F_n^{A,K_4}$
						F_n^{B,K_4}	$D_n F_n^{B,K_4}$
						F_n^{C,K_4}	$D_n F_n^{C,K_4}$
						F_n^{D,K_4}	$D_n F_n^{D,K_4}$
						F_n^{E,K_4}	$D_n F_n^{E,K_4}$
						$F_n^{K_5}$	$D_n F_n^{K_5}$
2	1	2	2	5	5	13	13

Table 3: Scalar solutions of the form factor equations. The first line specifies the level, the last line the number of solutions. $D_n \equiv \left(\sigma_1^{(n)}\bar{\sigma}_1^{(n)}\right)$ is m^{-2} times the eigenvalue of $\partial\bar{\partial}$ on the n-particle asymptotic state.

Kernel solution	Initial condition
$F_n^{K_N}$	$F_N^{K_N}$
F_n^{A,K_3}	$\left(\sigma_1^{(3)}\right)^2 \bar{\sigma}_2^{(3)} F_3^{K_3}$
F_n^{B,K_3}	$\sigma_2^{(3)} \left(\bar{\sigma}_1^{(3)}\right)^2 F_3^{K_3}$
F_n^{C,K_3}	$\left(\sigma_1^{(3)}\right)^4 \left(\bar{\sigma}_2^{(3)}\right)^2 F_3^{K_3}$
F_n^{D,K_3}	$\left(\sigma_2^{(3)}\right)^2 \left(\bar{\sigma}_1^{(3)}\right)^4 F_3^{K_3}$
F_n^{A,K_4}	$\left(\sigma_1^{(4)}\right)^2 \bar{\sigma}_2^{(4)} F_4^{K_4}$
F_n^{B,K_4}	$\sigma_2^{(4)} \left(\bar{\sigma}_1^{(4)}\right)^2 F_4^{K_4}$
F_n^{C,K_4}	$\sigma_2^{(4)} \bar{\sigma}_2^{(4)} F_4^{K_4}$
F_n^{D,K_4}	$\left(\sigma_1^{(4)}\right)^3 \bar{\sigma}_3^{(4)} F_4^{K_4}$
F_n^{E,K_4}	$\sigma_3^{(4)} \left(\bar{\sigma}_1^{(4)}\right)^3 F_4^{K_4}$

Table 4: The first column lists the names given to the form factor kernel solutions originating from the initial conditions specified in the second column.

(0,0)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	(6,6)	(7,7)
F_n^{Θ}	$F_n^{\partial\bar{\partial}\Theta}$	$F_n^{\partial^2\bar{\partial}^2\Theta}$	$F_n^{\partial^3\bar{\partial}^3\Theta}$	$F_n^{\partial^4\bar{\partial}^4\Theta}$	$F_n^{\partial^5\bar{\partial}^5\Theta}$	$F_n^{\partial^6\bar{\partial}^6\Theta}$	$F_n^{\partial^7\bar{\partial}^7\Theta}$
				$F_n^{R_4\bar{R}_4\Theta}$	$F_n^{Q_5\bar{Q}_5\Theta}$	$F_n^{\partial Q_5\bar{\partial}\bar{Q}_5\Theta}$	$F_n^{\partial^2 Q_5\bar{\partial}^2\bar{Q}_5\Theta}$
				$F_n^{\partial^4\bar{R}_4\Theta}$	$F_n^{\partial^5\bar{Q}_5\Theta}$	$F_n^{\partial^6\bar{\partial}\bar{Q}_5\Theta}$	$F_n^{\partial^7\bar{\partial}^2\bar{Q}_5\Theta}$
				$F_n^{R_4\bar{\partial}^4\Theta}$	$F_n^{Q_5\bar{\partial}^5\Theta}$	$F_n^{\partial Q_5\bar{\partial}^6\Theta}$	$F_n^{\partial^2 Q_5\bar{\partial}^7\Theta}$
						$F_n^{R_6\bar{\partial}^6\Theta}$	$F_n^{Q_7\bar{\partial}^7\Theta}$
						$F_n^{\partial^6\bar{R}_6\Theta}$	$F_n^{\partial^7\bar{Q}_7\Theta}$
						$F_n^{R_6\bar{R}_6\Theta}$	$F_n^{Q_7\bar{Q}_7\Theta}$
						$F_n^{\partial Q_5\bar{R}_6\Theta}$	$F_n^{\partial^2 Q_5\bar{Q}_7\Theta}$
						$F_n^{R_6\bar{\partial}\bar{Q}_5\Theta}$	$F_n^{Q_7\bar{\partial}^2\bar{Q}_5\Theta}$
1	1	1	1	4	4	9	9

Table 5: Scalar operators in the family of Θ up to level $(7,7)$. The first and last line are as in table 3.

(0,0)	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	(6,6)	(7,7)
$F_n^I = \delta_{n,0}$		$F_n^{T\bar{T}}$	$F_n^{\partial\bar{\partial}T\bar{T}}$	$F_n^{\partial^2\bar{\partial}^2T\bar{T}}$	$F_n^{\partial^3\bar{\partial}^3T\bar{T}}$	$F_n^{\partial^4\bar{\partial}^4T\bar{T}}$	$F_n^{\partial^5\bar{\partial}^5T\bar{T}}$
						$F_n^{S_4\bar{S}_4T\bar{T}}$	$F_n^{Q_5\bar{Q}_5T\bar{T}}$
						$F_n^{\partial^4\bar{S}_4T\bar{T}}$	$F_n^{\partial^5\bar{Q}_5T\bar{T}}$
						$F_n^{S_4\bar{\partial}^4T\bar{T}}$	$F_n^{Q_5\bar{\partial}^5T\bar{T}}$
1	0	1	1	1	1	4	4

Table 6: Scalar operators in the family of I up to level $(7,7)$. The first and last line are as in table 3.

6 Conclusion

This paper provides the first extensive study of the operator space of a massive two-dimensional quantum field theory aimed at the explicit construction of descendant operators. From the point of view of the S-matrix bootstrap, in particular, we have shown how the identification of operator subspaces can be realized supplementing the form factor equations with asymptotic conditions containing the information about the level of the descendants, thus extending what originally done in [14, 27, 24] for the primary operators.

The space of solutions generated in this way has been shown to coincide with the off-critical continuation of the conformal operator space obtained acting with the conserved quantities of the massive Lee-Yang model on the lowest non-trivial form factor solutions in the two operator families. The fact that not all the operators beyond level 7 can be generated using the conserved quantities is the model dependent feature that has determined the extent of our analysis of the operator space in this paper.

In constructing the composite descendant operators in the operator family of the identity we found that the proliferation of undetermined coefficients of resonant operators expected on purely dimensional grounds is actually limited by the structure of the operator space. More precisely, the requirement that some composite operators remain derivative operators also away from criticality can indeed be fulfilled by fixing in a unique way the coefficients of some resonant terms, with the consequence that only the "primitive" ambiguity contained in the definition of $T\bar{T}$ propagates through the operator family up to level 7. This as well as the previous points deserve to be further investigated in other models.

Acknowledgments. This work was partially supported by the European Commission programme HPRN-CT-2002-00325 (EUCLID) and by the COFIN "Teoria dei Campi, Meccanica Statistica e Sistemi Elettronici".

A Appendix

Let $Q_s\bar{Q}_r T\bar{T}$ be the off-critical continuation of the descendant $Q_s\bar{Q}_r L_{-2}\bar{L}_{-2}I$ obtained through the regularization

$$Q_s\bar{Q}_r T\bar{T}(x) = \lim_{x'\to x} \left\{ \left[Q_s, \left[\bar{Q}_r, T(x)\bar{T}(x')\right]\right] - \mathcal{R}_{Q_s\bar{Q}_r T\bar{T}}(x,x') \right\}, \tag{A.1}$$

with $\mathcal{R}_{Q_s\bar{Q}_r T\bar{T}}(x,x')$ such that the above limit is non-singular. Then the matrix elements of (A.1)

$$\begin{aligned}&\langle\theta'_m \dots \theta'_1|Q_s\bar{Q}_r T\bar{T}(x)|\theta_1 \dots \theta_n\rangle = \\ &= \lim_{x'\to x} \langle\theta'_m \dots \theta'_1| \left\{ \left[Q_s, \left[\bar{Q}_r, T(x)\bar{T}(x')\right]\right] - \mathcal{R}_{Q_s\bar{Q}_r T\bar{T}}(x,x') \right\} |\theta_1 \dots \theta_n\rangle\end{aligned} \tag{A.2}$$

must be finite for any x. Since

$$\langle\theta'_m\dots\theta'_1|\left[Q_s,\left[\bar{Q}_r,T(x)\bar{T}(x')\right]\right]|\theta_1\dots\theta_n\rangle=\left(\Lambda^{(s)}_m\left(\theta'_1,..,\theta'_m\right)-\Lambda^{(s)}_n\left(\theta_1,..,\theta_n\right)\right)\times$$
$$\times\left(\Lambda^{(-r)}_m\left(\theta'_1,..,\theta'_m\right)-\Lambda^{(-r)}_n\left(\theta_1,..,\theta_n\right)\right)\langle\theta'_m\dots\theta'_1|T(x)\bar{T}(x')|\theta_1\dots\theta_n\rangle \tag{A.3}$$

the regularization term is fixed to

$$\mathcal{R}_{Q_s\bar{Q}_rT\bar{T}}(x,x')=\left[Q_s,\left[\bar{Q}_r,\mathcal{R}_{T\bar{T}}(x,x')\right]\right]+\text{resonances}(x)\,, \tag{A.4}$$

where $\mathcal{R}_{T\bar{T}}(x,x')$ is the regularization term (4.8) of the operator $T\bar{T}$. Putting all together we have

$$Q_s\bar{Q}_rT\bar{T}(x)=Q_s\bar{Q}_r\left(T\bar{T}\right)(x)+\text{resonances}(x), \tag{A.5}$$

where $Q_s\bar{Q}_r\left(T\bar{T}\right)$ is the local field obtained acting with the conserved commuting quantities $Q_s\bar{Q}_r$ on the operator (4.7).

B Appendix

Consider a massive two-dimensional field theory without internal symmetries. It was argued in [24] that in such a case the form factors of a primary operator Φ_0 satisfy the asymptotic factorization property

$$\lim_{\alpha\longrightarrow+\infty}F^{\Phi_0}_n\left(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},..,\theta_n\right)=$$
$$=\frac{1}{\langle\Phi_0\rangle}F^{\Phi_0}_k\left(\theta_1,..,\theta_k\right)F^{\Phi_0}_{n-k}\left(\theta_{k+1},..,\theta_n\right), \tag{B.1}$$

for $k=0,1,\dots,n$. Here we consider the effect of the same limit on the descendant operators

$$\Phi^{\{s_1,..,s_a\}\{r_1,..,r_b\}}\left(x\right)=[\bar{Q}_{r_b},[...,[\bar{Q}_{r_1},[Q_{s_a},[...,[Q_{s_1},\Phi_0\left(x\right)]]...]]...]]$$

obtained acting on Φ_0 with the commuting conserved quantities Q_s and $\bar{Q}_r$ (r and s positive integers). The level of such operators is $(l,\bar{l})=\left(\sum_{i=1}^a s_i,\sum_{j=1}^b r_j\right)$.

The form factors for these operators take the form

$$F^{\Phi^{\{s_1,..,s_a\}\{r_1,..,r_b\}}}_n\left(\theta_1,..,\theta_n\right)=(-1)^{l+\bar{l}}\,\Lambda^{\{s_1,..,s_a\}\{r_1,..,r_b\}}_n\left(\theta_1,..,\theta_n\right)F^{\Phi_0}_n\left(\theta_1,..,\theta_n\right) \tag{B.2}$$

where

$$\Lambda^{\{s_1,..,s_a\}\{r_1,..,r_b\}}_n\left(\theta_1,..,\theta_n\right)=\prod_{\mu=1}^{a}\Lambda^{(s_\mu)}_n\left(\theta_1,..,\theta_n\right)\prod_{\upsilon=1}^{b}\Lambda^{(-r_\upsilon)}_n\left(\theta_1,..,\theta_n\right) \tag{B.3}$$

with $\Lambda^{(s)}_n$ defined in (2.23).

It follows from (B.1) and the asymptotic factorization properties

$$\lim_{\alpha\longrightarrow+\infty} \; e^{-s\alpha}\Lambda_n^{(s)}\left(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},..,\theta_n\right) = \lim_{\alpha\longrightarrow+\infty} \; e^{-s\alpha}m^s(\sum_{i=1}^{k} e^{s(\theta_i+\alpha)}+$$

$$+\sum_{i=k+1}^{n} e^{s\theta_i}) = m^s\sum_{i=1}^{k} e^{s\theta_i} = \Lambda_k^{(s)}\left(\theta_1,..,\theta_k\right), \tag{B.4}$$

$$\lim_{\alpha\longrightarrow+\infty} \; \Lambda_n^{(-r)}\left(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},..,\theta_n\right) = \lim_{\alpha\longrightarrow+\infty} \; m^r(\sum_{i=1}^{k} e^{-r(\theta_i+\alpha)}+$$

$$+\sum_{i=k+1}^{n} e^{-r\theta_i}) = m^r\sum_{i=k+1}^{n} e^{-r\theta_i} = \Lambda_{n-k}^{(-r)}\left(\theta_{k+1},..,\theta_n\right), \tag{B.5}$$

that the form factors of $\Phi^{\{s_1,..,s_a\}\{r_1,..,r_b\}}$ satisfy

$$\lim_{\alpha\longrightarrow+\infty} \; e^{-l\alpha}F_n^{\Phi^{\{s_1,..,s_a\}\{r_1,..,r_b\}}}\left(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},..,\theta_n\right) =$$

$$= \frac{1}{\langle\Phi_0\rangle}F_k^{\Phi^{\{s_1,..,s_a\}\{\}}}\left(\theta_1,..,\theta_k\right)F_{n-k}^{\Phi^{\{\}\{r_1,..,r_b\}}}\left(\theta_{k+1},..,\theta_n\right) \tag{B.6}$$

for $k = 0, 1, \ldots, n$. Hence, the form factors of the generic $(l, \bar{l})$ descendant $\Phi^{\{s_1,..,s_a\}\{r_1,..,r_b\}}$ factorizes into the form factors of the chiral descendants $\Phi^{\{s_1,..,s_a\}\{\}}$ and $\Phi^{\{\}\{r_1,..,r_b\}}$ of level $(l, 0)$ and $(0, \bar{l})$, respectively.

Denote by $\Phi_{(l,\bar{l})}$ a generic linear combination of the level $(l, \bar{l})$ descendants $\Phi^{\{s_1,..,s_a\}\{r_1,..,r_b\}}$ of a scalar primary other than the identity. Then (B.6) implies that the form factors of $\Phi_{(l,\bar{l})}$ are homogeneous functions, with respect to the variables $x_i \equiv e^{\theta_i}$, of degree $l - \bar{l}$ with further asymptotic behavior

$$F_n^{\Phi_{(l,\bar{l})}}\left(\theta_1+\alpha,..,\theta_k+\alpha,\theta_{k+1},...,\theta_n\right) \sim e^{l\alpha} \tag{B.7}$$

for $\alpha \to +\infty$, $n > 1$ and $1 \leq k \leq n-1$.

C Appendix

We give here the expansions of the kernel solutions appearing in table 3 in terms of the solutions of tables 5 and 6. The kernel solution $F_n^{K_3}$ arising at level $(2, 2)$ is related to $F_n^{T\bar{T}}$ by (4.9). Throughout this appendix we take $c = 0$ in (4.9). Then we have

$$F_n^{A,K_3} = \tfrac{1}{5m^8\langle\Theta\rangle}\left(F_n^{\partial^4\bar{\partial}^4\Theta} - iF_n^{\partial^4\bar{R}_4\Theta}\right) - \tfrac{1}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar{\partial}^3\Theta} + \tfrac{1}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar{\partial}^2\Theta} \tag{C.1}$$

$$F_n^{B,K_3} = \tfrac{1}{5m^8\langle\Theta\rangle}\left(F_n^{\partial^4\bar{\partial}^4\Theta} + iF_n^{R_4\bar{\partial}^4\Theta}\right) - \tfrac{1}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar{\partial}^3\Theta} + \tfrac{1}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar{\partial}^2\Theta} \tag{C.2}$$

$$F_n^{K_4} = \frac{1}{25m^8\langle\Theta\rangle}\left(F_n^{\partial^4\bar\partial^4\Theta} - iF_n^{\partial^4\bar R_4\Theta} + iF_n^{R_4\bar\partial^4\Theta} + F_n^{R_4\bar R_4\Theta}\right) + \frac{1}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar\partial^3\Theta} - \frac{2}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar\partial^2\Theta} + \frac{1}{\langle\Theta\rangle}F_n^{\Theta}$$
$$-\frac{1}{m^2\langle\Theta\rangle^2}F_n^{\partial\bar\partial T\bar T} + \frac{1}{\langle\Theta\rangle^2}F_n^{T\bar T} \tag{C.3}$$

for the kernels arising at level $(4,4)$, and

$$F_n^{C,K_3} = \frac{1}{175m^{12}\langle\Theta\rangle}\left(8F_n^{\partial^6\bar\partial^6\Theta} - 5F_n^{\partial Q_5\bar R_6\Theta} - 7F_n^{\partial Q_5\bar\partial\bar Q_5\Theta} - 28iF_n^{\partial^6\bar\partial\bar Q_5\Theta} - 20iF_n^{\partial^6\bar R_6\Theta} - 2iF_n^{\partial Q_5\bar\partial^6\Theta}\right)$$
$$+\frac{1}{5m^8\langle\Theta\rangle}\left(-6F_n^{\partial^4\bar\partial^4\Theta} + iF_n^{\partial^4\bar R_4\Theta}\right) + \frac{3}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar\partial^3\Theta} - \frac{3}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar\partial^2\Theta} + \frac{1}{m^2\langle\Theta\rangle}F_n^{\partial\bar\partial\Theta} \tag{C.4}$$

$$F_n^{D,K_3} = \frac{1}{175m^{12}\langle\Theta\rangle}\left(8F_n^{\partial^6\bar\partial^6\Theta} - 5F_n^{R_6\bar\partial\bar Q_5\Theta} - 7F_n^{\partial Q_5\bar\partial\bar Q_5\Theta} + 2iF_n^{\partial^6\bar\partial\bar Q_5\Theta} + 28iF_n^{\partial Q_5\bar\partial^6\Theta} + 20iF_n^{R_6\bar\partial^6\Theta}\right)$$
$$+\frac{1}{5m^8\langle\Theta\rangle}\left(-6F_n^{\partial^4\bar\partial^4\Theta} - iF_n^{R_4\bar\partial^4\Theta}\right) + \frac{3}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar\partial^3\Theta} - \frac{3}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar\partial^2\Theta} + \frac{1}{m^2\langle\Theta\rangle}F_n^{\partial\bar\partial\Theta} \tag{C.5}$$

$$F_n^{A,K_4} = \frac{1}{175m^{12}\langle\Theta\rangle}\left(2F_n^{\partial^6\bar\partial^6\Theta} + 5F_n^{\partial Q_5\bar R_6\Theta} + 7F_n^{\partial Q_5\bar\partial\bar Q_5\Theta} - 7iF_n^{\partial^6\bar\partial\bar Q_5\Theta} - 5iF_n^{\partial^6\bar R_6\Theta} + 2iF_n^{\partial Q_5\bar\partial^6\Theta}\right)$$
$$+\frac{1}{5m^{10}\langle\Theta\rangle}\left(-F_n^{\partial^5\bar\partial^5\Theta} + iF_n^{\partial^5\bar Q_5\Theta}\right) + \frac{1}{5m^8\langle\Theta\rangle}\left(6F_n^{\partial^4\bar\partial^4\Theta} - iF_n^{\partial^4\bar R_4\Theta}\right) - \frac{3}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar\partial^3\Theta} + \frac{3}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar\partial^2\Theta}$$
$$-\frac{1}{m^2\langle\Theta\rangle}F_n^{\partial\bar\partial\Theta} \tag{C.6}$$

$$F_n^{B,K_4} = \frac{1}{175m^{12}\langle\Theta\rangle}\left(2F_n^{\partial^6\bar\partial^6\Theta} + 5F_n^{R_6\bar\partial\bar Q_5\Theta} + 7F_n^{\partial Q_5\bar\partial\bar Q_5\Theta} - 2iF_n^{\partial^6\bar\partial\bar Q_5\Theta} + 7iF_n^{\partial Q_5\bar\partial^6\Theta} + 5iF_n^{R_6\bar\partial^6\Theta}\right)$$
$$+\frac{1}{5m^{10}\langle\Theta\rangle}\left(-F_n^{\partial^5\bar\partial^5\Theta} - iF_n^{Q_5\bar\partial^5\Theta}\right) + \frac{1}{5m^8\langle\Theta\rangle}\left(6F_n^{\partial^4\bar\partial^4\Theta} + iF_n^{R_4\bar\partial^4\Theta}\right) - \frac{3}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar\partial^3\Theta} + \frac{3}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar\partial^2\Theta}$$
$$-\frac{1}{m^2\langle\Theta\rangle}F_n^{\partial\bar\partial\Theta} \tag{C.7}$$

$$
\begin{aligned}
F_n^{C,K_4} &= \tfrac{1}{1225m^{12}\langle\Theta\rangle}\Big(4F_n^{\partial^6\bar\partial^6\Theta}+25F_n^{R_6\bar R_6\Theta}+35F_n^{R_6\bar\partial\bar Q_5\Theta}+35F_n^{\partial Q_5\bar R_6\Theta}+49F_n^{\partial Q_5\bar\partial\bar Q_5\Theta}-14iF_n^{\partial^6\bar\partial\bar Q_5\Theta}\\
&-10iF_n^{\partial^6\bar R_6\Theta}+14iF_n^{\partial Q_5\bar\partial^6\Theta}+10iF_n^{R_6\bar\partial^6\Theta}\Big)+\tfrac{1}{25m^{10}\langle\Theta\rangle}\Big(-F_n^{\partial^5\bar\partial^5\Theta}-F_n^{Q_5\bar Q_5\Theta}+iF_n^{\partial^5\bar Q_5\Theta}-iF_n^{Q_5\bar\partial^5\Theta}\Big)\\
&+\tfrac{1}{25m^8\langle\Theta\rangle}\Big(F_n^{\partial^4\bar\partial^4\Theta}+F_n^{R_4\bar R_4\Theta}-iF_n^{\partial^4\bar R_4\Theta}+iF_n^{R_4\bar\partial^4\Theta}\Big)+\tfrac{1}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar\partial^3\Theta}-\tfrac{4}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar\partial^2\Theta}+\tfrac{5}{m^2\langle\Theta\rangle}F_n^{\partial\bar\partial\Theta}\\
&-\tfrac{2}{\langle\Theta\rangle}F_n^{\Theta}
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
F_n^{D,K_4} &= \tfrac{1}{245m^{12}\langle\Theta\rangle}\Big(10F_n^{\partial^6\bar\partial^6\Theta}-25F_n^{R_6\bar R_6\Theta}-35F_n^{R_6\bar\partial\bar Q_5\Theta}-35F_n^{\partial Q_5\bar R_6\Theta}-84iF_n^{\partial^6\bar\partial\bar Q_5\Theta}\\
&+49F_n^{\partial Q_5\bar\partial\bar Q_5\Theta}+10iF_n^{\partial^6\bar R_6\Theta}+35iF_n^{\partial Q_5\bar\partial^6\Theta}+25iF_n^{R_6\bar\partial^6\Theta}\Big)+\tfrac{1}{35m^{10}\langle\Theta\rangle}\Big(-33F_n^{\partial^5\bar\partial^5\Theta}+14F_n^{Q_5\bar Q_5\Theta}\\
&+26iF_n^{\partial^5\bar Q_5\Theta}-33iF_n^{Q_5\bar\partial^5\Theta}\Big)+\tfrac{1}{5m^8\langle\Theta\rangle}\Big(-3F_n^{R_4\bar R_4\Theta}-2iF_n^{\partial^4\bar R_4\Theta}+5iF_n^{R_4\bar\partial^4\Theta}\Big)+\tfrac{1}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar\partial^3\Theta}\\
&+\tfrac{10}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar\partial^2\Theta}-\tfrac{20}{m^2\langle\Theta\rangle}F_n^{\partial\bar\partial\Theta}+\tfrac{10}{\langle\Theta\rangle}F_n^{\Theta}+\tfrac{1}{5m^8\langle\Theta\rangle^2}\Big(iF_n^{\partial^4\bar S_4T\bar T}-F_n^{S_4\bar S_4T\bar T}\Big)+\tfrac{5}{m^4\langle\Theta\rangle^2}F_n^{\partial^2\bar\partial^2T\bar T}\\
&-\tfrac{10}{m^2\langle\Theta\rangle^2}F_n^{\partial\bar\partial T\bar T}+\tfrac{5}{\langle\Theta\rangle^2}F_n^{T\bar T}
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
F_n^{E,K_4} &= \tfrac{1}{35m^{12}\langle\Theta\rangle}\Big(-12F_n^{\partial^6\bar\partial^6\Theta}+7iF_n^{\partial^6\bar\partial\bar Q_5\Theta}-5iF_n^{\partial^6\bar R_6\Theta}\Big)+\tfrac{1}{35m^{10}\langle\Theta\rangle}\Big(33F_n^{\partial^5\bar\partial^5\Theta}+7iF_n^{\partial^5\bar Q_5\Theta}\Big)\\
&+\tfrac{1}{5m^8\langle\Theta\rangle}\Big(-7F_n^{\partial^4\bar\partial^4\Theta}-3iF_n^{\partial^4\bar R_4\Theta}\Big)+\tfrac{1}{m^6\langle\Theta\rangle}F_n^{\partial^3\bar\partial^3\Theta}+\tfrac{1}{5m^8\langle\Theta\rangle^2}\Big(F_n^{\partial^4\bar\partial^4T\bar T}-iF_n^{\partial^4\bar S_4T\bar T}\Big)
\end{aligned}
\tag{C.10}
$$

$$
\begin{aligned}
F_n^{K_5} \;&= \tfrac{1}{1225m^{12}\langle\Theta\rangle}\Big(-94F_n^{\partial^6\bar\partial^6\Theta}+25F_n^{R_6\bar R_6\Theta}+35F_n^{R_6\bar\partial\bar Q_5\Theta}+35F_n^{\partial Q_5\bar R_6\Theta}+84iF_n^{\partial^6\bar\partial\bar Q_5\Theta}\\
&-49F_n^{\partial Q_5\bar\partial\bar Q_5\Theta}-10iF_n^{\partial^6\bar R_6\Theta}-84iF_n^{\partial Q_5\bar\partial^6\Theta}+10iF_n^{R_6\bar\partial^6\Theta}\Big)+\tfrac{1}{175m^{10}\langle\Theta\rangle}\Big(66F_n^{\partial^5\bar\partial^5\Theta}-14F_n^{Q_5\bar Q_5\Theta}\\
&-26iF_n^{\partial^5\bar Q_5\Theta}+26iF_n^{Q_5\bar\partial^5\Theta}\Big)+\tfrac{1}{25m^8\langle\Theta\rangle}\Big(-7F_n^{\partial^4\bar\partial^4\Theta}+3F_n^{R_4\bar R_4\Theta}+2iF_n^{\partial^4\bar R_4\Theta}-2iF_n^{R_4\bar\partial^4\Theta}\Big)\\
&-\tfrac{2}{m^4\langle\Theta\rangle}F_n^{\partial^2\bar\partial^2\Theta}+\tfrac{4}{m^2\langle\Theta\rangle}F_n^{\partial\bar\partial\Theta}-\tfrac{2}{\langle\Theta\rangle}F_n^{\Theta}+\tfrac{1}{25m^8\langle\Theta\rangle^2}\Big(F_n^{\partial^4\bar\partial^4T\bar T}+F_n^{S_4\bar S_4T\bar T}+iF_n^{S_4\bar\partial^4T\bar T}\\
&-iF_n^{\partial^4\bar S_4T\bar T}\Big)-\tfrac{1}{m^4\langle\Theta\rangle^2}F_n^{\partial^2\bar\partial^2T\bar T}+\tfrac{2}{m^2\langle\Theta\rangle^2}F_n^{\partial\bar\partial T\bar T}-\tfrac{1}{\langle\Theta\rangle^2}F_n^{T\bar T}
\end{aligned}
\qquad \text{(C.11)}
$$

for the kernels arising at level $(6,6)$.

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