

# Poisson Pencils, Integrability, and Separation of Variables

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## Abstract

In this paper we will review a recently introduced method for solving the Hamilton-Jacobi equations by the method of Separation of Variables. This method is based on the notion of pencil of Poisson brackets and on the bi-hamiltonian approach to integrable systems. We will discuss how separability conditions can be intrinsically characterized within such a geometrical set-up, the definition of the separation coordinates being encompassed in the bihamiltonian structure itself. We finally discuss these constructions studying in details a particular example, based on a generalization of the classical Toda Lattice.

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# 1 Introduction

The study of the separability of the Hamilton–Jacobi (HJ) equations associated with a given Hamiltonian function  $H$  is a very classical issue in Mechanics, dating back to the foundational works of Jacobi, Stäckel, Levi-Civita and others. It has recently received a strong renewed interests thanks to its applications to the theory of integrable PDEs of KdV type (namely, the theory of finite gap integration) and to the theory of quantum integrable systems (see, e.g., [11, 34]).

As it is well known, the problem can be formulated as follows. Let  $(M, \omega)$  be a  $2n$  dimensional symplectic manifold, and let  $(p_1, \dots, p_n, q_1, \dots, q_n) \equiv (\mathbf{p}, \mathbf{q})$  be canonical coordinates (in  $U \subset M$ ), i.e.,  $\omega|_U = \sum_{i=1}^n dp_i \wedge dq_i$ .

The (stationary) HJ equation reads

$$H(q_1, \dots, q_n; \frac{\partial S}{\partial p_1}, \dots, \frac{\partial S}{\partial p_n}) = E. \quad (1.1)$$

**Definition 1.1** *A complete integral  $S(\mathbf{q}; \alpha_1, \dots, \alpha_n)$  of HJ is a solution of (1.1), depending on  $n$  parameters  $(\alpha_1, \dots, \alpha_n)$  such that  $\text{Det} \frac{\partial^2 S}{\partial q_i \partial \alpha_j} \neq 0$ .  $H$  is said to be separable in the coordinates  $(\mathbf{p}, \mathbf{q})$  if HJ admits an additively separated complete integral, that is, a complete integral of the form*

$$S(\mathbf{q}; \alpha_1, \dots, \alpha_n) = \sum_{i=1}^n S_i(q_i; \alpha_1, \dots, \alpha_n). \quad (1.2)$$

In this paper we will focus on an equivalent ‘Definition–Theorem’ of separability, originally due to Jacobi and recently widely used by Sklyanin and his collaborators. Let us consider an *integrable* Hamiltonian  $H$ , that is, let us suppose that, along with  $H = H_1$  we have further  $n - 1$  mutually commuting integrals of the motion  $H_2, \dots, H_n$ , with  $dH_1 \wedge \dots \wedge dH_n \neq 0$ .

**Definition 1.2** *An integrable system  $(H_1, \dots, H_n)$  is separable in the coordinates  $(\mathbf{p}, \mathbf{q})$  if there exist  $n$  non-trivial relations*

$$\Phi_i(q_i, p_i; H_1, \dots, H_n) = 0, \quad i = 1, \dots, n, \quad (1.3)$$

*connecting single pairs  $(q_i, p_i)$  of canonical coordinates with the  $n$  Hamiltonians  $H_i$ .*

We called this a Definition–Theorem due to the fact that it is a *constructive* approach to separability, since the knowledge of the separation relations (1.3) allows one to reduce the problem of finding a separated solution of HJ to quadratures. Indeed, one can solve the relations  $\Phi_i(q_i, p_i; H_1, \dots, H_n) = 0$ ,  $i = 1, \dots, n$  for the  $p_i$  to get  $p_i = p_i(q_i; H_1, \dots, H_n)$  and then define:

$$S(\mathbf{q}; \alpha_1, \dots, \alpha_n) = \sum_{i=1}^n \int_{H_i=\alpha_i}^{q_i} p_i(q'_i; H_1, \dots, H_n) \Big|_{H_i=\alpha_i} dq'_i. \quad (1.4)$$

This is by construction a separated solution of HJ; the fact that it is a complete integral is equivalent to the fact that the integrals of the motion depend non trivially on the momenta.

In intrinsic terms, one notices that, the equations  $H_i = \alpha_i$ ,  $i = 1, \dots, n$  define a foliation  $\mathcal{F}$  of  $M$ . The leaves of  $\mathcal{F}$  are nothing but the (generalized) tori of the Arnol'd-Liouville theorem. The foliation is *Lagrangian*, that is, the restriction of the two-form  $\omega$  to  $\mathcal{F}$  vanishes. Hence the restriction to  $\mathcal{F}$  of the Liouville form  $\theta = \sum_{i=1}^n p_i dq_i$  to  $\mathcal{F}$  is (locally) exact. Indeed, the function  $S$  defined by (1.4) is a (local) potential for the restriction of  $\theta$  to  $\mathcal{F}$ . What is non intrinsic, and singles out the separation coordinates  $(\mathbf{p}, \mathbf{q})$  is that the separation relations (1.3), which are another set of defining equations for the foliation  $\mathcal{F}$ , have the very special property of containing a single pair of canonical coordinates at a time. The problem to find such a system of coordinates and relations is the core of the theory of SoV. In particular, a natural question arises:

*Is it possible to formulate intrinsic condition(s) on the Hamiltonians  $(H_1, \dots, H_n)$  to a priori ensure separability in a set of canonical coordinates?*

Actually, this is the main issue studied by both the ‘classical’ Eisenhart-Benenti theory of separability of natural systems defined on cotangent bundles to Riemannian manifolds  $(M, g)$ , as well as the ‘modern’ theory, due to the St. Petersburg and Montreal schools, of SoV for systems admitting a Lax representation. We notice that both such general approaches require the presence of *an additional structure* to solve the problem. Indeed, the Eisenhart-Benenti theory requires the existence of a conformal Killing tensor for the metric  $g$ , while the Lax theory requires – in addition to the knowledge of a Lax representation with spectral parameter for the Hamiltonian system under study – the existence of an  $r$ -matrix structure for such a Lax representation.

The method we will present/review in this paper, which has recently been exposed in the literature (see, e.g., [7, 30, 5, 14, 6, 20, 13, 31, 32, 16, 4]). can be seen as a kind

of bridge between the classical and the modern points of view, putting an emphasis on the geometrical structures of the Hamiltonian theory. Its ‘additional’ structure is simply the requirement of the existence, on the symplectic manifold  $(M, \omega)$  of a *second* Hamiltonian structure, compatible with the one defined by  $\omega$ . Namely, the bihamiltonian structure on  $M$  will allow us:

1. To encompass the definition of a special set of coordinates, to be called *Darboux–Nijenhuis (DN)* coordinates, within a well defined geometrical object.
2. To formulate intrinsic (i.e., tensorial) conditions for the separability of a Hamiltonian integrable system, in the DN coordinates associated with the bihamiltonian structure.
3. To give recipes to characterize, find and handle sets of DN coordinates.

The paper is organized as follows. In Section 2 we will briefly introduce the notions of bihamiltonian geometry relevant for the paper. In particular, we will discuss the notion of DN coordinates, as well as methods to find them. In Section 3 we will discuss the main theorems of the bihamiltonian set–up for SoV, namely, the tensorial conditions ensuring Separability of the HJ equations in DN coordinates.

Then we will discuss our constructions in a specific example, whose separability, to the best of our knowledge, has not yet been considered in the literature. It is a generalization of the periodic Toda lattice with four sites. In Section 4 we will recall its definition, and show how the “bihamiltonian recipe” for SoV can be applied to it. Although our constructions can be generalized to the generic  $N$ -site system, for the sake of concreteness and brevity we choose to consider the four-site system only, and sometimes rely on direct computations to prove some of its properties.

In the last subsection we will apply our geometrical scheme to study a specific reduction of this generalized Toda system, and find integrals of the motion which are not encompassed in the Lax representation. This result can be possibly as first step towards an alternative approach to the so–called chopping method of [8] for the full (non-periodic) Toda Lattice.

## 2 Some issues in the geometry of (bi)hamiltonian manifolds

We start this Section recalling some well known facts in the theory of Poisson manifolds.

**Definition 2.1** *A Poisson manifold  $(M, \{\cdot, \cdot\})$  is a manifold endowed with a Poisson bracket, that is a bilinear antisymmetric composition laws defined on the space  $C^\infty(M)$  satisfying:*

1. *The Leibnitz rule:  $\{fg, h\} = f\{g, h\} + g\{f, h\}$ ;*
2. *The Jacobi identity  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ .*

We recall that a Poisson bracket (or Poisson structure) can be equivalently described with a Poisson tensor, that is, with an application  $P : T^*M \rightarrow TM$ , smoothly varying with  $m \in M$  defined by:

$$\{f, g\} = \langle df, Pdg \rangle$$

where  $\langle \cdot, \cdot \rangle$  denotes the canonical pairing between  $T^*M$  and  $TM$ . In a given coordinate system  $(x^1, \dots, x^n)$  on  $M$ , the Poisson tensor  $P$  associated with the Poisson bracket  $\{\cdot, \cdot\}$  is represented as

$$P = \sum_{i,j} P^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad \text{with } P^{ij} = \{x^i, x^j\}.$$

The Jacobi identity is translated into a quadratic differential condition on the matrix  $P^{ij}$  known as vanishing Schouten bracket (see, for further details, e.g., [23, 35]), which in local coordinates reads

$$\sum_{s=1}^n P^{is} \frac{\partial P^{jk}}{\partial x_s} + P^{js} \frac{\partial P^{ki}}{\partial x_s} + P^{ks} \frac{\partial P^{ij}}{\partial x_s} = 0, \quad \forall i > j > k. \quad (2.1)$$

We notice that the existence of a Poisson structure does not give any restriction on the dimension of  $M$ . As an example, the reader should consider the dual  $\mathfrak{g}^*$  of a finite dimensional Lie algebra  $\mathfrak{g}$ , with Poisson brackets defined, thanks to the natural identification  $\mathfrak{g}^{**} \simeq \mathfrak{g}$ , by

$$\{f, g\}(X) = \langle X, [df(X), dg(X)] \rangle, \quad X \in \mathfrak{g}^*, \quad f, g \in C^\infty(\mathfrak{g}^*).$$

The most familiar instances of Poisson manifolds come from classical mechanics and are the cotangent bundles to smooth manifolds. As it is well known (see, e.g., [3]), they are equipped a canonical Poisson structure associated with a symplectic, that is, closed (indeed exact) and non degenerate, two-form  $\omega$ . A Poisson manifold  $(M, \{\cdot, \cdot\})$ , is called *symplectic* if the Poisson tensor  $P$  associated with  $\{\cdot, \cdot\}$  is everywhere invertible. In such a case, the connection between the Poisson tensor  $P$ , the Poisson brackets  $\{f, g\}$  and the symplectic two-form  $\omega$  can be described in the following equivalent ways:

a) In local coordinates  $(x_1, \dots, x_n)$  one has  $\omega = \sum_{i,j} \omega_{ij} dx_i \wedge dx_j$ , where the matrix  $\omega_{ij}$  is the inverse matrix to  $P^{ij}$ .

b) It holds the equality

$$\{f, g\} = \omega(X_f, X_g)$$

where, e.g.,  $X_f = PdF$  is the Hamiltonian vector field associated with  $f$ .

c) For every vector field  $Y$  and every function  $f$ , one has:

$$L_Y(f) \equiv \langle df, Y \rangle = \omega(Y, X_f)$$

The local structure of a Poisson manifold can be described as follows (see, e.g., [36, 11]). Under suitable assumptions on the regularity of the Poisson brackets, one sees that an open dense set of  $M$  the Poisson tensor has rank  $r = 2n$ . There,  $M$  is foliated in regular Poisson submanifolds, called generic symplectic leaves, that are the (generic) common level sets of  $k$  functions  $C_1, \dots, C_k$ , called Casimir functions of  $P$ . The distinguished property of the Casimirs of  $P$  is that their Poisson bracket with any other function on  $M$  vanishes, or, equivalently, their differential lies in the kernel of  $P$ . The dimension of  $M$  is related with the integers  $n$  and  $k$  by  $\dim M = k + 2n$ . As an example, we notice that, in the case of  $M = \mathfrak{sl}(n)^*$ , which we can identify with  $\mathfrak{sl}(n)$  itself by means, say, of the trace form, the Casimir functions are  $C_i = \text{Tr}(X^{i+1})$ ,  $i = 1, \dots, n-1$ , and the generic symplectic leaves consist of diagonalizable matrices with distinct eigenvalues that can be identified with the direct sum of the lower and upper nilpotent subalgebras.

Let us now come to the definition of bihamiltonian manifold.

**Definition 2.2** *A manifold  $M$  is called a bihamiltonian manifold if it is endowed with two Poisson brackets  $\{f, g\}$ ,  $\{f, g\}'$ , such that, for any  $\lambda \in \mathbb{R}$  (or  $\lambda \in \mathbb{C}$  if  $M$  is*

complex), the linear combination

$$\{f, g\}' - \lambda\{f, g\} \equiv \langle df, (P - \lambda P')dg \rangle \quad (2.2)$$

defines a Poisson bracket. This property is known as the compatibility condition between the two brackets.

The expression (2.2) will be referred to as *pencil* of Poisson brackets, and the sum  $P_\lambda = P' - \lambda P$  pencil of Poisson tensors. The most ‘popular’ property of bihamiltonian manifolds is contained in the following

**Proposition 2.3** *Let  $f$  and  $f'$  two functions on a bihamiltonian manifold  $M$ , which satisfy the characteristic condition  $Pdf = P'df'$ . Then the Poisson brackets  $\{f, f'\}$  and  $\{f, f'\}'$  vanish.*

**Proof.** It consists of a one-line computation. Let us consider, e. g.,  $\{f, f'\}$ :

$$\{f, f'\} = \langle df, Pdf' \rangle = -\langle df', Pdf \rangle = -\langle df' P' df' \rangle = 0.$$

□

**Definition 2.4** *A vector field  $X$  that can be written as  $X = Pdf = P'df'$  is called a bihamiltonian vector field.*

**Corollary 2.5** *Let  $f_k, k \in \mathbb{Z}$  be a sequence of (non-constant) functions satisfying*

$$Pdf_i = P'df_{i+1}. \quad (2.3)$$

*Then  $\{f_i, f_k\} = \{f_i, f_k\}' = 0, \quad \forall i, k \in \mathbb{Z}$ .*

**Proof.** Using twice equation (2.3) and the antisymmetry of the Poisson brackets we have, e.g.,

$$\begin{aligned} \{f_i, f_k\} &= \langle df_i, Pdf_k \rangle = \langle df_i P' df_{k-1} \rangle = -\langle df_{k-1}, P' df_i \rangle = \\ &= -\langle df_{k-1}, Pdf_{i+1} \rangle = \langle df_{i+1}, Pdf_{k-1} \rangle = \{f_{i+1}, f_{k-1}\}. \end{aligned}$$

Supposing  $k > i$  and iterating this procedure  $k - i$  times we get  $\{f_i, f_k\} = \{f_k, f_i\}$ .

□

Following [18], we can state the following

**Amplification 2.6** *Let  $f_n, n = 0, 1, \dots$  and  $g_n, n = 0, 1, \dots$  two sequences of (non-constant) functions satisfying*

$$Pdf_i = P'df_{i+1}; Pdf_0 = 0, \quad Pdg_i = P'dg_{i+1}; Pdg_0 = 0. \quad (2.4)$$

*Then, along with  $\{f_n, f_m\} = \{f_n, f_m\}' = \{g_n, g_m\} = \{g_n, g_m\}' = 0$ , it holds*

$$\{f_n, g_m\} = \{f_n, g_m\}' = 0, \quad \forall n, m \in \mathbb{N}.$$

The family of vector fields associated with a sequence of functions satisfying the recursion relations (2.3) are customarily said to form a Lenard-Magri sequence. Lenard-Magri sequences that start from the null vector field, as in Amplification 2.6 are pictorially called *anchored* Lenard-Magri sequences. Notice that anchored Lenard sequences can occur in bihamiltonian manifold where at least one of the Poisson brackets is non-symplectic (indeed, e.g.,  $df_0$  is a non-trivial element of the kernel of  $P$ ). We can compactly express equations (2.4) relative, say, to the sequence  $f_i$  by considering the formal Laurent series  $f(\lambda) = \sum_{i=0}^{\infty} f_i/\lambda^i$  via the equation

$$(P' - \lambda P)df(\lambda) = 0. \quad (2.5)$$

In analogy with the definition of Casimir of a Poisson bracket, Laurent series satisfying (2.5) are called Casimirs of the Poisson *pencil*. The reader should, however, bear in mind that while Casimir functions for a single Poisson bracket are, in a sense, uninteresting functions, Casimirs of a *pencil* of Poisson bracket compactly encode non-trivial dynamics and constants of the motion.

If, as it often happens in the applications, inside the family  $f_i$  we have an element  $f_n$  satisfying  $P'df_n = 0$ , we can form a *polynomial* Casimir of the pencil as

$$F(\lambda) = \lambda^n f_0 + \lambda^{n-1} f_1 + \dots + f_n. \quad (2.6)$$

Anchored Lenard sequences may give rise to families of integrable systems. Let us see how this happens in the case of a  $2n + 1$ -dimensional manifold endowed with a rank  $2n$  pencil of Poisson tensors. Let us suppose that we have found a polynomial Casimir of the form (2.6), with the  $n + 1$  functions  $f_0, \dots, f_n$  independent. Let  $\mathcal{S}_c$  be a generic symplectic leaf of  $P$  corresponding to  $f_0 = c$ . The vector fields  $X_{f_i}, i = 1, \dots, n$  are tangent to  $\mathcal{S}_c$ , are Hamiltonian on  $\mathcal{S}_c$  (w.r.t.  $P$ ) and the restriction of the functions  $f_1, \dots, f_n$  provide  $n$  commuting integrals for each of them. In general it holds [17, 18]:



**Proposition 2.7** *Let  $(M, P, P')$  a bihamiltonian manifold of dimension  $d = 2n + k$ , and let  $\dim(\text{Ker}(P' - \lambda P)) = k$ , for generic values of  $\lambda$ . Let us suppose that  $H^{(1)}(\lambda), \dots, H^{(k)}(\lambda)$  are  $k$  polynomial Casimirs of the pencil  $P_\lambda$  of the form*

$$H^{(i)}(\lambda) = \lambda^{n_i} H_0^{(i)} + \lambda^{n_i-1} H_1^{(i)} + \dots + H_{n_i}^{(i)}$$

*such that the collection of differentials  $\{dH_j^{(i)}\}_{j=0, \dots, n_i}^{i=1, \dots, k}$  defines a  $k + n$  dimensional distribution in  $T^*M$ . Then the vector fields defined by the anchored sequences associated with the  $H^{(i)}$ 's are integrable (in the Arnol'd-Liouville sense) on the generic symplectic leaves of  $P$ .*

## 2.1 Geometry of regular bihamiltonian manifolds and Darboux - Nijenhuis coordinates

An important class of bihamiltonian manifold occurs when one element of the Poisson pencil (which without loss of generality we will assume to be  $P$ ) is everywhere *invertible*, i.e., the Poisson bracket  $\{\cdot, \cdot\}$  associated with  $P$  is symplectic. The possibility of defining the inverse to one of the Poisson tensors lead us to introduce a fundamental object in the bihamiltonian theory of SoV: the Nijenhuis (or Hereditary, or Recursion) operator

$$N = P' P^{-1}, \tag{2.7}$$

(together with its dual  $N^* = P^{-1} P'$ ). By definition,  $N$  (resp.,  $N^*$ ) is an endomorphism of the tangent bundle to  $M$  (resp., of the cotangent bundle). As a remarkable consequence of the compatibility of  $P$  and  $P'$  the Nijenhuis torsion of  $N$ , defined by its action on a pair of vector fields  $X, Y$  as

$$T(N)(X, Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]) \tag{2.8}$$

*identically vanishes* [23]. So, from the classical Frölicher–Nijenhuis theory, we know that its eigenspaces are *integrable* distributions. Such distributions will be the building blocks of the bihamiltonian set-up for SoV.

To explain this, we have to make some remarks and a genericity assumption. It can be shown that, owing to the antisymmetry of the Poisson tensors defining  $N$ , pointwise the eigenspaces of  $N$  are even dimensional. Throughtout the paper, we will assume that for generic points  $m \in M$ ,  $N$  has the maximal number  $n = \frac{1}{2} \dim(M)$  of different eigenvalues  $\lambda_1, \dots, \lambda_n$ , so that the dimension of the eigenspace relative

to any eigenvalue is 2. Otherwise stated, the characteristic polynomial of  $N$  is the square of the degree  $n = \frac{1}{2}\dim(M)$  minimal polynomial  $\Delta_N(\lambda)$ , whose roots are pairwise distinct. We will call bihamiltonian manifolds endowed with a Poisson pencil with at least one of the elements of the Poisson pencil invertible, and such that the eigenvalues of the associated Nijenhuis tensor are maximally distinct, *regular bihamiltonian manifolds*.

**Theorem 2.8** *On a regular bihamiltonian manifold there exist a class of coordinates  $(y_i, x_i)$ , to be called Darboux–Nijenhuis (DN) coordinates, satisfying the two properties:*

**(Darboux)** *They are canonical, that is,  $\{x_i, y_j\} = \delta_{ij}$ ,  $\{x_i, x_j\} = \{y_i, y_j\} = 0$ .*

**(Nijenhuis)** *They diagonalize  $N$ , that is,  $N = \sum_i \lambda_i \left( \frac{\partial}{\partial y_i} \otimes dy_i + \frac{\partial}{\partial x_i} \otimes dx_i \right)$ .*

The proof of this Theorem can be found in [24, 17]. Here we will sketch it and discuss its meaning. In words, the assertion states that DN coordinates are defined by the spectral properties of  $N$ , as follows. For all  $m$  in the open set  $U$  where the eigenvalues  $\lambda_i$  of  $N$  (which are the same as the eigenvalues of  $N^*$ ) satisfy  $\lambda_i \neq \lambda_j$ ,  $i \neq j$ , the cotangent space  $T_m^*M$  admits the decomposition

$$T_m^*M = \bigoplus_{i=1}^n \mathcal{D}_{m, \lambda_i}, \quad \dim \mathcal{D}_{m, \lambda_i} = 2 \quad (2.9)$$

into eigenspaces of  $N^*$ . Thanks to the vanishing of the torsion of  $N$ , each eigenspace  $\mathcal{D}_{m, \lambda_i}$  is locally generated by differentials of pairs of independent functions  $(f_i, g_i)$ . This means that the pointwise decomposition (2.9) holds (in  $U' \subset U$ ) as

$$T^*M|_{U'} = \bigoplus_{i=1}^n \mathcal{D}_{\lambda_i},$$

where  $\mathcal{D}_{\lambda_i}$  is spanned by  $df_i$  and  $dg_i$ , with  $N^*df_i = \lambda_i df_i$  and  $N^*dg_i = \lambda_i dg_i$ .

Functions whose differential belong to *different* summands  $\mathcal{D}_{\lambda_i}$  are in involution with respect to the Poisson brackets defined both by  $P$  and  $P'$ . Indeed, suppose that  $f_1$  and  $f_2$  satisfy  $N^*df_1 = \lambda_1 df_1$ ,  $N^*df_2 = \lambda_2 df_2$ ,  $\lambda_1 \neq \lambda_2$ . The relation  $N^* = P^{-1}P'$  implies that  $P'df_1 = \lambda_1 Pdf_1$ ,  $P'df_2 = \lambda_2 Pdf_2$ . So,

$$\{f_1, f_2\}' = \begin{cases} \langle df_1, P'df_2 \rangle = \lambda_2 \langle df_1, Pdf_2 \rangle = \lambda_2 \{f_1, f_2\} \\ -\langle df_2, P'df_1 \rangle = -\lambda_1 \langle df_2, Pdf_1 \rangle = \lambda_1 \{f_1, f_2\} \end{cases}$$

whence the assertion. It is equally straightforward to realize that the only non vanishing Poisson brackets have the form

$$\{f_i, g_i\} = F_i(f_i, g_i), \quad \{f_i, g_i\}' = F_i'(f_i, g_i), \quad i = 1, \dots, n.$$

This means that, by quadratures, from the  $n$  pairs of functions  $\{f_i, g_i\}_{i=1, \dots, n}$  we can construct a set of canonical coordinates satisfying the Nijenhuis property of Theorem 2.8. Thus the class of coordinates where to frame the bihamiltonian set-up for SoV admits a clearcut and simple geometrical description. Admittedly, in the general case the computation of DN coordinates requires the integration of the two-dimensional distributions  $\mathcal{D}_{\lambda_i}$  associated with the eigenvalues  $\lambda_i$  of  $N^*$ . Fortunately enough there are instances (that frequently occur in the applications) in which DN coordinates can be found in an easier way.

## 2.2 On Darboux-Nijenhuis coordinates

In this subsection we will briefly discuss conditions and ‘recipes’ to algebraically find and/or characterize sets of Darboux-Nijenhuis coordinates on regular bihamiltonian manifolds. A very simplifying instance occurs whenever the eigenvalues  $\lambda_i$  of  $N$  (that are, in general, functions of the point  $m \in M$ ) are *functionally independent*. It holds (see, e.g., [25]):

**Proposition 2.9** *Let us define  $I_i := \frac{1}{2i} \text{Tr} N^i$ ,  $i = 1, \dots, n$ . In the open set  $U$  where  $dI_1 \wedge \dots \wedge \dots \wedge dI_n \neq 0$  the eigenvalues  $\lambda_i$ ,  $i = 1, \dots, n$  are functionally independent, satisfy  $N^* d\lambda_i = \lambda_i d\lambda_i$ , and so may be used to construct a set of Darboux-Nijenhuis coordinates.*

**Proof.** We express the normalized traces  $I_i$  of the Nijenhuis tensor  $N$  in terms of its eigenvalues as  $kI_k = \sum_{i=1}^n \lambda_i^k$ . Hence  $dI_k = \sum_{i=1}^n \lambda_i^{k-1} d\lambda_i$ , that is, in matrix terms:

$$\begin{bmatrix} dI_1 \\ dI_2 \\ \vdots \\ dI_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} d\lambda_1 \\ d\lambda_2 \\ \vdots \\ d\lambda_n \end{bmatrix} \quad (2.10)$$

So we have

$$dI_1 \wedge \dots \wedge dI_n = \prod_{i \neq j} (\lambda_i - \lambda_j) d\lambda_1 \wedge \dots \wedge d\lambda_n,$$

i.e., on the open set where the traces of the powers of the Nijenhuis tensor are functionally independent, we have that the eigenvalues  $\lambda_i$  are different and functionally independent.

To proceed further we need to recall that [23] that the normalized traces  $I_i$  of the powers of Nijenhuis operator satisfy the recursion relation:

$$N^*dI_k = dI_{k+1}. \quad (2.11)$$

This can be proved as follows. At first one notices that (2.11) is equivalent to the relation

$$L_{NX}(I_k) = L_X(I_{k+1}), \quad \forall \text{ vector field } X,$$

as it can be easily seen evaluating the equality of one-forms on a generic vector field  $X$ . Thanks to the Leibnitz property of the Lie derivative and the ciclicity of the trace, we see that

$$L_{NX}(I_k) = \text{Tr}(L_{NX}(N) \cdot N^{k-1}) \quad \text{and} \quad L_X(I_{k+1}) = \text{Tr}(L_X(N) \cdot N^k). \quad (2.12)$$

Since the vanishing of the Nijenhuis torsion of  $N$  implies that  $L_{NX}(N) = N \cdot L_X(N)$ ,  $\forall X$  the validity of (2.11) is proved..

We now express the relations (2.11) in terms of the eigenvalues  $\lambda_i$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} N^*d\lambda_1 - \lambda_1d\lambda_1 \\ N^*d\lambda_2 - \lambda_2d\lambda_2 \\ \vdots \\ N^*d\lambda_n - \lambda_nd\lambda_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (2.13)$$

Since the Vandermond matrix in the LHS of this equation is, by assumption, invertible, we conclude that, for  $i = 1, \dots, n$ , it holds  $N^*d\lambda_i = \lambda_id\lambda_i$ .

□

This proposition can be rephrased saying that “half of” the DN coordinates are algebraically provided by the Nijenhuis tensor itself. The remaining “half”  $y_1, \dots, y_n$  can always be found by quadratures. Actually, there is a condition leading to the algebraic solution of this problem too. To elucidate this, the following two considerations are crucial.

First of all one remarks that, thanks to the fact that, for DN coordinates, if  $i \neq j$  one has  $\{\lambda_i, y_j\} = 0$  one can replace the  $n(n-1)/2$  equations  $\{\lambda_i, y_j\} = \delta_{ij}$  with the  $n$  equations

$$\{\lambda_1 + \cdots + \lambda_n, y_j\} = 1, \quad j = 1, \dots, n,$$

that do not involve the individual coordinates  $\lambda_i$  but only their sum  $\sum_{i=1}^n \lambda_i = I_1$ , and the Hamiltonian vector field

$$Y = -PdI_1 = \sum_i \frac{\partial}{\partial y_i}.$$

The second argument goes as follows. Let us consider the distinguished functions  $I_k$  introduced in proposition 2.9, and trade them for the coefficients  $p_i$  of the minimal polynomial

$$\Delta_N(\lambda) = \lambda^n - p_1\lambda^{n-1} - p_2\lambda^{n-2} - \cdots - p_n$$

of  $N$ . The functions  $p_k$  and  $I_k$  are related by triangular Newton formulas

$$\begin{aligned} I_1 &= p_1; & I_2 &= p_2 + \frac{1}{2}p_1^2; & I_3 &= p_3 + p_2p_1 + \frac{1}{3}p_1^3; \\ I_4 &= p_4 + p_1p_3 + p_1^2p_2 + \frac{1}{2}p_2^2 + \frac{1}{4}p_1^4; & I_5 &= p_5 + \dots \end{aligned} \quad (2.14)$$

As a consequence of the recursion relations (2.11), it can be easily shown that the  $p_i$ 's satisfy the 'Frobenius' recursion relations

$$N^*dp_i = dp_{i+1} + p_idp_1, \quad \text{with } p_{n+1} \equiv 0. \quad (2.15)$$

We can compactly write these relations as a single relation for the polynomial  $\Delta_N(\lambda)$ ; indeed, a straightforward computation shows that they are equivalent to

$$N^*d\Delta_N(\lambda) = \lambda d\Delta_N(\lambda) + \Delta_N(\lambda)dp_1. \quad (2.16)$$

Actually relations of this kind are very important for our purposes. Indeed, in [16] we proved the following proposition:

**Proposition 2.10** *Let  $\Phi(\lambda)$  a smooth function defined on the manifold  $M$ , depending on an additional parameter  $\lambda$ . Suppose that there exists a one-form  $\alpha_\Phi$  such that*

$$N^*d\Phi(\lambda) = \lambda d\Phi(\lambda) + \Delta_N(\lambda)\alpha_\Phi. \quad (2.17)$$

*Then, the  $n$  functions  $\Phi_i$  obtained evaluating the "generating" function  $\Phi(\lambda)$  for  $\lambda = \lambda_i, i = 1, \dots, n$  are Nijenhuis functions, that is, they satisfy  $N^*d\Phi_i = \lambda_id\Phi_i$ .*

**Definition 2.11** *We will call a generating function  $\Phi(\lambda)$  satisfying equation (2.17) a Nijenhuis functions generator.*

The relevance of the notion of Nijenhuis functions generator in the search for DN coordinates stems from the fact that Nijenhuis functions generators form an algebra  $\mathcal{N}(M)$ , which is closed under the action of the vector field  $Y = -PdI_1$ . In this way, knowing a set of Nijenhuis functions generators, we can obtain further elements of the algebra  $\mathcal{N}(M)$  by repeated applications of the vector field  $Y$ . Clearly, in such an extended algebra, the characteristic equation

$$L_Y(\Psi(\lambda)) = 1 + \Delta_N(\lambda)\alpha_\psi$$

may be easier to be solved, thus yielding the missing Darboux-Nijenhuis coordinates  $y_i$  as  $y_i = \Psi(\lambda_i)$ . We will see an instance of this situation in the Example of Section 4.

For an analysis of Darboux-Nijenhuis coordinates within the theory of multi-hamiltonian structure on loop algebras, see [14, 9, 19]

### 3 The Separability Conditions

As we have briefly recalled in Section 2, on a bihamiltonian manifold one is usually lead to consider bihamiltonian vector fields, that is vector fields  $X$  admitting the twofold Hamiltonian representation  $X = Pdf = P'dg$ . Let us now suppose that  $(M, P, P')$  be a regular bihamiltonian manifold of dimension  $2n$ , and that we were able to construct, by means of the Lenard–Magri iteration procedure a sequence of functions  $H_1, H_2, \dots$  satisfying  $P'dH_i = PdH_{i+1}$ . Let us also suppose that the first  $n$  of them be functionally independent. Then one easily shows that the all the further Hamiltonians  $H_{n+1}, \dots$  are functionally dependent from the first  $n$ . Indeed this follows from the fact that a regular Poisson manifold of dimension  $2n$  cannot have more than  $n$  mutually commuting independent functions.

This means that, if we consider the Hamiltonian  $H_{n+1}$ , there must be a relation of the form

$$\psi(H_1, \dots, H_n; H_{n+1}) = 0, \quad \text{with } \psi_{H_{n+1}} \equiv \frac{\partial \psi}{\partial H_{n+1}} \neq 0. \quad (3.1)$$

relating it with the independent Hamiltonians  $H_i, i = 1, \dots, n$ .

Actually, the case of  $H_i = I_i \equiv \frac{1}{2i} \text{Tr} N^i$  is an instance of this situation. In fact, since by the Cayley–Hamilton theorem  $N$  annihilates its minimal polynomial, we have  $N^n - \sum_{i=1}^n p_i N^{n-1} = 0$ , yielding the relation

$$I_{n+1}/2(n+1) - \sum_{i=1}^n 2(n-i+1)p_i I_{n-i+1} = 0.$$

Differentiating equation (3.1) we see that, along with  $P'dH_i = PdH_{i+1}$ ,  $i = 1, \dots, n-1$  it holds:

$$P'dH_n = PdH_{n+1} = \frac{-1}{\psi_{H_{n+1}}} \left( \sum_{i=1}^n \frac{\partial \psi}{\partial H_i} PdH_i \right), \quad (3.2)$$

that is, the vector field  $X_{n+1} = PdH_{n+1} = P'dH_n$  is a linear combination of the vector fields  $X_1 = PdH_1, \dots, X_n = PdH_n$ .

This innocent looking observation is the clue for the bihamiltonian theory of SoV. Let  $\{H_1, H_2, \dots, H_n\}$  any integrable system on  $M$ , that is that is, the  $H_i$ s are mutually commuting (w.r.t.  $P$ ) independent functions. We can construct a  $n$ -dimensional distribution, namely the distribution  $\mathcal{D}_H$  spanned by the  $n$  mutually commuting vector fields  $X_i = PdH_i$ . This is nothing but the very classical tangent distribution to the invariant tori of the Liouville Arnol'd theory of integrable systems. Since  $M$  comes equipped with a second Poisson tensor  $P'$  we can as well consider the distribution  $\mathcal{D}'_H$  generated by the Hamiltonians  $H_i$  under the action of  $P'$ , that is, generated by the vector fields  $X'_i = P'dH_i$ . It holds:

**Theorem 3.1** *Let  $\{H_1, \dots, H_n\}$  define, as explained above, an integrable system on a regular bihamiltonian manifold  $(M, P, P')$ . The Hamilton-Jacobi equations associated with any of the Hamiltonians  $H_i$  are separable in the DN coordinates  $x, y_i$  defined by  $N = P'P^{-1}$  if and only if the distribution  $\mathcal{D}'_H$  is contained in  $\mathcal{D}_H$ , or, equivalently, if the distribution  $\mathcal{D}_H$  is invariant along  $N$ .*

**Proof.** We will first prove the equivalence of the invariance of  $\mathcal{D}_H$  under  $N$  and the inclusion  $\mathcal{D}'_H \subset \mathcal{D}_H$ . To say that  $\mathcal{D}'_H$  is contained in  $\mathcal{D}_H$  is tantamount to saying that there exists a matrix  $F_{ij}$ , whose entries are, in general, functions defined on  $M$  such that, for  $i = 1, \dots, n$  it holds:

$$X'_i \equiv P'dH_i = \sum_j F_{ij} PdH_j = \sum_j F_{ij} X_j. \quad (3.3)$$

Writing  $P' = NP$  we can translate these equalities into  $NX_i = \sum_j F_{ij}X_j$  for all  $i = 1, \dots, n$ .

The full proof of the fact that the invariance of  $\mathcal{D}_H$  insures separability in DN coordinates can be found in [16]. It goes as follows.

At first we notice that the translation in terms of the codistribution  $\mathcal{D}_H^*$  generated by the differentials of the Hamiltonians  $H_i$  of the invariance condition for  $\mathcal{D}_H$  is the invariance condition  $N^*\mathcal{D}_H^* \subset \mathcal{D}_H^*$ . This can be easily seen applying to (3.3) the operator  $P^{-1}$ , to get  $N^*dH_i = \sum_j F_{ij}dH_j$ .

Since all the Poisson brackets  $\{H_i, H_j\}$  vanish and  $M$  is a regular bihamiltonian manifold, the matrix  $F$  defined by (3.3) can be show to have simple eigenvalues, that coincide with eigenvalues  $\lambda_i$  of  $N$ . So there exists a matrix  $S$  satisfying

$$SF = \Lambda S, \text{ where } \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n).$$

If we define the  $n$  one forms  $\theta_i = \sum_j S_{ij}dH_j$ , we get:

$$N^*\theta_i = \sum_j S_{ij}N^*dH_j = \sum_{j,k} S_{ij}F_{j,k}dH_k = \sum_{j,k} \lambda_i \delta_{ij} S_{j,k}dH_k = \lambda_i \theta_i. \quad (3.4)$$

This means that  $\theta_i$  is an eigenvector of  $N^*$  relative to  $\lambda_i$ ; hence there must be functions  $F_i, G_i$  such that

$$\sum_j S_{ij}dH_j = F_i dx_i + G_i dy_i \quad (3.5)$$

whence the existence of a separation relation  $\Phi_i(x_i, y_i; H_1, \dots, H_h)$  for all  $i = 1, \dots, n$ . The converse statement can be trivially proved.

□

We would like to stress that the separability condition of Theorem 3.1 is a *tensorial one*. That is, given a regular bihamiltonian manifold  $(M, P, P')$  this separability criterion can be checked in *any system of coordinates*, without the a-priori calculation of the DN coordinates themselves. Notice, also, that the validity of the statement does not (as it should be!) depend on the choice of mutually commuting integrals  $\{H_1, \dots, H_n\}$ . That is, if we consider a “change of coordinates in the space of the actions”, that is we trade the  $H_i$ 's for another complete set of integrals of the motion  $K_i = K_i(H_1, \dots, H_n)$ , then the separability of the new Hamiltonians  $K_i$  will hold if and only if the separability of the original ones holds. Indeed, the dual distributions generated by the  $H_i$ 's and the  $K_i$ 's coincide.



A second remark is important and deserves to be explicitly spelled out. Although we have started our discussion considering the case of a family of bihamiltonian vector fields, that is the case of Lenard-Magri sequences, the hypotheses of Theorem 3.1 concern only the relations of the distributions generated respectively under the action of  $P$  and  $P'$  by the Hamiltonians  $H_i$ , *without any mention* of the fact that the generators of the distribution be bihamiltonian vector fields. Thus, although it might seem a somewhat odd statement, *the vector fields that are separable by means of the bihamiltonian approach are not necessarily bihamiltonian vector fields!* It is also important to notice that it is not only a matter of choice of generators. Indeed, in [27] it has been shown that the only bihamiltonian vector fields on a regular bihamiltonian manifold turn out to be associated with *separated functions of the eigenvalues of  $N$* , that is functions of the form  $H = \sum_{i=1}^n f_i(\lambda_i)$ . This means that, in such a case, the distribution  $\mathcal{D}_H$  coincides with that generated by the distinguished functions  $I_i$ . However, this is by far a very special example, that is, the range of applicability of the method is much wider than that, as it has already been quite widely shown in the literature.

The separation condition of Theorem 3.1 is based on the analysis of the behaviour of the characteristic distribution associated with an integrable system under the Nijenhuis tensor  $N$ . An equivalent criterion, based on the analysis of the Poisson brackets associated with the tensor  $P'$  can be formulated as follows.

**Theorem 3.2** *Let  $\{H = H_1, H_2, \dots, H_n\}$  an integrable system defined on regular bihamiltonian manifold  $(M, P, P')$ . The Hamiltonians  $H_i$  are separable in the DN coordinates defined by  $N = P'P^{-1}$  if and only if, along with the commutation relations  $\{H_i, H_j\} = 0$  there hold also*

$$\{H_i, H_j\}' \equiv \langle dH_i, P'dH_j \rangle = 0, \quad \text{for } i, j = 1, \dots, n. \quad (3.6)$$

**Proof.** The key formula is the relation between  $P, P'$  and  $N^*$ . Indeed, suppose that  $\mathcal{D}_H^*$  be invariant along  $N^*$ . Then:

$$\begin{aligned} \{H_i, H_j\}' &= \langle dH_i, P'dH_j \rangle = \langle dH_i, NPdH_j \rangle \\ \langle N^*dH_i, PdH_j \rangle &= \sum_k F_{ik} \langle dH_k, PdH_j \rangle = \sum_k F_{ik} \{H_k, H_j\} = 0, \end{aligned}$$

which, in view of Theorem 3.1 proves the statement in one direction. Now, let us suppose that (3.6) holds. Then, for every  $ij = 1, \dots, n$  we have:

$$0 = \{H_i, H_j\}' = \langle dH_i, P'dH_j \rangle = \langle dH_i, NPdH_j \rangle = \langle N^*dH_i, PdH_j \rangle,$$

meaning that, for all  $i = 1, \dots, n$ , the one-form  $N^*dH_i$  belongs to the annihilator (w.r.t.  $P$ ) of the distribution  $\mathcal{D}_H$ . Since such an annihilator coincides with  $\mathcal{D}_H^*$ , this means that, for all  $i = 1, \dots, n$ ,  $N^*dH_i \in \mathcal{D}_H^*$ .

□

This results leads to the following, somewhat daring, comparison. The Liouville-Arnol'd theorem on finite dimensional integrable Hamiltonian systems says that the geometrical structure underlying integrability of a Hamiltonian vector field defined on a symplectic manifold  $M, \omega$  is a *Lagrangian* foliation of  $M$ . We can rephrase the content of Theorem 3.2 saying that the geometrical structure underlying the separability of a system defined on a regular bihamiltonian manifold  $(M, P, P')$  is a *bilagrangian* foliation of  $M$ .

We end our presentation of the bihamiltonian set-up for SoV with the following remark. Theorem 3.1 concerns only the existence of the separation relations. In principle, one could try to find these relations in concrete examples by actually diagonalizing the matrix  $F$ , and explicitly finding and integrating the relations (3.5). However, there is a very simple tensorial criterion which can be used to determine the *functional form* of the separation relations  $\Phi_i(x_i, y_i; H_1, \dots, H_h)$ , whose proof can be found in [16].

**Proposition 3.3** *Let  $\{H_1, \dots, H_n\}$  be an integrable system defined on a regular bihamiltonian manifold, which is separable in the Darboux-Nijenhuis coordinates associated with  $N = P^{-1}P'$ . Consider the matrix  $F_{ij}$  fulfilling the relations (3.3). Then the separation relations are be affine in the Hamiltonians  $H$ , that is of the form*

$$\Phi_i(x_i, y_i; H_1, \dots, H_h) = \sum_j S_{ij}(x_i, y_i)H_j + U_i(x_i, y_i), \quad (3.7)$$

*if and only if the matrix  $F$  satisfies the relation  $N^*dF_{ij} = \sum_k F_{ik}dF_{kj}$ .*

The matrix  $S$  on (3.7) can be shown to be a suitably normalized matrix of eigenvectors of the matrix  $F$ . Its characteristic property is that, as expressed in the equation, the entries  $S_{ij}$  of the  $i$ -th row depend only on the pair  $(x_i, y_i)$  of Darboux-Nijenhuis coordinates. For this reason it can be called a Stäckel matrix.

## 4 Example: a generalized Toda Lattice

In this final Section we will apply the general scheme outlined in the previous Sections to a specific model, with the aim of showing how the recipes discussed so far from a theoretical standpoint can be concretely applied. We will study a generalization of the four site Toda lattice, to be termed *Toda<sub>3</sub><sup>4</sup> model*. This system is a member of a family introduced in [21] as reductions of the discrete KP hierarchy. It can be described as follows. We consider on  $M = \mathbb{C}^{12}$ , endowed with global coordinates  $\{b_i, a_i, c_i\}_{i=1..4}$  the Hamiltonian

$$H_{GT} = \frac{1}{2}(b_1^2 + b_2^2 + b_3 + b_4^2) - (a_1 + a_2 + a_3 + a_4), \quad (4.1)$$

and the linear Poisson tensor given by the matrix

$$P = \begin{bmatrix} \mathbf{0} & \mathbf{A}_1 & \mathbf{C}_1 \\ -\mathbf{A}_1^T & \mathbf{C}_2 & \mathbf{0} \\ -\mathbf{C}_1^T & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{with } \mathbf{A}_1 = \begin{bmatrix} -a_1 & 0 & 0 & a_4 \\ a_1 & -a_2 & 0 & 0 \\ 0 & a_2 & -a_3 & 0 \\ 0 & 0 & a_3 & -a_4 \end{bmatrix}, \quad (4.2)$$

$$\mathbf{C}_1 = \begin{bmatrix} -c_1 & 0 & c_3 & 0 \\ 0 & -c_2 & 0 & c_4 \\ c_1 & 0 & -c_3 & 0 \\ 0 & c_2 & 0 & -c_4 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 0 & -c_1 & 0 & c_4 \\ c_1 & 0 & -c_2 & 0 \\ 0 & c_2 & 0 & -c_3 \\ -c_4 & 0 & c_3 & 0 \end{bmatrix}.$$

where we denoted by  $\mathbf{0}$  the  $4 \times 4$  matrix with vanishing entries..

Using (here and in the sequel) the cyclic identifications  $a_{i+4} = a_i, b_{i+4} = b_i, c_{i+4} = c_i$ , the Hamiltonian vector field  $X_{H_{GT}} = PdH_{GT}$  can be written as follows

$$\begin{bmatrix} \dot{b}_i \\ \dot{a}_i \\ \dot{c}_i \end{bmatrix} = \begin{bmatrix} a_{i-1} - a_i \\ a_i(b_{i+1} - b_i) + c_{i-1} - c_i \\ c_i(b_{i-2} - b_i) \end{bmatrix}, \quad i = 1, \dots, 4, \quad (4.3)$$

The expert reader will notice that  $H_{GT}$  coincides with the Hamiltonian of the periodic four-site Toda lattice, written in Flaschka coordinates  $b_i = p_i, a_i = \exp(q_i - q_{i+1})$ . Indeed, on the hyperplane  $M_T \simeq \mathbb{C}^8$  defined by  $c_i = 0, i = 1, \dots, 4$  the vector field  $X_{H_{GT}}$  defines the periodic Toda flow.

**Proposition 4.1** *The Hamiltonian vector field  $X_{H_{GT}}$  admits the Lax representation  $\dot{L}(\mu) = [L(\mu), \Phi]$ , where*

$$L(\mu) = \begin{bmatrix} b_1 & -\mu & \frac{c_3}{\mu^2} & \frac{a_4}{\mu} \\ \frac{a_1}{\mu} & b_2 & -\mu & \frac{c_4}{\mu^2} \\ \frac{c_1}{\mu^2} & \frac{a_2}{\mu} & b_3 & -\mu \\ -\mu & \frac{c_2}{\mu^2} & \frac{a_3}{\mu} & b_4 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & 0 & \frac{c_3}{\mu^2} & \frac{a_4}{\mu} \\ \frac{a_1}{\mu} & 0 & 0 & \frac{c_4}{\mu^2} \\ \frac{c_1}{\mu^2} & \frac{a_2}{\mu} & 0 & 0 \\ 0 & \frac{c_2}{\mu^2} & \frac{a_3}{\mu} & 0 \end{bmatrix}. \quad (4.4)$$

The bihamiltonian aspects of this system have been discussed in [28]. In particular, it has been noticed that on  $M$  there exists a second Hamiltonian structure for the vector field  $X_{H_{GT}}$ . Namely one considers the bivector  $P'$  having the following form:

$$P' = \begin{bmatrix} \mathbf{A}_2 & \mathbf{B}_1 & \mathbf{C}_3 \\ -\mathbf{B}_1^T & \mathbf{A}_2 & \mathbf{C}_4 \\ -\mathbf{C}_3^T & -\mathbf{C}_4^T & \mathbf{A}_4 \end{bmatrix}, \quad \text{where } \mathbf{C}_3 = \begin{bmatrix} -b_1c_1 & 0 & b_1c_3 & 0 \\ 0 & -b_2c_2 & 0 & b_2c_4 \\ c_1b_3 & 0 & -b_3c_3 & 0 \\ 0 & c_2b_4 & 0 & -b_4c_4 \end{bmatrix}, \quad (4.5)$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & a_1 & 0 & -a_4 \\ -a_1 & 0 & a_2 & 0 \\ 0 & -a_2 & 0 & a_3 \\ a_4 & 0 & -a_3 & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} -b_1a_1 & c_1 & -c_3 & b_1a_4 \\ b_2a_1 & -b_2a_2 & c_2 & -c_4 \\ -c_1 & b_3a_2 & -b_3a_3 & c_3 \\ c_4 & -c_2 & b_4a_3 & -b_4a_4 \end{bmatrix},$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & -b_2c_1 - a_1a_2 & 0 & b_1c_4 + a_1a_4 \\ b_2c_1 + a_1a_2 & 0 & -b_3c_2 - a_2a_3 & 0 \\ 0 & b_3c_2 + a_2a_3 & 0 & -b_4c_3 - a_3a_4 \\ -b_1c_4 - a_1a_4 & 0 & b_4c_3 + a_3a_4 & 0 \end{bmatrix},$$

$$\mathbf{C}_4 = \begin{bmatrix} -a_1c_1 & -a_1c_2 & a_1c_3 & a_1c_4 \\ c_1a_2 & -a_2c_2 & -a_2c_3 & a_2c_4 \\ c_1a_3 & c_2a_3 & -a_3c_3 & -a_3c_4 \\ -c_1a_4 & c_2a_4 & c_3a_4 & -a_4c_4 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 0 & -c_1c_2 & 0 & c_1c_4 \\ c_1c_2 & 0 & -c_2c_3 & 0 \\ 0 & c_2c_3 & 0 & -c_3c_4 \\ -c_1c_4 & 0 & c_3c_4 & 0 \end{bmatrix}.$$

It can be easily noticed that  $X_{H_{GT}} = P'd(-\sum_{i=1}^4 b_i)$ . More in general, we have the following

**Proposition 4.2** *The pencil  $P' - \lambda P$  is a pencil of Poisson brackets. The rank of the generic element of the pencil is eight. The characteristic polynomial  $R(\lambda, \mu) = \text{Det}(\lambda \mathbf{1} - L(\mu))$ , written in terms of  $\rho = \mu^4$  can be expanded as:*

$$R(\lambda, \rho) = \lambda^4 - \rho + H(\lambda) + (K(\lambda) - \lambda^2 J_1)/\rho + J_2/\rho^2. \quad (4.6)$$

The functions  $J_1$  and  $J_2$  are common Casimirs of  $P$  and  $P'$ . The polynomials  $H(\lambda), K(\lambda)$  are polynomial Casimirs of the pencil  $P_\lambda = P' - \lambda P$ . They have the form

$$H(\lambda) = \lambda^3 H_0 - \lambda^2 H_1 + \lambda H_2 - H_3, \quad K(\lambda) = K_0 \lambda + K_1. \quad (4.7)$$

Explicitly,  $J_1 = c_1 c_3 + c_2 c_4$ ,  $J_2 = c_1 c_2 c_3 c_4$ , while the coefficients of  $H(\lambda)$  and  $K(\lambda)$  are given by:

$$\begin{aligned} H_0 &= \sum_{i=1}^4 b_i, \quad H_1 = \sum_{i>j=1}^4 b_i b_j + \sum_{i=1}^4 a_i, \quad H_2 = \sum_{i=1}^4 (c_i + b_i(a_{i+1} + a_{i+2}) + b_i b_{i+1} b_{i+2}) \\ H_3 &= \sum_{i=1}^4 b_i c_{i+1} + a_1 a_3 + a_2 a_4 + \text{cubic and quartic terms;} \\ K_0 &= \sum_{i=1}^4 b_i c_{i-1} c_{i+1} - c_i a_{i-1} a_{i+2}, \quad K_1 = \sum_{i=1}^4 a_i c_{i+1} c_{i+2} + \text{quartic terms} \end{aligned}$$

One can show that the eight functions  $H_0, H_1, H_2, H_3, K_0, K_1, J_1, J_2$  are functionally independent and, thanks to the fact that they fill in Lenard sequences, are mutually in involution. The kernel of  $P$  is generated (at generic points  $m \in M$ ) by the differentials of the four functions  $H_0, K_0, J_1, J_2$ . Hence, on the 8 dimensional manifold  $\mathcal{S}_\kappa$  defined by the equations  $H_0 = \kappa_1, K_0 = \kappa_2, J_1 = \kappa_3, J_2 = \kappa_4$ , that is, the generic symplectic leaf of  $P$ , the vector field  $X_{H_{GT}}$  is completely integrable. To realize this we simply have to notice that  $H_{GT}$  can be expressed as  $\frac{1}{2}H_0^2 - H_1$ , and apply the properties of anchored Lenard Magri sequences collected in Proposition 2.7

## 4.1 Separation of Variables

We will now show how to apply the ideas and recipes if the bihamiltonian set-up for SoV to the Toda<sub>3</sub><sup>4</sup> model introduced above. The first problem to deal with is that the

Poisson tensor  $P'$  does not restrict to  $\mathcal{S}_\kappa$ , but must be projected. This can be done as follows (see [9, 15, 16, 29] for details and the geometric background), by means of a kind of Dirac reduction process.

We consider the vector fields  $Z_1 = -\frac{\partial}{\partial b_4}$  and  $Z_2 = \frac{\partial}{\partial a_4}$ , notice that the matrix

$$G = \begin{pmatrix} L_{Z_1}(H_0) & L_{Z_1}(K_1) \\ L_{Z_2}(H_0) & L_{Z_2}(K_1) \end{pmatrix} = \begin{pmatrix} 1 & -c_1c_3 \\ 0 & -c_1a_3 - a_1c_2 \end{pmatrix}$$

is invertible, and form the bivector

$$\Delta = \sum_{i,j=1}^2 [G_{ij}^{-1}] Z_i \wedge X_1^j, \text{ where } X_1^1 = P'dH_0, X_1^2 = P'dK_0. \quad (4.8)$$

**Lemma 1** *The modified bivector  $Q = P' - \Delta$  defines a Poisson bracket, compatible with  $P$ ;  $Q$  restricts to  $\mathcal{S}_\kappa$ .*

**Proof.** The proof of the fact that  $Q_\lambda = Q - \lambda P$  is a Poisson pencil follows (see, e.g., [15]), from the equalities:

$$\begin{aligned} L_{Z_1}P &= 0, & L_{Z_1}P' &= W_1^1 \wedge Z_1 - c_3 \frac{\partial}{\partial a_3} \wedge Z_2 \\ L_{Z_2}P &= \left( \frac{\partial}{\partial b_1} - \frac{\partial}{\partial a_1} \right) \wedge Z_2, & L_{Z_2}P' &= \left( b_4 \frac{\partial}{\partial a_4} + \frac{\partial}{\partial b_1} \right) \wedge Z_1 + W_2^2 \wedge Z_2. \end{aligned} \quad (4.9)$$

with  $W_1^1 = \left( a_3 \frac{\partial}{\partial a_3} - a_4 \frac{\partial}{\partial a_4} + c_2 \frac{\partial}{\partial c_2} - c_4 \frac{\partial}{\partial c_4} \right)$  and

$$W_2^2 = \left( b_4 \frac{\partial}{\partial b_4} - b_1 \frac{\partial}{\partial b_1} - a_3 \frac{\partial}{\partial a_3} - a_4 \frac{\partial}{\partial a_4} - c_1 \frac{\partial}{\partial c_1} + c_2 \frac{\partial}{\partial c_2} + c_3 \frac{\partial}{\partial c_3} - c_4 \frac{\partial}{\partial c_4} \right),$$

as well as from the fact that

$$QdH_0 = QdK_0 = dJ_1 = QdJ_2 = 0. \quad (4.10)$$

To show that (4.9) holds true is a matter of an explicit computation, while (4.10) follows from the definition of  $Q$ . In fact, the last two equations hold since  $J_1$  and  $J_2$

are Casimirs of  $P'$  invariant under  $Z_1$  and  $Z_2$ . For, e.g.,  $H_0$  one computes

$$\begin{aligned} QdH_0 &= P'dH_0 - \Delta dH_0 = X_1^1 - \sum_{i,j=1}^2 ([G_{i,j}^{-1}]L_{Z_j}(H_0)) \cdot X_1^i \\ &= X_1^1 - \sum_{i,j=1}^2 ([G_{i,j}^{-1}]G_{j1}) \cdot X_1^i = X_1^1 - \sum_i \delta_{i,1} \cdot X_1^i = 0, \end{aligned}$$

where the second equality follows from the fact that all the functions  $H_i, K_\alpha, J_\alpha$  are in involution w.r.t.  $P$ .

□

Thanks to the above lemma, the generic symplectic leaf  $\mathcal{S}_\kappa$  is endowed with the structure of a regular bihamiltonian manifold. It is easy to show that the non trivial Hamiltonians  $H_1, H_2, H_3, K_1$  (more precisely, the restriction to  $\mathcal{S}_\kappa$  of these Hamiltonians) satisfy the hypothesis of Theorem 3.1 w.r.t the (restriction to  $\mathcal{S}_\kappa$ ) of the pencil  $Q - \lambda P$ . Indeed we have:

$$QdH_i = P'dH_i - \sum_{i,j=1}^2 [G_{i,j}^{-1}]Z_i \wedge X_1^j(dH_i) = PdH_{i+1} - \sum_{i,j=1}^2 [G_{i,j}^{-1}]L_{Z_i}(H_i)X_1^j$$

(where we understand  $H_4 = 0$ ) and

$$QdK_2 = P'dK_1 - \sum_{i,j=1}^2 [G_{i,j}^{-1}]Z_i \wedge X_1^j(dK_2) = \sum_{i,j=1}^2 [G_{i,j}^{-1}]L_{Z_i}(dK_1)X_1^j$$

So we proved that, for generic values of the Casimirs  $\kappa_i, i = 1, \dots, 4$ , the system obtained by restriction of the Toda<sub>3</sub><sup>4</sup> flows on  $\mathcal{S}_\kappa$  is separable in the DN coordinates associated with the restriction to  $\mathcal{S}_\kappa$  of pencil  $Q - \lambda P$ . To finish our job we finally have to:

- a) explicitly define the DN coordinates;
- b) find the separation relations.

To solve the first problem, we will use the tools briefly described in Section 2.2. We rely on a result of [16], as well as on explicit computations, to state the following

**Proposition 4.3** *Let  $\mathcal{G}(\lambda)$  be the matrix*

$$\mathcal{G} = \begin{pmatrix} L_{Z_1}H(\lambda) & L_{Z_2}H(\lambda) \\ L_{Z_1}K(\lambda) & L_{Z_2}K(\lambda) \end{pmatrix}; \quad (4.11)$$

*The roots of the degree 4 polynomial  $\text{Det}(\mathcal{G})$  are the roots of the minimal polynomial  $\Delta(\lambda) = \lambda^4 - \sum_{i=1}^4 p_i \lambda^{4-i}$  of the Nijenhuis tensor  $N = P^{-1}Q$  associated with the regular Poisson pencil  $Q_\lambda$ . The coefficients  $p_i$  are well defined functions on the generic symplectic leaf  $\mathcal{S}_\kappa$ , and are functionally independent. Furthermore, the ratios  $\rho(\lambda) = -\mathcal{G}_{2,2}/\mathcal{G}_{1,2}$ ,  $\sigma(\lambda) = -\mathcal{G}_{2,1}/\mathcal{G}_{1,1}$  are Nijenhuis function generators.*

Thus, one half of the Darboux-Nijenhuis coordinates will be given by the roots of  $\text{Det}(\mathcal{G})$ . To find the remaining half we consider vector field  $Y = -Pdp_1$  whose role has been discussed in Section 2.2. Since an explicit computations shows that  $L_Y \log(\rho(\lambda)) = 1$ , hence we can state the following

**Proposition 4.4** *A set of Darboux-Nijenhuis coordinates for the restriction to the generic symplectic leaf  $\mathcal{S}_\kappa$  of the Toda<sub>3</sub><sup>4</sup> flows are given by the four roots  $\lambda_i$  of  $\text{Det}(\mathcal{G})(\lambda)$  and by the values  $\mu_i$  of the function  $\log(\rho(\lambda))$  for  $\lambda = \lambda_i$ , where*

$$\rho(\lambda) = \frac{(-c_1 a_3 - a_1 c_2) \lambda + c_2 a_1 b_3 - a_1 a_2 a_3 + c_1 b_2 a_3 + c_1 c_2}{c_1 c_3 \lambda + a_1 a_2 c_3 - c_1 b_2 c_3}$$

To find the separation relations we reconsider the Lax matrix (4.4), and notice that the pairs, e.g.,  $\lambda_i, \rho(\lambda_i)$  are common solutions to the system

$$\begin{aligned} \rho \mathcal{G}_{1,1} + \mathcal{G}_{21} &= 0 \\ \rho \mathcal{G}_{1,2} + \mathcal{G}_{22} &= 0. \end{aligned} \quad (4.12)$$

Since the Lie derivative of the matrix  $\mathcal{L}(\lambda, \mu) = \lambda \mathbf{1} - L(\mu)$  along the vector fields  $Z_i$  are given by:

$$L_{Z_1}(\mathcal{L}(\lambda, \mu)) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad L_{Z_2}(\mathcal{L}(\lambda, \mu)) = \begin{bmatrix} 0 & 0 & 0 & -\mu^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



we see that the solutions of the system (4.12) are nothing but the solutions of the equations

$$[\mathcal{L}(\lambda, \mu)^\vee]_{44} = 0, [\mathcal{L}(\lambda, \mu)^\vee]_{41} = 0,$$

$\mathcal{L}(\lambda, \mu)^\vee$  being the classical adjoint to  $\mathcal{L}(\lambda, \mu) = \lambda \mathbf{1} - L(\mu)$ . Using standard arguments of linear algebra, and the results collected in Proposition 4.4, we can state

**Proposition 4.5** *The separation relations connecting pairs of Darboux-Nijenhuis coordinates  $\lambda_i, \mu_i$ , the Hamiltonians  $H_1, H_2, H_3, K_2$  and the Casimirs  $H_0, K_1, K_2, J_0$  are, on the generic symplectic leaf  $\mathcal{S}_\kappa$ , given by the evaluation of the characteristic polynomial  $\text{Det}(\mathcal{L}(\lambda, \mu))$  in  $\lambda = \lambda_i, \mu = \mu_i = \log(\rho(\lambda_i))$ .*

We notice that, *a posteriori*, the separation coordinates for the Toda<sub>3</sub><sup>4</sup> system fall in the class described in, e.g., [33, 1, 10, 34, 2, 22]. Namely, the DN coordinates that separate the Toda<sub>3</sub><sup>4</sup> system are Algebro-geometrical Darboux coordinates associated with the spectral curve (4.6), and fulfill the so-called Sklyanin's 'magical recipe'.

As a final remark, in connection with the discussion on the relation between the bihamiltonian property of an integrable vector field and the separability of the associated HJ equations of Section 3, we notice that the Hamiltonians  $H_1, H_2, H_3, K_2$  are functionally independent from the coefficients of the minimal polynomial of the Nijenhuis tensor obtained from  $Q - \lambda P$ . So, this is a further instance of a system which is not bihamiltonian on a regular manifold, but turns out to be separable via the bihamiltonian method of SoV.

## 4.2 A remarkable subsystem: the open Toda<sub>3</sub><sup>4</sup> system

In this last subsection we will discuss a remarkable reduction of the periodic Toda<sub>3</sub><sup>4</sup> system, leading to the corresponding generalization of the open (or non-periodic) one. In the manifold  $M \simeq \mathbb{C}^{12}$  we consider the nine-dimensional submanifold  $M_0$  defined by the equations:

$$a_4 = c_3 = c_4 = 0. \tag{4.13}$$

One can easily verify that the restriction  $X_{H_{GT}}^0$  to  $M_0$  of the vector field  $X_{H_{GT}}$  is tangent to  $M_0$ . Also, the tensor  $P$  can be restricted to  $M_0$ ; indeed, the expression of its restriction  $P_0$  w.r.t. the natural coordinates  $\{b_1, \dots, b_4, a_1, \dots, a_3, c_1, c_2\}$  of  $M_0$  is obtained from (4.2) simply by removing the 9th, 11th, 12th rows and columns.

One can easily check that  $X_{HGT}^0 = P_0 dH_{GT,0}$  with

$$H_{GT,0} = \frac{1}{2}(b_1^2 + b_2^2 + b_3 + b_4^2) - (a_1 + a_2 + a_3), \quad (4.14)$$

and recognize that this function is the Hamiltonian of the open Toda lattice. Also, a Lax pair for  $X_{HGT}^0$  is

$$L_0 = \begin{bmatrix} b_1 & -\mu & 0 & 0 \\ \frac{a_1}{\mu} & b_2 & -\mu & 0 \\ \frac{c_1}{\mu^2} & \frac{a_2}{\mu} & b_3 & -\mu \\ -\mu & \frac{c_2}{\mu^2} & \frac{a_3}{\mu} & b_4 \end{bmatrix}, \quad \Phi_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{a_1}{\mu} & 0 & 0 & 0 \\ \frac{c_1}{\mu^2} & \frac{a_2}{\mu} & 0 & 0 \\ 0 & \frac{c_2}{\mu^2} & \frac{a_3}{\mu} & 0 \end{bmatrix}. \quad (4.15)$$

It should be clear from the form of the Lax pair that the vector field  $X_{HGT}^0$  on  $M_0$  is an extension of the standard open Toda lattice towards the so-called full open Toda lattice, which is a system describing a flow on the lower Borel subgroup of  $sl(N)$ . The integrability of the full open Toda lattice was established in [8] (see, also, [12]). The idea was to complement the integrals of the motion coming from the Lax representation with additional integrals obtained by means of the ‘chopping method’, within the group–theoretical point of view.

The need to supply the standard results of the Lax theory with further methods should be clear from the following considerations. The only Casimir function of  $P_0$  is  $h_0 = \sum_{i=1}^4 b_i$ . Hence, its symplectic leaves  $\mathcal{S}_\xi \subset (M_0)$  are the eight dimensional manifolds defined by  $h_0 = \xi$ , and  $X_{HGT}^0$  can be seen as a Hamiltonian system with *four* degrees of freedom.

The characteristic polynomial of the matrix  $L_0$  is

$$\text{Det}(\lambda \mathbf{1} - L_0(\mu)) = -\mu^4 + \lambda^4 - h_0 \lambda^3 + h_1 \lambda^2 - h_2 \lambda + h_3, \quad (4.16)$$

that is, it provides us with only *three* non trivial Hamiltonians

$$h_1 = \sum_{i>j=1}^4 b_i b_j + \sum_{i=1}^3 a_i, \quad h_2 = \sum_{i>j>k=1}^4 b_i b_j b_k + \sum_{i=1}^3 a_i (b_{i+2} + b_{i+3}) + c_1 + c_2 \quad (4.17)$$

$$h_3 = b_1 a_2 b_4 + a_1 b_3 b_4 + b_1 b_2 a_3 + b_1 c_2 + a_1 a_3 + c_1 b_4 + b_1 b_2 b_3 b_4.$$

We will now show how the the tools we previously introduced can be used to geometrically prove the complete integrability of such a system and, moreover, yield the existence of an additional integral of the motion .

The main property is that, along with  $P$ , the tensor  $Q$  restricts to  $M_0$ . This can be proven as follows: one checks by direct inspection that this is true for  $P'$ ; the assertion follows from the fact that the vector field  $X_1^2$  (defined in (4.8)) vanishes on  $M_0$ , while  $Z_1$  and  $X_1^1$ , which coincides with  $X_{GT}$ , are tangent to  $M_0$  at the points of  $M_0$ .

Furthermore, we add two observations. The first one concerns the restriction  $\mathcal{G}_0$  to  $M_0$  of the matrix  $\mathcal{G}$ . It has the form

$$\mathcal{G}_0 = \begin{bmatrix} \mathcal{G}_{11}^0 & \lambda^2 - (b_2 + b_3)\lambda + b_2b_3 + a_2 \\ 0 & -(c_1a_3 + a_1c_2)\lambda - a_1a_2a_3 + c_1c_2 + c_1b_2a_3 + a_1c_2b_3 \end{bmatrix} \quad (4.18)$$

with  $\mathcal{G}_{11}^0 = \lambda^3 - \pi_1\lambda^2 - \pi_2\lambda - \pi_3$  a degree three polynomial. Hence its determinant (that is, the minimal polynomial of the Nijenhuis tensor  $N_0$  induced by the pair  $Q_0 - \lambda P_0$  on  $\mathcal{S}_\xi$ ) factors as  $\mathcal{G}_{11}^0\mathcal{G}_{22}^0$ .

The second observation consists in the fact that the three surviving Hamiltonians  $h_1, h_2, h_3$  given by (4.17) satisfy the conditions:

$$Q_0dh_i = \sum_{j=1}^3 F_{ij}^0 P_0dh_j, \quad \text{with } F_{ij}^0 = \begin{bmatrix} \pi_1 & 1 & 0 \\ \pi_2 & 0 & 1 \\ \pi_3 & 0 & 0 \end{bmatrix}. \quad (4.19)$$

We notice that the functions  $\pi_i, i = 1, \dots, 3$  and the root

$$\lambda_4 = \frac{-a_1a_2a_3 + c_1c_2 + a_1c_2b_3 + c_1b_2a_3}{c_1a_3 + a_1c_2} \quad (4.20)$$

of  $\mathcal{G}_{2,2}^0$  are still functionally independent and hence (generically) different on  $\mathcal{S}_\xi$ .

**Lemma 2** *Let  $\sigma$  any function satisfying  $Q_0d\sigma = \lambda_4P_0d\sigma$ . Under the above hypotheses, the brackets  $\{\sigma, h_i\}_{P_0}$  and  $\{\sigma, h_i\}_{Q_0}$  vanish.*

**Proof.** Evaluating both sides of  $Q_0d\sigma = \lambda_4P_0d\sigma$  on the differentials  $(dh_1, dh_2, dh_3)$ , and switching the action of the Poisson tensors on the  $dh_i$ 's, we get

$$\langle d\sigma, Q_0dh_i \rangle = \lambda_4 \langle d\sigma, P_0dh_i \rangle, \quad i = 1, \dots, 3$$

Inserting (4.19) we get the equation  $\sum_{j=1}^3 (F_{ij}^0 - \lambda_4\delta_{ij}) \langle d\sigma, P_0dh_j \rangle = 0$ . Since  $\lambda_4$  is *not* an eigenvalue of  $F_{ij}^0$ , the Lemma is proved.

□

So a fourth integral of the motion, that commutes with the Hamiltonian  $H_{GT}^0$  for the open Toda<sub>3</sub><sup>4</sup> lattice is given indeed by the distinguished root  $\lambda_4$  of equation (4.20); this constructively proves the integrability of the system.

Finally, we notice that this method proves the existence of a *fifth* integral of the motion. Indeed, we know that, along with  $\lambda_4$ , there must exist another independent function  $\mu_4$ , satisfying the hypotheses of Lemma 2 and functionally independent of  $\lambda_4$  and of the  $h_i$ 's. In such a comparatively low dimensional case, such a function can be explicitly found to be

$$\mu_4 = \frac{c_2 (a_1 b_2 c_1 - a_2 a_1^2 - c_1^2 - c_1 b_3 a_1)}{c_1 (c_1 a_3 + a_1 c_2) (\lambda_4^3 - \pi_1 \lambda_4^2 - \pi_2 \lambda_4 - \pi_3)}.$$

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