SISSA preprint 16/2017/MATE

Friedrichs systems in a Hilbert space framework: Solvability and multiplicity [☆]

N. Antonić [a], M. Erceg [a,*], A. Michelangeli [b]

[a] *Department of mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia*
[b] *SISSA – International School for Advanced Studies, via Bonomea 265, 34136 Trieste, Italy*

10 April 2017

Abstract

The Friedrichs (1958) theory of positive symmetric systems of first order partial differential equations encompasses many standard equations of mathematical physics, irrespective of their type. This theory was recast in an abstract Hilbert space setting by Ern, Guermond and Caplain (2007), and by Antonić and Burazin (2010). In this work we make a further step, presenting a purely operator-theoretic description of abstract Friedrichs systems, and proving that any pair of abstract Friedrichs operators admits bijective extensions with a signed boundary map. Moreover, we provide sufficient and necessary conditions for existence of infinitely many such pairs of spaces, and by the universal operator extension theory (Grubb, 1968) we get a complete identification of all such pairs, which we illustrate on two concrete one-dimensional examples.

MSC: 35F45; 46C05; 46C20; 47A05; 47B38

Keywords: Symmetric positive first-order system of partial differential equations; Kreĭn space; Universal parametrisation of extensions

☆ This work was partially supported by the Croatian Science Foundation grant No. IP-2013-11-9780 (WeConMApp) and by the 2014–2017 MIUR-FIR grant No. RBFR13WAET.

E-mail addresses: nenad@math.hr (N. Antonić), maerceg@math.hr (M. Erceg), alemiche@sissa.it (A. Michelangeli).

1. Introduction

Following his research on symmetric hyperbolic systems [19], Friedrichs [20] introduced the concept of 'positive symmetric system', today customarily referred to as the *Friedrichs system*, encompassing a wide variety of equations of mathematical physics, including classical elliptic, parabolic and hyperbolic equations, which can be adapted, or rewritten, in the required form.

More precisely, for a given open and bounded set $\Omega \subseteq \mathbb{R}^d$ with Lipschitz boundary Γ, let the matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r)$ and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r)$ satisfy $\mathbf{A}_k = \mathbf{A}_k^*$ and

$$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad \text{a.e. on } \Omega\,.$$

Then the first-order differential operator $T : \mathrm{L}^2(\Omega)^r \longrightarrow \mathcal{D}'(\Omega)^r$ defined by

$$T\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

is called *the (classical) Friedrichs operator* or *the symmetric positive operator*, while (for given $\mathsf{f} \in \mathrm{L}^2(\Omega)^r$) the first-order system of partial differential equations $T\mathsf{u} = \mathsf{f}$ is called *the (classical) Friedrichs system* or *the symmetric positive system*.

However, as pointed out explicitly by Friedrichs himself in [20], the main motivation of his approach 'was not the desire for a unified treatment of elliptic and hyperbolic equations, but the desire to handle equations which are partly elliptic, partly hyperbolic, such as the Tricomi equation'. Friedrichs devised a clever way of representing different boundary (or initial) conditions by using a matrix field on the boundary, which – unfortunately – was not intrinsic (i.e., unique) for a given set of boundary conditions. Finally, he was only able to prove the existence of weak solutions, and the uniqueness of strong ones, leaving the general question open on the joint existence and uniqueness of either a weak or a strong solution.

A number of improvements were made to the theory, and in some special examples the gap towards the joint existence and uniqueness was closed, mostly by Friedrichs' collaborators and former students in the following years. Whereas there was some progress in very specific points, the topic appeared to be less active from mid 1960s to the late 1990s.

New interest in Friedrichs systems arose from numerical analysis (see, for example, [24]), thanks to their feature of providing a convenient unified framework for numerical solutions to partial differential equations of different type (a comprehensive overview from this perspective can be found in [25]). In turn, this prompted further theoretical investigations of the properties of Friedrichs systems [15]. In [18] an abstract Hilbert space approach was introduced, with an intrinsic formulation of the boundary conditions, followed by an extensive study in the course of the last ten years. Three different, albeit equivalent, abstract representations of boundary conditions were motivated by their classical analogues (as proposed by Friedrichs [20], Friedrichs and Lax [21], or Phillips and Sarason [29]). Besides the well-posedness result (Theorem 5 below), important recent results include the equivalence of different representations of boundary conditions [2], its relationship with the classical theory [3–6], applications to various (initial-)boundary value problems of elliptic, hyperbolic, and parabolic type [7,12,14,15,28], and the development of different numerical schemes [10,11,13,15–17].

In the above-mentioned literature, for each concrete example the goal was to find at least one feasible set of boundary conditions such that the corresponding problem is well-posed. However, to our best knowledge, that was done without any general result ensuring the existence of such boundary conditions. This point constitutes the motivation for our present analysis.

In this work we are concerned with the following general questions, for a given classical Friedrichs system $T\mathsf{u} = \mathsf{f}$:

(1) the existence of boundary conditions for which the problem is well-posed;
(2) the possibility that there exist infinitely many different boundary conditions of well-posedness;
(3) an efficient classification of such boundary conditions for which the problem is well-posed.

By reformulating these questions into the abstract intrinsic setting of [18,2], which is taken as the starting point in this paper, we provide complete answers to (1) and (2) (our first main result: Theorem 13), and also a convenient approach to (3), producing in particular a complete and operational classification of the most relevant sub-class of Friedrichs operators (our second main result: Theorem 18).

In order to do so, in Section 2 we move from the abstract intrinsic construction of [18,2] and we re-write it in a purely Hilbert space operator-theoretic language. In Section 3 we use the Kreĭn space theory to state rigorously and to prove our main result on existence and classification of Friedrichs operators. In Sections 4 and 5 we make a fruitful connection between abstract Friedrichs operators and the operator extension theory arising from the so-called universal classification of extensions [22,23], which allows us to classify all relevant isomorphic realisations of Friedrichs operators. Finally, in Section 6 we present concrete applications of our theoretical results to two instructive examples of first order ordinary differential operators.

Notation. Most of our notation is standard, let us only emphasise the following. By L we denote a complex Hilbert space with scalar product $\langle\cdot\mid\cdot\rangle_L$, which we take to be linear in the first and anti-linear in the second entry. The topological (anti)dual L' will be identified with L by means of the usual duality (the Riesz representation theorem). For any Banach space W by ${}_{W'}\langle\cdot,\cdot\rangle_W$ we denote the corresponding dual product between W and its (anti)dual W'. For a densely defined linear operator $A : L \to L$ we denote by $\operatorname{dom} A$, $\overline{A}$, A^* its *domain*, *closure* (if it exists), and *adjoint*, respectively. For $V \subseteq L$, the *restriction* of A to V is denoted by $A|_V$. By $\langle\cdot\mid\cdot\rangle_A := \langle\cdot\mid\cdot\rangle_L + \langle A\cdot\mid A\cdot\rangle_L$ we denote the *graph scalar product*, while the corresponding norm $\|\cdot\|_A := \sqrt{\langle\cdot\mid\cdot\rangle_A}$ is called the *graph norm*. If $A = A^*$, then A is said to be *self-adjoint*, while the infimum of its spectrum is called the *bottom*. The *identity* operator is denoted by $\mathbb{1}$. For a *direct* sum between two vector spaces we use the symbol $\dotplus$. We write $\ominus$ for the *orthogonal difference* in order to express in which Hilbert space the orthogonal complement is taken.

2. Abstract Friedrichs systems and their Hilbert space formulation

The process of inference of those essential features that qualify abstract Friedrichs operators, following the ideas of concrete realisation of pointwise conditions for a first-order system of partial differential equations introduced in [29], has been carried out in [18,2] for real vector spaces, while the required differences for complex vector space have been supplemented more recently in [4]. Let us first briefly review this procedure, and then present a more natural purely Hilbert space formalism for the abstract formulation.

Definition 1. A (densely defined) linear operator T on a complex Hilbert space L is called an *abstract Friedrichs operator* if it admits another (densely defined) linear operator $\widetilde{T}$ on L with the following properties:

(T1) T and $\widetilde{T}$ have a common domain $\mathcal{D}$, which is dense in L, satisfying

$$\langle T\phi \mid \psi \rangle_L = \langle \phi \mid \widetilde{T}\psi \rangle_L , \qquad \phi, \psi \in \mathcal{D};$$

(T2) there is a constant $c > 0$ for which

$$\|(T + \widetilde{T})\phi\|_L \leqslant c\|\phi\|_L , \qquad \phi \in \mathcal{D};$$

(T3) there exists a constant $\mu_0 > 0$ such that

$$\langle (T + \widetilde{T})\phi \mid \phi \rangle_L \geqslant 2\mu_0\|\phi\|_L^2 , \qquad \phi \in \mathcal{D}.$$

The pair $(T, \widetilde{T})$ is usually referred to as a *joint pair of abstract Friedrichs operators* (the definition is indeed symmetric in T and $\widetilde{T}$).

Abstract Friedrichs operators are modelled on classical concrete examples of boundary value problems for first order systems of partial differential equations, as mentioned in the Introduction, and in that setting the main problem is the well-posedness of the boundary value problem, namely, the existence and uniqueness of the solution, as well as its continuous dependence on given data.

The *abstract boundary value problem* for a Friedrichs operator is determined by an extended domain $V \supseteq \mathcal{D}$ on which the problem $Tu = f$ has a unique solution $u \in V$, for each given $f \in L$.

More precisely, let us consider the abstract construction introduced in [18], which starts with the given joint pair of abstract Friedrichs operators $(T, \widetilde{T})$:

- Pair $(T_2, \widetilde{T}_2)$: one equips $\mathcal{D}$ with the usual graph-norm topology associated to T, induced by the scalar product

$$\langle \phi \mid \psi \rangle_T := \langle \phi \mid \psi \rangle_L + \langle T\phi \mid T\psi \rangle_L , \qquad \phi, \psi \in \mathcal{D};$$

  the completion of $\mathcal{D}$ in the graph norm $\|\cdot\|_T := \sqrt{\langle \cdot \mid \cdot \rangle_T}$ (which is equivalent to $\|\cdot\|_{\widetilde{T}}$ due to (T2)) yields a complete space, which we denote by W_0: as a consequence of condition (T1), $W_0 \subseteq L$ and both T and $\widetilde{T}$ extend uniquely to bounded linear operators defined on the whole W_0, and with values in L; we shall denote this new pair of abstract Friedrichs operators by $(T_2, \widetilde{T}_2)$.
- Pair $(T_3, \widetilde{T}_3)$: with respect to the Gel'fand triplet

$$W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0'$$

  (the first embedding is continuous and dense, where W_0 is equipped with graph-norm topology), let $T_3 := \widetilde{T}_2' : L \to W_0'$ and $\widetilde{T}_3 := T_2' : L \to W_0'$ be the Banach adjoints (respectively) to $\widetilde{T}_2$ and T_2.

- Pair $(T_4, \widetilde{T}_4)$: the following two expressions determine the same space

$$W \;:=\; \{u \in L \,|\, T_3 u \in L\} \;=\; \{u \in L \,|\, \widetilde{T}_3 u \in L\} \;\subseteq\; L\,,$$

  called the *graph space*, which in turn identifies the restrictions $T_4 := T_3|_W$ and $\widetilde{T}_4 := \widetilde{T}_3|_W$.

This produces the *maximal* pair of abstract Friedrichs operators $(T_4, \widetilde{T}_4)$, i.e., the maps $T_4, \widetilde{T}_4 : W \to L$ are associated to the initial pair $(T, \widetilde{T})$. In particular, $T \subseteq T_4$ and $\widetilde{T} \subseteq \widetilde{T}_4$ (see Theorem 7(iii) below). It is worth remarking that so far only the properties (T1) and (T2) from the definition above have been exploited.

The well-posedness problem for a given abstract Friedrichs operator T can now be formulated as follows:

to find restrictions of T_4 to a suitable subspace V, with $W_0 \subseteq V \subseteq W$, such that $T_4|_V : V \to L$ is an isomorphism, namely a continuous bijection, when V is equipped with the graph-norm topology.

In fact, the continuity (in the graph norm) holds for any restriction of T_4 to *a closed subspace* of W, so the actual core of the problem is bijectivity. It also turns out that the above question for T_4 is intimately related with the same question for $\widetilde{T}_4$.

In [18,4] sufficient conditions on V were produced for the restriction $T_4|_V : V \to L$ to be an isomorphism. They are formulated in terms of the *boundary operator* associated with the pair $(T_4, \widetilde{T}_4)$, namely the continuous map

$$\begin{aligned} D : (W, \langle \cdot \mid \cdot \rangle_{T_4}) \;&\to\; (W, \langle \cdot \mid \cdot \rangle_{T_4})' \\ {}_{W'}\langle\, Du, v \,\rangle_W \;&:=\; \langle\, T_4 u \mid v \,\rangle_L - \langle\, u \mid \widetilde{T}_4 v \,\rangle_L\,, \qquad u, v \in W\,. \end{aligned} \tag{1}$$

It can be easily seen [4, Lemma 1] that D is symmetric, i.e.,

$${}_{W'}\langle\, Du, v \,\rangle_W = \overline{{}_{W'}\langle\, Dv, u \,\rangle_W}\,, \qquad u, v \in W\,, \tag{2}$$

and that

$$\ker D \;=\; W_0\,. \tag{3}$$

In particular, for any $u \in W$ we have

$$\langle\, T_4 u \mid u \,\rangle_L - \langle\, u \mid \widetilde{T}_4 u \,\rangle_L \;\in\; \mathbb{R}\,. \tag{4}$$

Definition 2. For a given joint pair of abstract Friedrichs operators $(T, \widetilde{T})$, a pair $(V, \widetilde{V})$ of linear subspaces of W is said to *allow the* (V)-*boundary conditions* relative to $(T, \widetilde{T})$ when the following properties are satisfied:

(V1) the boundary operator has opposite sign on V and on $\widetilde{V}$, in the sense that

$$\begin{aligned} (\forall\, u \in V) \qquad & {}_{W'}\langle\, Du, u \,\rangle_W \;\geqslant\; 0\,, \\ (\forall\, v \in \widetilde{V}) \qquad & {}_{W'}\langle\, Dv, v \,\rangle_W \;\leqslant\; 0\,; \end{aligned}$$

(V2) the image, via D, of either space has as annihilator the other space, namely,

$$V = D(\widetilde{V})^0 \qquad \text{and} \qquad \widetilde{V} = D(V)^0\,,$$

where ^{0} stands for the annihilator.

Remark 3. The above conditions (V1) and (V2) can be naturally reformulated in terms of the corresponding indefinite inner product [2].

Remark 4. It is worth observing that condition (V2) has two relevant consequences. First, since taking the annihilator produces a closed subspace of W, both V and $\widetilde{V}$ are closed in W when (V2) holds. Moreover, even though V and $\widetilde{V}$ are not required to be supersets of W_0, one deduces from (V2) that

$$\ker D = W_0 \subseteq V \cap \widetilde{V}\,.$$

Let us also remark, as was pointed out in [4, Remark 2], that if V is a closed space (in the graph-norm) such that $W_0 \subseteq V \subseteq W$, then $V = D(D(V)^0)^0$ (see Lemma 11), thus implying that the sole condition $\widetilde{V} = D(V)^0$ is enough to ensure (V2).

Theorem 5 *(Sufficient criterion for well-posedness of a Friedrichs system, [18,4]). Let $(T, \widetilde{T})$ be a joint pair of abstract Friedrichs operators on the Hilbert space L. If $(V, \widetilde{V})$ is a pair of linear subspaces of W allowing the* (V)*-boundary conditions, then the restrictions $T_4|_V : V \to L$ and $\widetilde{T}_4|_{\widetilde{V}} : \widetilde{V} \to L$ are isomorphisms, where V and $\widetilde{V}$ are both equipped with the graph norm.*

In short, if the pair of operators $(T, \widetilde{T})$ satisfies both conditions (T1)–(T3) and (V1)–(V2), then $T_4|_V$ and $\widetilde{T}_4|_{\widetilde{V}}$ are isomorphisms.

Remark 6. We stress that *isomorphism* refers to the map between the Hilbert space $(V, \langle \cdot \mid \cdot \rangle_{T_4})$, namely V equipped with the graph norm topology, and the Hilbert space L equipped with its usual norm topology. In fact, as we shall often do in the following, this is tantamount to saying that the operator is closed and acts bijectively between V and L. Indeed, a densely defined and closed operator $T : X \to Y$ between Banach spaces X and Y, which is also a bijection from $\operatorname{dom} T$ to Y, is a $(\operatorname{dom} T, \|\cdot\|_T) \to Y$ continuous map whose inverse is a $Y \to (\operatorname{dom} T, \|\cdot\|_T)$ continuous map, and as such T is an isomorphism between $(\operatorname{dom} T, \|\cdot\|_T)$ and Y. For future purposes it is convenient to highlight also the next elementary property, that follows from the previous one: a densely defined and closed operator $T : X \to Y$ between Banach spaces X and Y, which is also a bijection from $\operatorname{dom} T$ to Y, has an inverse T^{-1} which is necessarily everywhere defined and bounded (with respect to the norm topologies of Y and X). Thus, in the present case, Theorem 5 produces bijections $T_4|_V : V \to L$ and $\widetilde{T}_4|_{\widetilde{V}} : \widetilde{V} \to L$ whose inverses are everywhere defined in L and bounded on L.

As mentioned already, the above procedure was motivated by concrete realisations of Friedrichs systems as boundary value problems for partial differential operators with suitable boundary conditions; as such, the overall formulation is naturally expressed by duality arguments.

On the other hand, further insight in abstract Friedrichs operators on Hilbert spaces is to be expected when the previous construction is rephrased solely in the Hilbert space operator-theoretic language. As we shall see in the sequel, this will allow us to formulate and successfully address additional questions related to the well-posedness of Friedrichs systems.

With this perspective, let us first revisit the abstract construction of Friedrichs operators, characterising the intermediate pairs of operators $(T_j, \widetilde{T}_j)$.

Theorem 7 *(Hilbert space construction of Friedrichs operators). Let* $(T, \widetilde{T})$ *be a joint pair of abstract Friedrichs operators on Hilbert space* L*, and let* $(T_2, \widetilde{T}_2)$*,* $(T_3, \widetilde{T}_3)$ *and* $(T_4, \widetilde{T}_4)$ *be the pairs of operators obtained from* $(T, \widetilde{T})$ *as in the preceding construction.*

(i) *The operators* T *and* $\widetilde{T}$ *are closable and their closures have the common domain* W_0*. Moreover, the pair* $(T_2, \widetilde{T}_2)$ *satisfies*

$$T_2 = \overline{T}\,, \qquad \widetilde{T}_2 = \overline{\widetilde{T}}\,.$$

Furthermore, the pair $(\overline{T}, \overline{\widetilde{T}})$ *satisfies conditions* (T1)–(T3) *on* W_0 *and the corresponding graph norms* $\|\cdot\|_{\overline{T}}$ *and* $\|\cdot\|_{\overline{\widetilde{T}}}$ *are equivalent Banach norms on* W_0*.*

(ii) *The pair* $(T_3, \widetilde{T}_3)$ *satisfies*

$$\begin{aligned} T_3|_{W_0} &= T_2 = \overline{T}\,, \\ \widetilde{T}_3|_{W_0} &= \widetilde{T}_2 = \overline{\widetilde{T}}\,, \end{aligned}$$

whereas

$$T_3 + \widetilde{T}_3 = \overline{T + \widetilde{T}}$$

is a (everywhere defined) bounded operator in L*. The graph norms* $\|\cdot\|_{T_3}$ *and* $\|\cdot\|_{\widetilde{T}_3}$ *are equivalent on* W*.*

(iii) *The pair* $(T_4, \widetilde{T}_4)$ *satisfies*

$$\begin{aligned} T &\subseteq T_4 = \widetilde{T}^*\,, \\ \widetilde{T} &\subseteq \widetilde{T}_4 = T^*\,. \end{aligned}$$

Thus, in particular, T^* *and* $\widetilde{T}^*$ *are defined on the common domain* W*. Moreover,* $\overline{T + \widetilde{T}}$ *is a bounded self-adjoint operator in* L *with strictly positive bottom.*

From Theorem 7 above, for a joint pair of abstract Friedrichs operators $(T, \widetilde{T})$ one has $T \subseteq \widetilde{T}^*$ and $\widetilde{T} \subseteq T^*$, and $\overline{T + \widetilde{T}}$ is an everywhere defined, bounded self-adjoint operator in L with strictly positive bottom. Since the converse of this statement is trivially satisfied, we have the following *characterisation* of the original (T1)–(T3) conditions, which for its relevance we cast in separate theorem.

Theorem 8 *(Hilbert space formulation for abstract Friedrichs operators).*

(i) *For a pair of operators* $(T, \widetilde{T})$ *on the Hilbert space* L*,*

$$\textit{condition } (\mathrm{T1}) \quad \Longleftrightarrow \quad \begin{cases} T \subseteq \widetilde{T}^* \\ \widetilde{T} \subseteq T^* . \end{cases}$$

(ii) *Therefore, a pair of operators* $(T, \widetilde{T})$ *on the Hilbert space* L *is a joint pair of abstract Friedrichs operators on* L *if and only if* $T \subseteq \widetilde{T}^*$*,* $\widetilde{T} \subseteq T^*$*, and* $\overline{T + \widetilde{T}}$ *is an everywhere defined, bounded, self-adjoint operator in* L *with strictly positive bottom.*

Theorem 7 re-does the construction of Friedrichs operators and Theorem 8 characterises Friedrichs operators *in a purely Hilbert space language*. The next result does the same concerning condition (V2).

Theorem 9. *Let* $(T, \widetilde{T})$ *be a pair of operators on the Hilbert space* L *satisfying conditions* (T1)–(T2)*, and let* $(V, \widetilde{V})$ *be a pair of subspaces of* L*. Then*

$$\textit{condition } (\mathrm{V2}) \quad \Longleftrightarrow \quad \begin{cases} W_0 \subseteq V \subseteq W, \;\; W_0 \subseteq \widetilde{V} \subseteq W \\ V \textit{ and } \widetilde{V} \textit{ closed in } W \\ (\widetilde{T}^*|_V)^* = T^*|_{\widetilde{V}} \\ (T^*|_{\widetilde{V}})^* = \widetilde{T}^*|_V . \end{cases}$$

That is, condition (V2) *is equivalent to the fact that the spaces* V *and* $\widetilde{V}$ *are included between* W_0 *and* W*, and are closed in* W*, with the restrictions* $\widetilde{T}^*|_V$ *and* $T^*|_{\widetilde{V}}$ *being mutually adjoint.*

Remark 10. By Theorem 7(i), if $(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators, then so is $(\overline{T}, \overline{\widetilde{T}})$.

With Theorems 7, 8 and 9 above we have thus completed the programme of translating the notions of abstract Friedrichs operators and the well-posedness of boundary value problems from the form in which they are discussed so far in the literature [2,18,4] into an equivalent, purely Hilbert space operator-theoretic form.

Let us present the proofs of the above results in the remaining part of this Section.

Proof of Theorem 7. Part (i). From (T1) we can deduce that $\widetilde{T} \subseteq T^*$ and $T \subseteq \widetilde{T}^*$. In particular, both adjoints are densely defined, and T and $\widetilde{T}$ are closable. It is easy to check that by (T2) the norms $\|\cdot\|_T$ and $\|\cdot\|_{\widetilde{T}}$ are equivalent on $\mathcal{D}$, which gives rise to the same completion $W_0 = \operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}}$, having used that the domain of the closure is the completion of operator domain with respect to the graph norm. Identities $T_2 = \overline{T}$ and $\widetilde{T}_2 = \overline{\widetilde{T}}$ also follow immediately. For any $\phi \in W_0$ there is a sequence (ϕ_n) in $\mathcal{D}$ such that $\phi_n \to \phi$, $\overline{T}\phi_n \to \overline{T}\phi$ and $\overline{\widetilde{T}}\phi_n \to \widetilde{T}\phi$ in L. Thus, after passing to the limit, it is easy to verify that (T1)–(T3) extend to W_0, where the operators are replaced by their closures.

Part (ii). For any $u \in L$, the action of T_3 is given by

$$(\forall \phi \in W_0) \qquad {}_{W_0'}\langle T_3 u, \phi \rangle_{W_0} := \langle u \mid \widetilde{T}_2 \phi \rangle_L ,$$

and analogously for $\widetilde{T}_3$

$$(\forall \phi \in W_0) \qquad {}_{W_0'}\langle \widetilde{T}_3 u, \phi \rangle_{W_0} := \langle u \mid T_2 \phi \rangle_L ,$$

thus obtaining continuous operators $T_3, \widetilde{T}_3 : (L, \langle \cdot \mid \cdot \rangle_L) \longrightarrow (W_0, \langle \cdot \mid \cdot \rangle_{T_2})'$. Let now $\phi \in W_0$: then, owing to (i), for any $\psi \in W_0$ one has

$${}_{W_0'}\langle T_3 \phi, \psi \rangle_{W_0} = \langle \phi \mid \widetilde{T}_2 \psi \rangle_L = \langle T_2 \phi \mid \psi \rangle_L = {}_{W_0'}\langle T_2 \phi, \psi \rangle_{W_0} ,$$

having made use of the embedding $L \hookrightarrow W_0'$. Therefore, $T_3 \phi$ and $T_2 \phi$ are the same functional in W_0' and, by density of W_0 in L, $T_3|_{W_0} = T_2 = \overline{T}$. In the same manner one proves the statement for $\widetilde{T}_3$. In particular, the restriction to $\mathcal{D}$ yields $T_3|_{\mathcal{D}} = T$ and $\widetilde{T}_3|_{\mathcal{D}} = \widetilde{T}$, whence the fact that $(T_3 + \widetilde{T}_3)|_{\mathcal{D}} = T + \widetilde{T}$. Last, it is clear from (T2) that $\overline{T + \widetilde{T}}$ is bounded and everywhere defined, while it coincides with $T_3 + \widetilde{T}_3$ on the dense subspace $\mathcal{D}$. Both such maps are continuous from $(L, \langle \cdot \mid \cdot \rangle_L)$ to $(W_0, \langle \cdot \mid \cdot \rangle_{T_2})'$, because L is continuously embedded in W_0', thus $T_3 + \widetilde{T}_3 = \overline{T + \widetilde{T}}$ on L. In particular, the graph norms $\| \cdot \|_{T_3}$ and $\| \cdot \|_{\widetilde{T}_3}$ are equivalent on W.

Part (iii). From part (ii) and $W_0 \subseteq W$ we have $T \subseteq \overline{T} = T_3|_{W_0} \subseteq T_4$, and similarly $\widetilde{T} \subseteq \widetilde{T}_4$. By the definition of $\widetilde{T}_3$ and W, for any $u \in W$ and $\phi \in \mathcal{D}$ we have

$$\langle T_4 u \mid \phi \rangle_L = {}_{W_0'}\langle T_3 u, \phi \rangle_{W_0} = \langle u \mid \widetilde{T}_2 \phi \rangle_L = \langle u \mid \widetilde{T} \phi \rangle_L ,$$

thus implying that $\widetilde{T}_4 \subseteq T^*$ (and analogously $T_4 \subseteq \widetilde{T}^*$). Conversely, for any $u \in \operatorname{dom} T^*$ we have $\widetilde{T}_3 u = T^* u \in L$, because for any $\phi \in \mathcal{D}$

$${}_{W_0'}\langle \widetilde{T}_3 u, \phi \rangle_{W_0} = \langle u \mid T_2 \phi \rangle_L = \langle u \mid T \phi \rangle_L = \langle T^* u \mid \phi \rangle_L .$$

Thus, $u \in W$ and we have obtained that $\operatorname{dom} T^* \subseteq W$. The overall conclusion is that $\widetilde{T}_4 = T^*$, and analogously $T_4 = \widetilde{T}^*$. Moreover, $T + \widetilde{T} \subseteq T_4 + \widetilde{T}_4 = T^* + \widetilde{T}^*$, while by the density of $\operatorname{dom}(T^* + \widetilde{T}^*) = W$ we have $T^* + \widetilde{T}^* \subseteq (T + \widetilde{T})^*$. Therefore, $\overline{T + \widetilde{T}}$ is an everywhere defined, bounded, and symmetric operator, hence it is self-adjoint. By the boundedness and (T3), $\overline{T + \widetilde{T}}$ has strictly positive bottom, thus concluding the proof. □

The proof of Theorem 9 is in turn based on the following result.

Lemma 11. *Let $(T, \widetilde{T})$ be a pair operators on the Hilbert space L satisfying conditions* (T1)–(T2)*, and let $D : W \longrightarrow W'$ be the associated boundary operator defined by* (1)*. Then for each pair $(V, \widetilde{V})$ of spaces between W_0 and W one has*

$$\big(\widetilde{T}^*|_V\big)^* = T^*|_{D(V)^0} \qquad \textit{and} \qquad \big(T^*|_{\widetilde{V}}\big)^* = \widetilde{T}^*|_{D(\widetilde{V})^0} . \tag{5}$$

In particular, for any space V such that $W_0 \subseteq V \subseteq W$ and V is closed in W, one has $V = D(D(V)^0)^0$.

Proof. First we see that $(\widetilde{T}^*|_V)^*$ is a restriction of T^*; indeed, by Theorem 7(iii) we have $T \subseteq \widetilde{T}^*|_V$, implying $(\widetilde{T}^*|_V)^* \subseteq T^*$. Thus, we only need to show that $\mathrm{dom}(\widetilde{T}^*|_V)^* = D(V)^0$, and we shall do it by proving the two opposite inclusions that lead to such identity. For $v \in D(V)^0$ and $u \in V$ we have ${}_{W'}\langle Du, v\rangle_W = 0$, which together with the very definition of D and Theorem 7(iii) yields

$$\begin{aligned}\langle (\widetilde{T}^*|_V)u \mid v\rangle_L &= \langle \widetilde{T}^* u \mid v\rangle_L \\ &= {}_{W'}\langle Du, v\rangle_W + \langle u \mid T^* v\rangle_L = \langle u \mid T^* v\rangle_L\,,\end{aligned}$$

which implies $D(V)^0 \subseteq \mathrm{dom}(\widetilde{T}^*|_V)^*$. On the other hand, for $v \in \mathrm{dom}(\widetilde{T}^*|_V)^*$ and $u \in V$ we have

$$\begin{aligned}\langle (\widetilde{T}^*|_V)u \mid v\rangle_L &= \langle u \mid (\widetilde{T}^*|_V)^* v\rangle_L \\ &= \langle u \mid T^* v\rangle_L = \langle (\widetilde{T}^*|_V)u \mid v\rangle_L - {}_{W'}\langle Du, v\rangle_W\,,\end{aligned}$$

thus obtaining ${}_{W'}\langle Du, v\rangle_W = 0$. By the arbitrariness of $u \in V$ we get $v \in D(V)^0$, and hence $\mathrm{dom}(\widetilde{T}^*|_V)^* \subseteq D(V)^0$. This completes the proof of $(\widetilde{T}^*|_V)^* = T^*|_{D(V)^0}$, and the same can be argued for $(T^*|_{\widetilde{V}})^* = \widetilde{T}^*|_{D(\widetilde{V})^0}$. If in addition V is closed in W, then $\widetilde{T}^*|_V$ is closed, whence

$$\widetilde{T}^*|_V = \big(\widetilde{T}^*|_V\big)^{**} = \big(T^*|_{D(V)^0}\big)^*\,.$$

Formula (5) is applicable to $D(V)^0$ as well, since $W_0 \subseteq D(V)^0 \subseteq W$ (Remark 4), hence

$$\big(T^*|_{D(V)^0}\big)^* = \widetilde{T}^*|_{D(D(V)^0)^0}\,,$$

and combining the last two identities we finally obtain $V = D(D(V)^0)^0$. □

Proof of Theorem 9. Let us first assume that the pair of spaces $(V, \widetilde{V})$ satisfies condition (V2). Then, in particular (Remark 4), both V and $\widetilde{V}$ are included between W_0 and W, and are closed in W. Moreover, owing to the identities $V = D(\widetilde{V})^0$ and $\widetilde{V} = D(V)^0$, one deduces at once from Lemma 11 that $\widetilde{T}^*|_V$ and $T^*|_{\widetilde{V}}$ are mutually adjoint. On the other hand, let $(V, \widetilde{V})$ be a pair of spaces between W_0 and W, closed in W, such that $\widetilde{T}^*|_V$ and $T^*|_{\widetilde{V}}$ are mutually adjoint. Then by Lemma 11 and the uniqueness of adjoints, the pair of subspaces $(V, \widetilde{V})$ satisfies (V2). □

3. Main results: existence, infinity, and classification of Friedrichs operators with signed boundary map

Based on the Hilbert space formulation discussed in the previous Section, we shall now focus our attention on pairs of *closed* operators (A_0, A_0') on L satisfying

$$A_0 \subseteq (A_0')^* =: A_1 \qquad \text{and} \qquad A_0' \subseteq (A_0)^* =: A_1'\,, \tag{6}$$

and such that $A_0 + A_0'$ is bounded on L and extends to an everywhere defined, bounded, self-adjoint operator in L with strictly positive bottom. In view of Theorem 8, we shall refer to any such (A_0, A_0') as a *joint pair of closed abstract Friedrichs operators*. Observe that by parts (i) and (iii) of Theorem 7 this definition implies that

$$\operatorname{dom} A_0 = \operatorname{dom} A_0' =: W_0 \qquad \text{and} \qquad \operatorname{dom} A_1 = \operatorname{dom} A_1' =: W \,. \tag{7}$$

In particular, we are interested in restrictions $A_1|_V$ and $A_1'|_{\widetilde{V}}$ onto suitable subspaces V and $\widetilde{V}$ of L which satisfy conditions (V1)–(V2). As may be argued by using Theorem 9, this is precisely the class of restrictions $A_1|_V$ and $A_1'|_{\widetilde{V}}$ onto spaces V and $\widetilde{V}$ such that

$$W_0 \subseteq V \subseteq W \qquad \text{and} \qquad W_0 \subseteq \widetilde{V} \subseteq W \,, \tag{8}$$

and satisfying the property that $A_1|_V$ and $A_1'|_{\widetilde{V}}$ are mutually adjoint (thus, in particular, $A_1|_V$ and $A_1'|_{\widetilde{V}}$ are closed operators) and

$$\begin{aligned} (\forall u \in V) \qquad {}_{W'}\langle Du, u\rangle_W &= \langle A_1 u \mid u\rangle_L - \langle u \mid A_1' u\rangle_L \geqslant 0\,, \\ (\forall v \in \widetilde{V}) \qquad {}_{W'}\langle Dv, v\rangle_W &= \langle A_1 v \mid v\rangle_L - \langle v \mid A_1' v\rangle_L \leqslant 0\,. \end{aligned} \tag{9}$$

We shall refer to any such pair $(A_1|_V, A_1'|_{\widetilde{V}})$ as an *adjoint pair of bijective realisations with signed boundary map* relative to the given joint pair of closed abstract Friedrichs operators (A_0, A_0').

Remark 12. If $V = \widetilde{V}$, i.e., if the domains of the two operators of the considered adjoint pair of bijective realisations with signed boundary map are equal, then in (9) we have equalities.

As commented already in Remark 10, one may think of (A_0, A_0') as a pair of closed operators

$$A_0 := \overline{T} \qquad \text{and} \qquad A_0' := \overline{\widetilde{T}}\,, \tag{10}$$

where $(T, \widetilde{T})$ is a joint pair of abstract Friedrichs operators. Moreover, for their adjoints we have

$$A_1 := (A_0')^* = \widetilde{T}^* \qquad \text{and} \qquad A_1' := (A_0)^* = T^*\,.$$

It is immediate that there is a one-to-one correspondence between all pairs of isomorphisms induced by Theorem 5 with respect to $(T, \widetilde{T})$, and all adjoint pairs of bijective realisations with signed boundary map relative to (A_0, A_0'), i.e. $\big(\overline{T}, \overline{\widetilde{T}}\big)$.

As observed in Remark 6, since $A_1|_V$ is closed and bijective onto L, then $(A_1|_V)^{-1}$ is necessarily everywhere defined and bounded, so we may also speak of $A_1|_V$ as of an *isomorphic realisation of* A_0 *with signed boundary map*.

It is also worth stressing that the fact that a closed operator S satisfies $A_0 \subseteq S \subseteq A_1$ is *equivalent* to $A_0' \subseteq S^* \subseteq A_1'$.

The interest towards such pairs $(A_1|_V, A_1'|_{\widetilde{V}})$ is two-fold: first, when (V1)–(V2) hold, $A_1|_V$ and $A_1'|_{\widetilde{V}}$ are bijections onto L (Theorem 5), thus providing a sufficient criterion of well-posedness of the abstract Friedrichs system; moreover, (V1)–(V2) encode the most relevant class of boundary conditions, as it may be seen from a large variety of concrete examples of boundary value problems on which such conditions are modelled (we surveyed the related literature in the Introduction).

Our main goal is to address the so far unanswered problem, given a joint pair of closed abstract Friedrichs operators (A_0, A_0'), of whether bijective realisations of A_0 with signed boundary map *do* exist, with which multiplicity, and possibly according to which general classification. Whereas in this respect one can infer some information from many examples of Friedrichs operators, a general discussion is surely still missing.

In fact, the essentially complete answer is contained in our main result, as follows.

Theorem 13. *Let* (A_0, A_0') *be a joint pair of closed abstract Friedrichs operators on the Hilbert space* L *and denote by* (A_1, A_1') *the corresponding pair of adjoint operators* (6).

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0')*. Moreover, there is an adjoint pair* $(A_\mathrm{r}, A_\mathrm{r}^*)$ *of bijective realisations with signed boundary map relative to* (A_0, A_0') *such that*

$$W_0 + \ker A_1' \subseteq \operatorname{dom} A_\mathrm{r} \qquad \text{and} \qquad W_0 + \ker A_1 \subseteq \operatorname{dom} A_\mathrm{r}^* .$$

(ii) *If both* $\ker A_1 \neq \{0\}$ *and* $\ker A_1' \neq \{0\}$*, then the pair* (A_0, A_0') *admits uncountably many adjoint pairs of bijective realisations with signed boundary map. On the other hand, if either* $\ker A_1 = \{0\}$ *or* $\ker A_1' = \{0\}$*, then there is exactly one adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0')*. Such a pair is precisely* (A_1, A_0') *when* $\ker A_1 = \{0\}$*, and* (A_0, A_1') *when* $\ker A_1' = \{0\}$.

(iii) *An explicit (constructive) classification of all adjoint pairs of bijective realisations with signed boundary map relative to* (A_0, A_0') *is possible, the precise formulation of which we defer to Theorem 18 below.*

In this Section we shall prove parts (i) and (ii). The proof of part (iii) is the object of Sections 4 and 5.

In preparation for the proof, we recall a few useful facts from Kreĭn space theory (we refer to [9] for a standard reference). As recognised first in [2, Lemma 8], the quotient space $\widehat{W} := W/W_0$ equipped with the indefinite inner product

$$[\,\hat{u} \mid \hat{v}\,]_{\widehat{W}} \;:=\; {}_{W'}\langle\, Du, v\,\rangle_W \;=\; \langle\, A_1 u \mid v\,\rangle_L - \langle\, u \mid A_1' v\,\rangle_L \,,$$

where $\hat{u} \equiv u + W_0$ is the element of $\widehat{W}$ with representative $u \in W$, is a Kreĭn space. The topology on $\widehat{W}$ induced by the graph norm $\|\cdot\|_{A_1}$ on W (which is equivalent to $\|\cdot\|_{A_1'}$ by Theorem 7(ii)) is *admissible*. Therefore, there exist subspaces $X_+, X_- \subseteq \widehat{W}$ such that

$$\widehat{W} \;=\; X_+ \,[\dot{+}]_{\widehat{W}}\, X_- \,,$$

and both $(X_+, [\cdot \mid \cdot]_{\widehat{W}})$ and $(X_-, -[\cdot \mid \cdot]_{\widehat{W}})$ are Hilbert spaces. Here $[\dot{+}]_{\widehat{W}}$ denotes the $[\cdot \mid \cdot]_{\widehat{W}}$-orthogonal direct sum, i.e. $X_+ [\dot{+}]_{\widehat{W}} X_-$ if and only if $X_+ \cap X_- = \{0\}$, and for any $\hat{u}_+ \in X_+$ and $\hat{u}_- \in X_-$ one has $[\,\hat{u}_+ \mid \hat{u}_-\,]_{\widehat{W}} = 0$. The decomposition above is the so-called *canonical* (or *fundamental*) decomposition of the corresponding Kreĭn space. One says that the Kreĭn space $\widehat{W}$ is *non-trivial* if both $X_+ \neq \{0\}$ and $X_- \neq \{0\}$.

The main idea behind the proof of parts (i) and (ii) of Theorem 13 is to show that each canonical decomposition of the Kreĭn space $\widehat{W}$ identifies a pair of subspaces $(V, \widetilde{V})$ satisfying (V1)–(V2) (Proposition 14). This immediately ensures the existence of an adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0'). The multiplicity of such pairs follows from the well known fact that any (non-trivial) Kreĭn space admits infinitely many canonical decompositions. For its relevance in the proof of Theorem 13, we present a proof of this property in Lemma 15. As the last preparatory step, in Lemma 16 we provide a sufficient condition for $\widehat{W}$ to be a non-trivial Kreĭn space in terms of (A_0, A_0') (or, more precisely, in terms of their adjoints (A_1, A_1')).

Proposition 14. *Let $\widehat{W} = X_+[\dot{+}]_{\widehat{W}} X_-$ be a canonical decomposition of the Kreĭn space $\widehat{W} = W/W_0$. Then the spaces*

$$\begin{aligned} V &:= \{u \in W : \hat{u} \in X_+\} \\ \widetilde{V} &:= \{u \in W : \hat{u} \in X_-\} \end{aligned}$$

satisfy (V1)–(V2). *In particular, $W = V + \widetilde{V}$.*

Proof. (V1) is satisfied because $u \in V$ implies $\hat{u} \in X_+$, so

$$\langle A_1 u \mid u \rangle_L - \langle u \mid A_1' u \rangle_L = [\hat{u} \mid \hat{u}]_{\widehat{W}} \geqslant 0,$$

and similarly for $v \in \widetilde{V}$. Moreover, as $u \in W_0$ implies $\hat{u} = 0$, we have that $W_0 \subseteq V$ and by standard results on quotient spaces (see, e.g., [2, Lemma 7]) this implies that V is closed in W. Therefore, in order to check (V2), by Lemma 11 it is sufficient to show that $\widetilde{V} = D(V)^0$. To this aim, let $v \in \widetilde{V}$, and hence $\hat{v} \in X_-$: then, for any $u \in V$ we have ${}_{W'}\langle Du, v \rangle_W = [\hat{u} \mid \hat{v}]_{\widehat{W}} = 0$, because $\hat{u} \in X_+$; this implies that $\widetilde{V} \subseteq D(V)^0$. Conversely, if $v \in D(V)^0$, then $\hat{v} \in \widehat{W}$ and there exist unique $\hat{v}_+ \in X_+$ and $\hat{v}_- \in X_-$ such that $\hat{v} = \hat{v}_+ + \hat{v}_-$. For any $u \in \hat{v}_+$, one has $u \in V$, whence also $\hat{u} = \hat{v}_+$, and

$$0 = {}_{W'}\langle Du, v \rangle_W = [\hat{u} \mid \hat{v}]_{\widehat{W}} = [\hat{u} \mid \hat{v}_+]_{\widehat{W}} + [\hat{u} \mid \hat{v}_-]_{\widehat{W}} = [\hat{v}_+ \mid \hat{v}_+]_{\widehat{W}}.$$

This forces $\hat{v}_+ = 0$, and hence $\hat{v} = \hat{v}_- \in X_-$, which implies $v \in \widetilde{V}$. Last, $W = V + \widetilde{V}$ is a direct consequence of $\widehat{W} = X_+ + X_-$. $\square$

Lemma 15. *Let $K = K_+[\dot{+}]_K K_-$ be a canonical decomposition of a given Kreĭn space $(K, [\cdot \mid \cdot]_K)$ (over $\mathbb{R}$ or $\mathbb{C}$). If both $K_+, K_- \neq \{0\}$ then there are uncountably many distinct canonical decompositions, whereas if either $K_+ = \{0\}$ or $K_- = \{0\}$, then there is only one canonical decomposition.*

Proof. Clearly, if $K_- = \{0\}$ (or $K_+ = \{0\}$), then there exists only one decomposition $K = K[\dot{+}]\{0\}$ (or $K = \{0\}[\dot{+}]K$). Let us now assume that $K_+, K_- \neq \{0\}$, and let $\{e_\gamma^+\}_{\gamma \in \Gamma}$ and $\{e_{\gamma'}^-\}_{\gamma' \in \Gamma'}$ be orthonormal bases respectively of $(K_+, [\cdot \mid \cdot]_K)$ and $(K_-, -[\cdot \mid \cdot]_K)$. For fixed $\gamma_0 \in \Gamma$ and $\gamma_0' \in \Gamma'$ ($\Gamma, \Gamma' \neq \emptyset$ since K_+ and K_- are non-trivial), and for $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$, let us define orthogonal systems

$$\tilde{e}_\gamma^+ := \begin{cases} e_\gamma^+, & \gamma \neq \gamma_0 \\ e_{\gamma_0}^+ + \alpha e_{\gamma_0'}^-, & \gamma = \gamma_0 \end{cases} \qquad \tilde{e}_{\gamma'}^- := \begin{cases} e_{\gamma'}^-, & \gamma' \neq \gamma_0' \\ e_{\gamma_0'}^- + \beta e_{\gamma_0}^+, & \gamma' = \gamma_0', \end{cases}$$

where $\alpha, \beta \in \mathbb{C}$. It is easy to see that taking

$$\alpha := re^{\mathrm{i}\theta}, \qquad \beta := re^{-\mathrm{i}\theta}$$

for some $r \in (0,1)$ and $\theta \in (-\pi, \pi]$ ensures that

$$[\,\tilde{e}^+_{\gamma_0} \mid \tilde{e}^+_{\gamma_0}\,]_K > 0, \qquad [\,\tilde{e}^-_{\gamma'_0} \mid \tilde{e}^-_{\gamma'_0}\,]_K < 0, \qquad [\,\tilde{e}^+_{\gamma_0} \mid \tilde{e}^-_{\gamma'_0}\,]_K = 0,$$

as well as the fact that $\tilde{e}_{\gamma_0^+}$ and $\tilde{e}^-_{\gamma'_0}$ are linearly independent. In turn, this clearly implies that

- $(\mathrm{span}\{\tilde{e}^+_\gamma\}_{\gamma\in\Gamma}, [\cdot\mid\cdot]_K)$ and $(\mathrm{span}\{\tilde{e}^-_{\gamma'}\}_{\gamma'\in\Gamma'}, -[\cdot\mid\cdot]_K)$ are inner product spaces;
- $\mathrm{span}\{\tilde{e}^+_\gamma\}_{\gamma\in\Gamma}$ and $\mathrm{span}\{\tilde{e}^-_{\gamma'}\}_{\gamma'\in\Gamma'}$ are $[\cdot\mid\cdot]_K$-orthogonal;
- $\mathrm{span}\{\tilde{e}^+_\gamma\}_{\gamma\in\Gamma} + \mathrm{span}\{\tilde{e}^-_{\gamma'}\}_{\gamma'\in\Gamma'} = \mathrm{span}\{e^+_\gamma\}_{\gamma\in\Gamma} + \mathrm{span}\{e^-_{\gamma'}\}_{\gamma'\in\Gamma'}$.

For such r and θ, let us denote by $\tilde{K}_+$ and $\tilde{K}_-$ the completions of $(\mathrm{span}\{\tilde{e}^+_\gamma\}_{\gamma\in\Gamma}, [\cdot\mid\cdot]_K)$ and $(\mathrm{span}\{\tilde{e}^-_{\gamma'}\}_{\gamma'\in\Gamma'}, -[\cdot\mid\cdot]_K)$, respectively. It is straightforward to see that $K = \tilde{K}_+[\dot{+}]_K\tilde{K}_-$ is another canonical decomposition of K, and there are as many as (r,θ). □

Lemma 16. *There exists a canonical decomposition* $\widehat{W} = X_+[\dot{+}]_{\widehat{W}}X_-$ *such that* $\ker A'_1/W_0 \subseteq X_+$ *and* $\ker A_1/W_0 \subseteq X_-$.

Proof. We start by proving that $(\widehat{\ker A'_1}, [\cdot\mid\cdot]_{\widehat{W}})$ and $(\widehat{\ker A_1}, -[\cdot\mid\cdot]_{\widehat{W}})$ are Hilbert spaces, where $\widehat{\ker A'_1} := \ker A'_1/W_0$ and $\widehat{\ker A_1} := \ker A_1/W_0$. Since $A_1 + A'_1$ has strictly positive bottom, there exists $\mu_0 > 0$ such that, for any $u \in \ker A'_1$,

$$[\,\hat{u} \mid \hat{u}\,]_{\widehat{W}} = \langle\, A_1 u \mid u\,\rangle_L = \langle\,(A_1 + A'_1)u \mid u\,\rangle_L \geqslant \mu_0\|u\|_L^2. \tag{11}$$

Thus, $[\,\hat{u}\mid\hat{u}\,]_{\widehat{W}} \geqslant 0$ and $[\,\hat{u}\mid\hat{u}\,]_{\widehat{W}} = 0$ if and only if $\hat{u} = 0$ $(= W_0)$, which implies that $(\widehat{\ker A'_1}, [\cdot\mid\cdot]_{\widehat{W}})$ is a (definite) inner product space. Concerning its completeness, let $(\hat{u}_n)$ be a Cauchy sequence in $(\widehat{\ker A'_1}, [\cdot\mid\cdot]_{\widehat{W}})$, where $u_n \in \ker A'_1$. Then (11) implies that (u_n) is a Cauchy sequence in L, hence $u_n \to u$ in L for some $u \in L$. Since A'_1 is closed and for any n we have $A'_1 u_n = 0$, we get that $u \in W$ and $A'_1 u = 0$. Thus, $u \in \ker A'_1$, which together with the boundedness of $A_1 + A'_1$ yields

$$\begin{aligned}[\,\hat{u} - \hat{u}_n \mid \hat{u} - \hat{u}_n\,]_{\widehat{W}} &= \langle\,(A_1 + A'_1)(u - u_n) \mid u - u_n\,\rangle_L \\ &\leqslant \|A_1 + A'_1\|\|u_n - u\|_L^2.\end{aligned}$$

Hence, $\hat{u}_n \to \hat{u}$ in $(\widehat{\ker A'_1}, [\cdot\mid\cdot]_{\widehat{W}})$, and the completeness is proved. Analogously, one shows that $(\widehat{\ker A_1}, -[\cdot\mid\cdot]_{\widehat{W}})$ is a Hilbert space too. Now, since $(\ker A'_1 + \ker A_1)/W_0 = \widehat{\ker A'_1}[\dot{+}]_{\widehat{W}}\widehat{\ker A_1}$, then this space is a Kreĭn space, hence a closed subspace of $\widehat{W}$. Then a standard argument in Kreĭn space theory [9, Theorems V.3.4 and V.3.5] shows that $\widehat{W}$ allows a canonical decomposition of the form $\widehat{W} = X_+[\dot{+}]_{\widehat{W}}X_-$, such that $\widehat{\ker A'_1} \subseteq X_+$ and

$\widehat{\ker A_1} \subseteq X_-$, thus completing the proof. Alternatively we could have also argued by deducing from (11) the fact that $\widehat{\ker A'_1}$ is *uniformly positive* [9, Section V.5], and similarly that $\widehat{\ker A_1}$ is *uniformly negative*, and then applying [9, Theorem V.5.6]. $\square$

Proof of Theorem 13(i)–(ii). Since any Kreĭn space admits a canonical decomposition, by Proposition 14 and Theorem 5 we obtain the first statement of part (i), while the second statement follows from Lemma 16.

Concerning part (ii), if $\ker A_1 \neq \{0\}$ and $\ker A'_1 \neq \{0\}$, then by Lemma 16 there exists a non-trivial canonical decomposition $\widehat{W} = X_+[\dot{+}]_{\widehat{W}} X_-$ such that $X_+, X_- \neq \{0\}$. By Lemma 15 there are uncountably many distinct canonical decompositions of $\widehat{W}$, and each of them defines one distinct adjoint pair of bijective realisation with signed boundary map relative to (A_0, A'_0), as can be deduced from Proposition 14.

On the other hand, let $\ker A_1 = \{0\}$ and let $(A_{\mathrm{r}}, A^*_{\mathrm{r}})$ be an arbitrary adjoint pair of bijective realisations with signed boundary map relative to (A_0, A'_0), which exists by (i). We have that $A_{\mathrm{r}} \subseteq A_1$, and let us prove that in fact $A_{\mathrm{r}} = A_1$. If that was not the case, then there would exist $u \in W \setminus \mathrm{dom}\, A_{\mathrm{r}}$, $u \neq 0$. Owing to the bijectivity of A_{r} there would exist a unique $v \in \mathrm{dom}\, A_{\mathrm{r}} \subseteq W$ such that $A_{\mathrm{r}} v = A_1 u$. As $A_{\mathrm{r}} \subseteq A_1$, $A_1(u - v) = 0$. Thus, $u - v \in \ker A_1 = \{0\}$, and this implies that $u = v \in \mathrm{dom}\, A_{\mathrm{r}}$, which is in contradiction with the actual choice of u. Therefore, $A_{\mathrm{r}} = A_1$ and $A^*_{\mathrm{r}} = A^*_1 = A'_0$. Due to the arbitrariness of $(A_{\mathrm{r}}, A^*_{\mathrm{r}})$ we conclude that (A_1, A'_0) is the only joint pair of bijective realisations with signed boundary map relative to (A_0, A'_0).

The case $\ker A'_1 = \{0\}$ is completely analogous. $\square$

In particular, we have shown a slightly stronger result than what stated in Theorem 13(i)–(ii). Indeed, owing Proposition 14 there is always an adjoint pair of bijective realisations with signed boundary map $(A_{\mathrm{r}}, A^*_{\mathrm{r}})$ relative to (A_0, A'_0) such that $\mathrm{dom}\, A_{\mathrm{r}} + \mathrm{dom}\, A^*_{\mathrm{r}} = W$, and, if $\ker A_1 \neq \{0\}$ and $\ker A'_1 \neq \{0\}$, then there are infinitely many bijective realisations with signed boundary map with such property.

4. Infinite multiplicity of bijective realisations with signed boundary map: a classification

In this Section we develop the tools needed in order to formulate rigorously and to prove part (iii) of Theorem 13.

In fact, one main advantage of our Hilbert space reformulation of abstract Friedrichs operators made in Section 2 is the possibility to connect such theory to the general extension theory of (closed and) densely defined operators on Hilbert spaces, in particular for those extensions covered by Theorem 5.

The latter is an operator-theoretic framework that has (and had historically) an autonomous development and relevance, independently of the special case when the starting pair of operators to extend is a joint pair (A_0, A'_0) of closed abstract Friedrichs operators. In turn, the extension problem for abstract Friedrichs operators displays peculiar features that we are going to discuss in detail.

For the present general formulation, we follow the exposition of [23, Chapter 13] – the main results were already obtained in [22].

Theorem 17. *Let (A_0, A'_0) and (A_1, A'_1) be two pairs of mutually adjoint, closed and densely defined operators in L satisfying*

$$A_0 \subseteq (A_0')^* = A_1 \qquad \textit{and} \qquad A_0' \subseteq (A_0)^* = A_1',$$

which admit a further pair $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ *of reference operators that are closed, satisfy* $A_0 \subseteq A_{\mathrm{r}} \subseteq A_1$, *equivalently* $A_0' \subseteq A_{\mathrm{r}}^* \subseteq A_1'$, *and are invertible with everywhere defined bounded inverses* A_{r}^{-1} *and* $(A_{\mathrm{r}}^*)^{-1}$.

(i) *There are decompositions*

$$\operatorname{dom} A_1 = \operatorname{dom} A_{\mathrm{r}} \dot{+} \ker A_1 \qquad \textit{and} \qquad \operatorname{dom} A_1' = \operatorname{dom} A_{\mathrm{r}}^* \dot{+} \ker A_1', \tag{12}$$

the corresponding projections

$$\begin{aligned} p_{\mathrm{r}} &: \operatorname{dom} A_1 \to \operatorname{dom} A_{\mathrm{r}}, & p_{\mathrm{k}} &: \operatorname{dom} A_1 \to \ker A_1, \\ p_{\mathrm{r}'} &: \operatorname{dom} A_1' \to \operatorname{dom} A_{\mathrm{r}}^*, & p_{\mathrm{k}'} &: \operatorname{dom} A_1' \to \ker A_1', \end{aligned} \tag{13}$$

satisfying

$$\begin{aligned} p_{\mathrm{r}} &= A_{\mathrm{r}}^{-1} A_1, & p_{\mathrm{r}'} &= (A_{\mathrm{r}}^*)^{-1} A_1', \\ p_{\mathrm{k}} &= \mathbb{1} - p_{\mathrm{r}}, & p_{\mathrm{k}'} &= \mathbb{1} - p_{\mathrm{r}'}, \end{aligned} \tag{14}$$

and being continuous with respect to the graph norms.

(ii) *There is a one-to-one correspondence between all pairs of mutually adjoint operators* (A, A^*) *with* $A_0 \subseteq A \subseteq A_1$, *equivalently* $A_0' \subseteq A^* \subseteq A_1'$, *and all pairs of densely defined mutually adjoint operators* $B : \mathcal{V} \to \mathcal{W}$ *and* $B^* : \mathcal{W} \to \mathcal{V}$, *with domains* $\operatorname{dom} B \subseteq \mathcal{V}$ *and* $\operatorname{dom} B^* \subseteq \mathcal{W}$, *where* $\mathcal{V}$ *and* $\mathcal{W}$ *run through the closed subspaces of* $\ker A_1$ *and* $\ker A_1'$. *The correspondence is given by*

$$\begin{aligned} \operatorname{dom} A &= \Big\{ u \in \operatorname{dom} A_1 \; : \; p_{\mathrm{k}} u \in \operatorname{dom} B\,, \; P_{\mathcal{W}}(A_1 u) = B(p_{\mathrm{k}} u) \Big\}, \\ \operatorname{dom} A^* &= \Big\{ v \in \operatorname{dom} A_1' \; : \; p_{\mathrm{k}'} v \in \operatorname{dom} B^*\,, \; P_{\mathcal{V}}(A_1' v) = B^*(p_{\mathrm{k}'} v) \Big\}, \end{aligned} \tag{15}$$

and conversely, by

$$\begin{aligned} \operatorname{dom} B &= p_{\mathrm{k}} \operatorname{dom} A, & \mathcal{V} &= \overline{\operatorname{dom} B}, & B(p_{\mathrm{k}} u) &= P_{\mathcal{W}}(A_1 u), \\ \operatorname{dom} B^* &= p_{\mathrm{k}'} \operatorname{dom} A^*, & \mathcal{W} &= \overline{\operatorname{dom} B^*}, & B^*(p_{\mathrm{k}'} v) &= P_{\mathcal{V}}(A_1' v), \end{aligned} \tag{16}$$

where $P_{\mathcal{V}}$ *and* $P_{\mathcal{W}}$ *are the* orthogonal *projections from* L *onto* $\mathcal{V}$ *and* $\mathcal{W}$.

(iii) *In the correspondence above,* A *is injective, resp. surjective, resp. bijective, if and only if so is* B.

(iv) *When A_B corresponds to B as above, then*

$$\begin{aligned} \operatorname{dom} A_B \;&=\; \left\{ w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu \;\middle|\; \begin{array}{c} w_0 \in \operatorname{dom} A_0 \\ \nu \in \operatorname{dom} B \\ \nu' \in \ker A_1' \ominus \mathcal{W} \end{array} \right\}, \\ A_B(w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu) \;&=\; A_0 w_0 + B\nu + \nu' \end{aligned} \tag{17}$$

and

$$\begin{aligned} \operatorname{dom}(A_B)^* \;&=\; \left\{ w_0' + (A_{\mathrm{r}}^*)^{-1}(B^*\mu' + \mu) + \mu' \;\middle|\; \begin{array}{c} w_0' \in \operatorname{dom} A_0' \\ \mu' \in \operatorname{dom} B^* \\ \mu \in \ker A_1 \ominus \mathcal{V} \end{array} \right\}, \\ (A_B)^*(w_0' + (A_{\mathrm{r}}^*)^{-1}(B^*\mu' + \mu) + \mu') \;&=\; A_0' w_0' + B^*\mu' + \mu\,, \end{aligned} \tag{18}$$

and, moreover,

$$(A_B)^* \;=\; A_{B^*}\,.$$

We stress that our notation is chosen in such a way that p denotes the projection induced by a direct sum decomposition, whereas P denotes the orthogonal projection onto a closed subspace. Furthermore, the non-primed ν's or μ's all belong to $\ker A_1$, whereas their primed counterparts belong to $\ker A_1'$. Let us also remark that $\ker A_1' \ominus \mathcal{W}$ denotes the orthogonal complement of $\mathcal{W}$ in $\ker A_1'$, and respectively for $\ker A_1 \ominus \mathcal{V}$.

For the trivial choice $\mathcal{V} = \mathcal{W} = \{0\}$ one can easily see that $A_B = A_{\mathrm{r}}$.

It is worth mentioning that the problem addressed to in Theorem 17 above has a long history, which unfolded through different mathematical contexts along the time. It first arose as a self-adjoint extension problem for a given densely defined symmetric A_0, where the question is to find and characterise the operators S with $A_0 \subseteq S = S^* \subseteq A_0^*$. If A_0 is semi-bounded, say, with a strictly positive bottom, then a distinguished self-adjoint extension surely exists which is invertible with everywhere defined and bounded inverse – the Friedrichs extension $A_0^{(F)}$ of A_0 – and assumptions of Theorem 17 are thus satisfied with

$$A_0 \,=\, A_0' \subseteq A_1 \,=\, A_1' \,=\, (A_0)^* \qquad \text{and} \qquad A_{\mathrm{r}} \,=\, A_{\mathrm{r}}' \,=\, A_0^{(F)}\,.$$

For this *symmetric case*, Kreĭn [26], Višik [31], and Birman [8] produced a complete classification of the self-adjoint (and more generally symmetric) extensions in the spirit of Theorem 17: modern formulations can be found, for instance, in [23, Section 13.2], in [30, Section 14.8], and in [27]. The generalisation of the previous classification to a wider class of closed extensions A with $A_0 \subseteq A \subseteq A_1$ was established by Grubb [22] by means of Hilbert space and operator graph methods resembling very closely those used by Kreĭn, Višik, and Birman. Today the extension problem, at least in the symmetric case, is customarily formulated in the modern language of boundary triplets theory – an updated survey of which may be found, for instance, in [30, Chapter 14].

With the abstract result of Theorem 17 above at hand, let us assume from now on that (A_0, A_0') is a joint pair of closed abstract Friedrichs operators. The existence of one reference pair $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$

is always guaranteed by Theorem 13(i). Then Theorem 17 classifies (constructively) *all* other closed bijective realisations of A_0, indexing them with the closed and densely defined bijections $B : \mathrm{dom}\, B \to \mathcal{W}$, where $\overline{\mathrm{dom}\, B} = \mathcal{V}$, and $\mathcal{V}$ and $\mathcal{W}$ run through the closed subspaces of $\ker A_1$ and $\ker A_1'$, respectively. Thus, *the whole class of adjoint pairs of bijective realisations of a given joint pair of abstract Friedrichs operators is fully characterised.*

What remains non-trivial is the qualification of the special (and relevant) sub-class of bijective realisations of (A_0, A_0') *with signed boundary map*. For this problem, the parametrisation $A_B \leftrightarrow B$ of Theorem 17 is of no advantage and we need to develop a further operator-theoretic analysis.

In this respect comes our second main result, Theorem 18 below. In particular, part (ii) of this Theorem completes the rigorous formulation of the classification announced in Theorem 13(iii).

Theorem 18. *Let (A_0, A_0') be a joint pair of closed abstract Friedrichs operators, and let $(A_\mathrm{r}, A_\mathrm{r}^*)$ be an adjoint pair of bijective realisations of (A_0, A_0') with signed boundary map. Let (A_B, A_B^*) be a generic pair of closed extensions $A_0 \subseteq A_B \subseteq A_1$, $A_0' \subseteq A_B^* \subseteq A_1'$, according to the notation of the parametrisation given in Theorem 17. Let p_k and $p_{\mathrm{k}'}$ be the projections (13) identified by direct decompositions (12). With reference to the following two sets of* 'mirror' *conditions, namely*

$$(\forall\, \nu \in \mathrm{dom}\, B) \quad (\forall\, \nu' \in \ker A_1' \ominus \mathcal{W}) \qquad \begin{cases} \langle\, \nu \mid A_1' \nu \,\rangle_L - 2\,\mathfrak{Re}\langle\, p_{\mathrm{k}'} \nu \mid B\nu \,\rangle_L \leqslant 0 \\ \langle\, p_{\mathrm{k}'} \nu \mid \nu' \,\rangle_L = 0 \end{cases} \tag{19}$$

and

$$(\forall\, \mu' \in \mathrm{dom}\, B^*) \quad (\forall\, \mu \in \ker A_1 \ominus \mathcal{V}) \qquad \begin{cases} \langle\, A_1 \mu' \mid \mu' \,\rangle_L - 2\,\mathfrak{Re}\langle\, B^* \mu' \mid p_\mathrm{k} \mu' \,\rangle_L \leqslant 0 \\ \langle\, \mu \mid p_\mathrm{k} \mu' \,\rangle_L = 0\,, \end{cases} \tag{20}$$

one has these conclusions.

(i) *Any of the following three facts,*
  (a) *conditions (19) and (20) hold true, or*
  (b) *condition (19) holds true and $B : \mathrm{dom}\, B \to \mathcal{W}$ is a bijection, or*
  (c) *condition (20) holds true and $B^* : \mathrm{dom}\, B^* \to \mathcal{V}$ is a bijection,*
  is sufficient *for (A_B, A_B^*) to be another adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0').*

(ii) *Assume further that* $\mathrm{dom}\, A_\mathrm{r} = \mathrm{dom}\, A_\mathrm{r}^*$. *Then the following properties are* equivalent*:*
  (a) *(A_B, A_B^*) is another adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0');*
  (b) *the mirror conditions (19) and (20) are satisfied.*

We present the proof Theorem 18 in the next Section. Before that, let us conclude the discussion of this Section with a number of relevant remarks.

Remark 19. One case in which the assumptions (i)–(b) or (i)–(c) of Theorem 18 are surely not empty is when $\ker A_1 \nsubseteq \mathrm{dom}\, A_\mathrm{r}^*$, respectively when $\ker A_1' \nsubseteq \mathrm{dom}\, A_\mathrm{r}$, in which case there are actually infinitely many pairs (A_B, A_B^*) satisfying (i)–(b), respectively (i)–(c).

To see this, let $\ker A_1 \nsubseteq \mathrm{dom}\, A_\mathrm{r}^*$. Then there exists $\nu_0 \in \ker A_1$ such that $p_{\mathrm{k}'} \nu_0 \neq 0$, where $p_{\mathrm{k}'}$ is the projection (13) identified by direct decompositions (12) with respect to A_r^*. For, otherwise, for any $v \in \ker A_1$ we would have $v = p_{\mathrm{r}'}v + p_{\mathrm{k}'}v = p_{\mathrm{r}'}v \in \operatorname{dom} A_{\mathrm{r}}^*$, thus obtaining a contradiction. Correspondingly, let $v_0' := p_{\mathrm{k}'}v_0 \neq 0$, $\mathcal{V} := \operatorname{span}\{v_0\}$, and $\mathcal{W} := \operatorname{span}\{v_0'\}$. For any $z \in \mathbb{C}$ with $\mathfrak{Re}z \geqslant \frac{\langle v_0 | A_1' v_0 \rangle_L}{2\|v_0'\|_L^2}$ the operator $B_z : \mathcal{V} \to \mathcal{W}$, $v_0 \mapsto z v_0'$, turns out to satisfy (19) and to be a bijection. Indeed, concerning (19), since $p_{\mathrm{k}'}v_0 = v_0' \in \mathcal{W}$ the condition

$$(\forall\, v' \in \ker A_1' \ominus \mathcal{W}) \qquad \langle\, p_{\mathrm{k}'}v_0 \mid v' \,\rangle_L \;=\; 0$$

trivially holds, whereas our choice of z ensures

$$\langle\, v_0 \mid A_1' v_0 \,\rangle_L - 2\,\mathfrak{Re}\langle\, p_{\mathrm{k}'}v_0 \mid B_z v_0 \,\rangle_L \;=\; \langle\, v_0 \mid A_1' v_0 \,\rangle_L - 2\|v_0'\|_L^2\,\mathfrak{Re}z \;\leqslant\; 0\,.$$

The bijectivity of B_z is due to

$$\frac{\langle\, v_0 \mid A_1' v_0 \,\rangle_L}{2\|v_0'\|_L^2} \;=\; \frac{\langle\, v_0 \mid (A_1 + A_1')v_0 \,\rangle_L}{2\|v_0'\|_L^2} \;\geqslant\; \frac{\mu_0\|v_0\|_L^2}{2\|v_0'\|_L^2} \;>\; 0$$

($\mu_0 > 0$ by assumption), since then $z \neq 0$.

Remark 20. The conclusion of Theorem 18 which follows from the conditions (i)–(a) or from the conditions (i)–(b) is the same, and in either case the argument is based on the universal classification given by Theorem 17. However, as emerges from the proof, in order to deduce from (i)–(a) that (A_B, A_B^*) is another adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0'), we need to appeal to Theorem 5, that is, a result from earlier literature. On the other hand, in order to deduce the same from (i)–(b) we do not need Theorem 5: the result rather follows from our intermediate technical result, Lemma 24 of Section 5.

Remark 21. Any of the assumptions (i)–(a), or (i)–(b), or (i)–(c) of Theorem 18 may be exploited in concrete examples so as to produce other pairs of bijective and signed realisations of the given abstract Friedrichs operators (A_0, A_0'), in analogy to what Theorem 5 already does. The case when the *reference* operators A_{r} and A_{r}^* have the same domain is of relevance in many concrete situations (see, e.g., [4,12,18]). Noticeably, in this case we have produced conditions (namely, (19) and (20)) that identify the whole family of bijective and signed realisations of (A_0, A_0').

5. Proofs of the classification scheme (Theorem 18)

In this Section we present the proof of Theorem 18. At the core of the analysis are the following three Lemmas.

Lemma 22. *Let (A_0, A_0') be a joint pair of closed abstract Friedrichs operators, and let $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ be an adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0'). Then for any $v, \mu \in \ker A_1$ and for any $v', \mu' \in \ker A_1'$ one has*

$$\begin{aligned}\langle\, v \mid A_1'(A_{\mathrm{r}})^{-1}v' \,\rangle_L \;&=\; -\langle\, p_{\mathrm{k}'}v \mid v' \,\rangle_L \\ \langle\, A_1(A_{\mathrm{r}}^*)^{-1}\mu \mid \mu' \,\rangle_L \;&=\; -\langle\, \mu \mid p_{\mathrm{k}}\mu' \,\rangle_L\,,\end{aligned} \tag{21}$$

where $p_{\mathrm{k}} : \operatorname{dom} A_1 \to \ker A_1$ *and* $p_{\mathrm{k}'} : \operatorname{dom} A_1' \to \ker A_1'$ *are the projections* (13) *identified by direct decompositions* (12)*.*

Proof. We present the proof for the first identity of (21), the proof for the second being analogous. For $\nu \in \ker A_1$ and $\nu' \in \ker A_1'$ we find

$$\begin{aligned}
\langle \nu \mid A_1'(A_{\mathrm{r}})^{-1}\nu' \rangle_L &= \langle \nu \mid (A_1 + A_1')(A_{\mathrm{r}})^{-1}\nu' \rangle_L - \langle \nu \mid \nu' \rangle_L \\
&= \langle (A_{\mathrm{r}}^*)^{-1}(A_1 + A_1')\nu \mid \nu' \rangle_L - \langle \nu \mid \nu' \rangle_L \\
&= \langle p_{\mathrm{r}'}\nu \mid \nu' \rangle_L - \langle \nu \mid \nu' \rangle_L \\
&= -\langle p_{\mathrm{k}'}\nu \mid \nu' \rangle_L \,,
\end{aligned}$$

where we used the identity $A_1(A_{\mathrm{r}})^{-1}\nu' = \nu'$ in the first step, the self-adjointness of $A_1 + A_1'$ in the second step, $(A_{\mathrm{r}}^*)^{-1}A_1' = p_{\mathrm{r}'}$ in the third step, and $p_{\mathrm{r}'} + p_{\mathrm{k}'} = \mathbb{1}$ in the fourth step. □

Lemma 23. *Let* (A_0, A_0') *be a joint pair of closed abstract Friedrichs operators, and let* $(A_{\mathrm{r}}, A_{\mathrm{r}}^*)$ *be an adjoint pair of bijective realisations with signed boundary map relative to* (A_0, A_0')*. Let* A_B *be a closed operator with* $A_0 \subseteq A_B \subseteq A_1$*, parametrised by a closed densely defined operator* $B : \mathcal{V} \to \mathcal{W}$ *according to the classification of Theorem 17.*

(i) *For an arbitrary* $u = w_0 + (A_{\mathrm{r}})^{-1}(B\nu + \nu') + \nu \in \operatorname{dom} A_B$*, in the notation of* (17)*, after setting* $u_{\mathrm{r}} := (A_{\mathrm{r}})^{-1}(B\nu + \nu') \in \operatorname{dom} A_{\mathrm{r}}$*, one has*

$$\begin{aligned}
\langle A_1 u \mid u \rangle_L - \langle u \mid A_1' u \rangle_L &= \langle A_1 u_{\mathrm{r}} \mid u_{\mathrm{r}} \rangle_L - \langle u_{\mathrm{r}} \mid A_1' u_{\mathrm{r}} \rangle_L \\
&\quad - (\langle \nu \mid A_1'\nu \rangle_L - 2\,\mathfrak{Re}\langle p_{\mathrm{k}'}\nu \mid B\nu + \nu' \rangle_L)\,.
\end{aligned}$$

(ii) *For an arbitrary* $v = w_0' + (A_{\mathrm{r}}^*)^{-1}(B^*\mu' + \mu) + \mu' \in \operatorname{dom}(A_B)^*$*, in the notation of* (18)*, setting* $v_{\mathrm{r}} := (A_{\mathrm{r}}^*)^{-1}(B^*\mu' + \mu) \in \operatorname{dom} A_{\mathrm{r}}^*$*, one has*

$$\begin{aligned}
\langle A_1 v \mid v \rangle_L - \langle v \mid A_1' v \rangle_L &= \langle A_1 v_{\mathrm{r}} \mid v_{\mathrm{r}} \rangle_L - \langle v_{\mathrm{r}} \mid A_1' v_{\mathrm{r}} \rangle_L \\
&\quad + (\langle A_1 \mu' \mid \mu' \rangle_L - 2\,\mathfrak{Re}\langle B^*\mu' + \mu \mid p_{\mathrm{k}}\mu' \rangle_L)\,.
\end{aligned}$$

Proof. Let us prove (i); the proof of (ii) follows in the same manner. Using the symmetry of D and the identity $\ker D = W_0$ (see (2) and (3) above), we compute

$$\begin{aligned}
\langle A_1 u \mid u \rangle_L - \langle u \mid A_1' u \rangle_L &= {}_{W'}\langle Du, u \rangle_W \\
&= {}_{W'}\langle D(u_{\mathrm{r}} + \nu), (u_{\mathrm{r}} + \nu) \rangle_W \\
&= {}_{W'}\langle Du_{\mathrm{r}}, u_{\mathrm{r}} \rangle_W + 2\mathfrak{Re}\,{}_{W'}\langle D\nu, u_{\mathrm{r}} \rangle_W + {}_{W'}\langle D\nu, \nu \rangle_W \\
&= \langle A_1 u_{\mathrm{r}} \mid u_{\mathrm{r}} \rangle_L - \langle u_{\mathrm{r}} \mid A_1' u_{\mathrm{r}} \rangle_L - (2\mathfrak{Re}\langle \nu \mid A_1' u_{\mathrm{r}} \rangle_L + \langle \nu \mid A_1'\nu \rangle_L)\,,
\end{aligned}$$

where in the last identity we used that $\nu \in \ker A_1$. Since $B\nu \in \mathcal{W} \subseteq \ker A_1'$, Lemma 22 is applicable and (21) yields

$$\langle\, \nu \mid A_1' u_\mathrm{r} \,\rangle_L = \langle\, \nu \mid A_1'(A_\mathrm{r})^{-1}(B\nu + \nu')\,\rangle_L = -\langle\, p_{\mathrm{k}'}\nu \mid B\nu + \nu' \,\rangle_L\,.$$

Combining the last two identities proves the claim. □

Lemma 24. *Let (A_0, A_0') be a joint pair of closed abstract Friedrichs operators, and let the operator $A : L \to L$, with $A_0 \subseteq A \subseteq A_1$, be closed, bijective, and satisfy*

$$(\forall u \in \operatorname{dom} A) \qquad \langle\, A_1 u \mid u \,\rangle_L - \langle\, u \mid A_1' u \,\rangle_L \;\geqslant\; 0\,. \tag{22}$$

Then $A^* : L \to L$ is such that $A_0' \subseteq A^* \subseteq A_1'$, it is closed, bijective, and satisfies

$$(\forall v \in \operatorname{dom} A^*) \qquad \langle\, A_1 v \mid v \,\rangle_L - \langle\, v \mid A_1' v \,\rangle_L \;\leqslant\; 0\,. \tag{23}$$

Proof. Since A is closed and bijective, and it is included between A_0 and A_1, then it is invertible with bounded and everywhere defined inverse (Remark 4), thus we can take it as the reference operator in the classification provided by Theorem 17. In particular, A^* too is closed and bijective, it is included between A_0' and A_1', and moreover one can decompose an arbitrary $v \in \operatorname{dom} A^* \subseteq \operatorname{dom} A_1' = \operatorname{dom} A_1$ as $v = u + z$ with $u \in \operatorname{dom} A$ and $z \in \ker A_1$, according to the decomposition (12). Also, $A_0 + A_0'$ is bounded and self-adjoint. From

$$\begin{aligned}
\langle\, A_1 v \mid v \,\rangle_L &= \langle\, Au \mid v \,\rangle_L = \langle\, u \mid A^* v \,\rangle_L \\
&= \langle\, u \mid A_1' u \,\rangle_L + \langle\, u \mid (A_1 + A_1') z \,\rangle_L \\
&= \langle\, u \mid A_1' u \,\rangle_L + \langle\, (A_1 + A_1') u \mid z \,\rangle_L \\
&= \langle\, u \mid A_1' u \,\rangle_L + \langle\, A_1 u \mid z \,\rangle_L + \langle\, A_1' u \mid z \,\rangle_L
\end{aligned}$$

and

$$\begin{aligned}
\langle\, v \mid A_1' v \,\rangle_L &= \langle\, u \mid A^* v \,\rangle_L + \langle\, z \mid A_1' v \,\rangle_L \\
&= \langle\, Au \mid v \,\rangle_L + \langle\, z \mid A_1' u \,\rangle_L + \langle\, z \mid A_1' z \,\rangle_L \\
&= \langle\, A_1 u \mid u \,\rangle_L + \langle\, A_1 u \mid z \,\rangle_L + \langle\, z \mid A_1' u \,\rangle_L + \langle\, z \mid A_1' z \,\rangle_L\,,
\end{aligned}$$

we get

$$\begin{aligned}
&\langle\, A_1 v \mid v \,\rangle_L - \langle\, v \mid A_1' v \,\rangle_L = \\
&\quad = -(\langle\, A_1 u \mid u \,\rangle_L - \langle\, u \mid A_1' u \,\rangle_L) - \langle\, z \mid A_1' z \,\rangle_L + (\langle\, A_1' u \mid z \,\rangle_L - \overline{\langle\, A_1' u \mid z \,\rangle_L}) \\
&\quad = -(\langle\, A_1 u \mid u \,\rangle_L - \langle\, u \mid A_1' u \,\rangle_L) - \langle\, (A_1 + A_1') z \mid z \,\rangle_L + 2\mathrm{i}\,\mathfrak{Im}\langle\, A_1' u \mid z \,\rangle_L\,.
\end{aligned}$$

Since, owing to (4), both the l.h.s. above and the quantities $\langle\, A_1 u \mid u \,\rangle_L - \langle\, u \mid A_1' u \,\rangle_L$ and $\langle\, (A_1 + A_1') z \mid z \,\rangle_L$ are real, then necessarily $\mathfrak{Im}\langle A_1' u, z \rangle = 0$. Moreover, $\langle\, A_1 u \mid u \,\rangle_L - \langle\, u \mid A_1' u \,\rangle_L \geqslant 0$ by assumption, and similarly $\langle\, (A_1 + A_1') z \mid z \,\rangle_L \geqslant 0$, because $A_1 + A_1'$ has a strictly positive bottom. Thus we conclude $\langle\, A_1 v \mid v \,\rangle_L - \langle\, v \mid A_1' v \,\rangle_L \leqslant 0$. □

We are now ready to prove Theorem 18.

Proof of Theorem 18. We start with proving that condition (19) implies that A_B satisfies (22), and that condition (20) implies that A_B^* satisfies (23). Moreover, we shall see that when $\mathrm{dom}\, A_\mathrm{r} = \mathrm{dom}\, A_\mathrm{r}^*$ the converse of such implications is valid too.

To this aim we pick $u \in \mathrm{dom}\, A_B$ and decompose it as $u = w_0 + (A_\mathrm{r})^{-1}(B\nu + \nu') + \nu$ according to (17). Owing to Lemma 23(i),

$$\begin{aligned}\langle A_1 u \mid u \rangle_L - \langle u \mid A_1' u \rangle_L \;&=\; \langle A_1 u_\mathrm{r} \mid u_\mathrm{r} \rangle_L - \langle u_\mathrm{r} \mid A_1' u_\mathrm{r} \rangle_L \\ &\quad - \big(\langle \nu \mid A_1' \nu \rangle_L - 2\,\mathfrak{Re}\langle p_{\mathrm{k}'}\nu \mid B\nu + \nu' \rangle_L\big)\end{aligned}$$

with $u_\mathrm{r} := (A_\mathrm{r})^{-1}(B\nu + \nu') \in \mathrm{dom}\, A_\mathrm{r}$. By assumption, $\langle A_1 u_\mathrm{r} \mid u_\mathrm{r} \rangle_L - \langle u_\mathrm{r} \mid A_1' u_\mathrm{r} \rangle_L \geqslant 0$. Therefore, if $\langle \nu \mid A_1' \nu \rangle_L - 2\,\mathfrak{Re}\langle p_{\mathrm{k}'}\nu \mid B\nu \rangle_L \leqslant 0$ and $\langle p_{\mathrm{k}'}\nu \mid \nu' \rangle_L = 0$, then $\langle A_1 u \mid u \rangle_L - \langle u \mid A_1' u \rangle_L \geqslant 0$, which proves that (19) implies (22). The other implication follows from a completely analogous argument.

Let us now assume (i)–(a). We proved above that conditions (19)–(20) imply (22)–(23) and so the spaces $V := \mathrm{dom}\, A_B$ and $\widetilde{V} := \mathrm{dom}\, A_B^*$ satisfy condition (V1). Condition (V2) is satisfied too, owing to Theorem 9 and the fact that A_B and A_B^* are mutually adjoint. Therefore, the conditions of Theorem 5 are fulfilled and as a consequence (A_B, A_B^*) is an adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0').

Next, let us consider the case (i)–(b) (the case (i)–(c) being analogous). Since $B : \mathrm{dom}\, B \to \mathcal{W}$ is a bijection, so is $A_B : \mathrm{dom}\, A_B \to L$ (Theorem 17(iii)). Thus, A_B is a closed bijection satisfying (19), and hence also (22) by the argument used at the beginning of the proof: this, owing to Lemma 24, implies that A_B^* is a closed bijection satisfying (23). This yields again the conclusion for (A_B, A_B^*).

Concerning part (ii), let us consider the converse implications to (19) $\Rightarrow$ (22) and (20) $\Rightarrow$ (23) established above, now making the additional assumption that $\mathrm{dom}\, A_\mathrm{r} = \mathrm{dom}\, A_\mathrm{r}^*$. In this case A_r has a vanishing boundary term (Remark 12), and the conclusion of Lemma 23(i) takes the form

$$\begin{aligned}\langle A_1 u \mid u \rangle_L - \langle u \mid A_1' u \rangle_L \;&=\; 2\,\mathfrak{Re}\langle p_{\mathrm{k}'}\nu \mid B\nu \rangle_L \\ &\quad - \langle \nu \mid A_1' \nu \rangle_L + 2\,\mathfrak{Re}\langle p_{\mathrm{k}'}\nu \mid \nu' \rangle_L\,. \end{aligned} \tag{*}$$

Because of (22), $\langle A_1 u \mid u \rangle_L - \langle u \mid A_1' u \rangle_L \geqslant 0$ for any $u \in \mathrm{dom}\, A_B$; by replacing ν' with $\lambda\nu'$, for $\lambda \in \mathbb{R}$ and $|\lambda|$ large enough, the r.h.s. of (*) fails to be non-negative, unless $\mathfrak{Re}\langle p_{\mathrm{k}'}\nu \mid \nu' \rangle_L = 0$ for any $\nu \in \mathrm{dom}\, B$ and $\nu' \in \ker A_1' \ominus \mathcal{W}$. In turn, the latter condition must hold also for $\mathrm{i}\nu'$ in place of ν', thus implying that $\mathfrak{Im}\langle p_{\mathrm{k}'}\nu \mid \nu' \rangle_L = 0$ as well. Then $\langle p_{\mathrm{k}'}\nu \mid \nu' \rangle_L = 0$ and (*) gives the other inequality of (19).

Thus, the mirror conditions (19)–(20) are *equivalent* to (22)–(23), and hence to the fact that the spaces $V := \mathrm{dom}\, A_B$ and $\widetilde{V} := \mathrm{dom}\, A_B^*$ satisfy condition (V1). This, together with Theorem 9, completes the proof of part (ii). □

6. Examples

In order to discuss concrete applications of the theory developed so far, we identified two different instructive examples that we work out in this Section. They are both one-dimensional realisations of classical Friedrichs systems, i.e., first order ordinary differential operators. Despite their simplicity, they display enough richness so as to illustrate the essential features of our main results, Theorems 13 and 18, and in particular the usage of conditions (19)–(20).

6.1. Equation on an interval

For this example we take $L := \mathrm{L}^2(a,b)$, $\mathcal{D} := \mathrm{C}_c^\infty(a,b)$, and define $T, \widetilde{T} : \mathcal{D} \longrightarrow L$ by

$$T\phi := \frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi \qquad \text{and} \qquad \widetilde{T}\phi := -\frac{\mathrm{d}}{\mathrm{d}x}\phi + \phi\,.$$

It is immediate to see that T and $\widetilde{T}$ satisfy (T1)–(T3) and to recognise that

$$W_0 = \mathrm{H}_0^1(a,b) \qquad \text{and} \qquad W = \mathrm{H}^1(a,b)\,.$$

This example was already thoroughly studied in the real valued setting in [1, Example 1]; we are now going to revisit it in view of our discussion of Sections 3-4, focusing on bijective realisations of $(T, \widetilde{T})$ with or without signed boundary map.

Let us set $A_0 := \overline{T}$, $A_0' := \overline{\widetilde{T}}$, $A_1 := \widetilde{T}^*$, and $A_1' := T^*$. As weak differential operators they have the same formal action $\pm\frac{\mathrm{d}}{\mathrm{d}x} + \mathbb{1}$. Clearly, $\operatorname{dom} A_0 = \operatorname{dom} A_0' = W_0$ and $\operatorname{dom} A_1 = \operatorname{dom} A_1' = W$. Since

$${}_{W'}\langle Du, v\rangle_W = u(b)\bar{v}(b) - u(a)\bar{v}(a)\,,$$

owing to Theorem 5 the operators $A_\mathrm{r} := A_1|_V$, and its adjoint A_r^*, where

$$V := \widetilde{V} := \{u \in \mathrm{H}^1(a,b) \,|\, u(a) = u(b)\}\,,$$

form an adjoint pair of bijective realisations relative to (A_0, A_0'), with $\operatorname{dom} A_\mathrm{r}^* = \operatorname{dom} A_\mathrm{r} = V$. Then by means of Theorem 17 one controls the whole class of adjoint pairs of bijective realisations, and in particular Theorem 18(ii) selects those that have signed boundary map.

Next, the kernels of A_1 and A_1' are given by

$$\ker A_1 = \operatorname{span}\{e^{-x}\} \qquad \text{and} \qquad \ker A_1' = \operatorname{span}\{e^{x}\}\,.$$

This, together with the above choice for $(A_\mathrm{r}, A_\mathrm{r}^*)$, implies that the (non-orthogonal) projections defined in (13) act in this case as

$$p_\mathrm{k} u = -\frac{u(b) - u(a)}{e^{-a} - e^{-b}}e^{-x} \qquad \text{and} \qquad p_{\mathrm{k}'} u = \frac{u(b) - u(a)}{e^{b} - e^{a}}e^{x}\,.$$

Since both $\ker A_1$ and $\ker A_1'$ are finite dimensional, by Theorem 17(ii) $\operatorname{dom} B = \overline{\operatorname{dom} B} = \mathcal{V}$ and $\operatorname{dom} B^* = \overline{\operatorname{dom} B^*} = \mathcal{W}$, and by Theorem 17(iii) the isomorphic realisations of A_0 are those indexed by bijections $B : \mathcal{V} \to \mathcal{W}$. The trivial choice $\mathcal{V} = \{0\}$, $\mathcal{W} = \{0\}$ produces the reference operators A_r and A_r^*.

Let us consider the generic choice $\mathcal{V} = \ker A_1$, $\mathcal{W} = \ker A_1'$, $Be^{-x} := (\alpha + \mathrm{i}\beta)e^{x}$, where $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}$ (for $(\alpha,\beta) = (0,0)$ B is not a bijection). In order to determine for which $\alpha, \beta \in \mathbb{R}$ the corresponding pair (A_B, A_B^*) is an adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0'), we apply part (i)–(b) and part (ii) of Theorem 18. Since $\ker A_1' \ominus \mathcal{W} = \{0\}$ and $\dim \operatorname{dom} B = 1$, we only need to check that

$$\langle e^{-x} \mid A_1' e^{-x} \rangle_L - 2\,\mathfrak{Re}\langle p_{\mathrm{k}'} e^{-x} \mid B e^{-x} \rangle_L \leqslant 0 .$$

Explicitly,

$$\langle e^{-x} \mid A_1' e^{-x} \rangle_L = 2\|e^{-x}\|_L^2 = 2e^{-2(a+b)}\|e^x\|_L^2$$

and (using $p_{\mathrm{k}'} e^{-x} = -e^{-(a+b)} e^x$)

$$2\,\mathfrak{Re}\langle p_{\mathrm{k}'} e^{-x} \mid B e^{-x} \rangle_L = -2\,\alpha\, e^{-(a+b)}\|e^x\|_L^2 ,$$

and the inequality above is satisfied if and only if $\alpha \leqslant -e^{-(a+b)}$. Therefore, denoting by $A_{\alpha,\beta}$ the operator A_B associated to $Be^{-x} = (\alpha + \mathrm{i}\beta)e^x$,

$$\left\{(A_{\alpha,\beta}, A_{\alpha,\beta}^*) \,|\, \alpha \leqslant -e^{-(a+b)} ,\ \beta \in \mathbb{R}\right\} \cup \{(A_{\mathrm{r}}, A_{\mathrm{r}}^*)\}$$

is the family of *all* adjoint pairs of bijective realisations with signed boundary map relative to (A_0, A_0').

The domains of $A_{\alpha,\beta}$ and $A_{\alpha,\beta}^*$ are determined by means of Theorem 17(ii): since $\operatorname{dom} B = \ker A_1$, $\operatorname{dom} B^* = \ker A_1'$, and

$$B^* e^x = \frac{\|e^x\|_L^2}{\|e^{-x}\|_L^2} (\alpha - \mathrm{i}\beta)\, e^{-x} = e^{2(a+b)}(\alpha - \mathrm{i}\beta)e^{-x} ,$$

then (15) yields

$$\begin{aligned}
\operatorname{dom} A_{\alpha,\beta} &= \left\{u \in \mathrm{H}^1(a,b) \,|\, P_{\ker A_1'}(A_1 u) = (\alpha + \mathrm{i}\beta)\, p_{\mathrm{k}} u\right\} \\
\operatorname{dom} A_{\alpha,\beta}^* &= \left\{u \in \mathrm{H}^1(a,b) \,|\, P_{\ker A_1}(A_1' u) = e^{2(a+b)}(\alpha - \mathrm{i}\beta)\, p_{\mathrm{k}'} u\right\} .
\end{aligned}$$

Since $P_{\ker A_1'} u = \|e^x\|_L^{-2}\langle u \mid e^x \rangle_L e^x = \frac{2}{e^{2b}-e^{2a}}\langle u \mid e^x \rangle_L e^x$, then

$$\begin{aligned}
P_{\ker A_1'}(A_1 u) &= \frac{2}{e^{2b} - e^{2a}} \langle \frac{\mathrm{d}}{\mathrm{d}x} u + u \mid e^x \rangle_L e^x \\
&= \frac{2}{e^{2b} - e^{2a}} \big(- \langle u \mid e^x \rangle_L + (e^x u)|_a^b + \langle u \mid e^x \rangle_L \big)\, e^x \\
&= \frac{2(e^b u(b) - e^a u(a))}{e^{2b} - e^{2a}}\, e^x .
\end{aligned}$$

Thus, $u \in \mathrm{H}^1(a,b)$ belongs to $\operatorname{dom} A_{\alpha,\beta}$ if and only if

$$\frac{2(e^b u(b) - e^a u(a))}{e^{2b} - e^{2a}} = (\alpha + \mathrm{i}\beta)\, \frac{u(b) - u(a)}{e^{-b} - e^{-a}} ,$$

equivalently,

$$\begin{aligned}\big(2e^{-a} + \alpha(e^a + e^b) + \mathrm{i}\beta(e^a + e^b)\big)\, u(b) \;&= \\ = \big(2e^{-b} + \alpha(e^a + e^b) + \mathrm{i}\beta(e^a + e^b)\big)\, u(a)\,.\end{aligned}$$

Analogously, for $u \in \mathrm{H}^1(a,b)$ one has

$$P_{\ker A_1}(A_1' u) \;=\; \frac{2(e^{-a}u(a) - e^{-b}u(b))}{e^{-2a} - e^{-2b}}\, e^{-x}\,,$$

so $u \in \operatorname{dom} A^*_{\alpha,\beta}$ if and only if

$$\begin{aligned}\big(2e^{-b} + \alpha(e^a + e^b) - \mathrm{i}\beta(e^a + e^b)\big)\, u(b) \;&= \\ = \big(2e^{-a} + \alpha(e^a + e^b) - \mathrm{i}\beta(e^a + e^b)\big)\, u(a)\,.\end{aligned}$$

When $\alpha = -2e^{-a}(e^a + e^b)^{-1} (\leqslant -e^{-(a+b)})$ and $\beta = 0$,

$$\begin{aligned}\operatorname{dom} A_{-\frac{2e^{-a}}{e^a+e^b},0} \;&=\; \{u \in \mathrm{H}^1(a,b) \,|\, u(a) = 0\} \\ \operatorname{dom} A^*_{-\frac{2e^{-a}}{e^a+e^b},0} \;&=\; \{u \in \mathrm{H}^1(a,b) \,|\, u(b) = 0\}\,,\end{aligned}$$

whereas for all other realisations, except for the reference one,

$$\begin{aligned}\operatorname{dom} A_{\alpha,\beta} \;&=\; \{u \in \mathrm{H}^1(a,b) \,|\, u(b) = z_{\alpha,\beta}\, u(a)\} \\ \operatorname{dom} A^*_{\alpha,\beta} \;&=\; \{u \in \mathrm{H}^1(a,b) \,|\, \overline{z_{\alpha,\beta}}\, u(b) = u(a)\}\end{aligned}$$

with

$$z_{\alpha,\beta} \;=\; \frac{2e^{-b} + \alpha(e^a + e^b) + \mathrm{i}\beta(e^a + e^b)}{2e^{-a} + \alpha(e^a + e^b) + \mathrm{i}\beta(e^a + e^b)}$$

for $\alpha \leqslant -e^{-(a+b)}$, $\beta \in \mathbb{R}$, and $(\alpha, \beta) \neq (-2e^{-a}(e^a + e^b)^{-1}, 0)$.

Let us comment on some specific values of α and β. When $\beta = 0$, $\alpha \leqslant -e^{-(a+b)}$, and $\alpha \neq -2e^{-a}(e^a + e^b)^{-1}$, the number $z_{\alpha,0}$ takes all values in $(-\infty, -1] \cup (1, \infty)$, the value $z_{\alpha,0} = 1$ not being included because it corresponds to the reference pair. This agrees with results in [1, Example 1]. In particular, for $\alpha = -\frac{2(e^{b-a} - e^{-(b-a)})}{e^{2b} - e^{2a}} (\leqslant -e^{-(a+b)})$ we have

$$\operatorname{dom} A_{\alpha,0} \;=\; W_0 + \ker A_1' \qquad \text{and} \qquad \operatorname{dom} A^*_{\alpha,0} \;=\; W_0 + \ker A_1\,.$$

Furthermore, $\operatorname{dom} A_{\alpha,\beta} = \operatorname{dom} A^*_{\alpha,\beta}$ if and only if $|z_{\alpha,\beta}| = 1$, that is, if and only if $\alpha = -e^{-(a+b)}$. In fact, when $\beta \in \mathbb{R}$ the number $z_{-e^{-(a+b)},\beta}$ takes all values on the unite circle in the complex plane except for the value 1 that corresponds to the reference pair.

Besides such realisations $(A_{\alpha,\beta}, A^*_{\alpha,\beta})$ there are many other bijective realisations which are not with a signed boundary map. For example, when $\alpha = -2e^{-b}(e^a + e^b)^{-1} (> -e^{-(a+b)})$ and $\beta = 0$,

$$\operatorname{dom} A_B = \{u \in \mathrm{H}^1(a,b) \mid u(b) = 0\}$$
$$\operatorname{dom} A_B^* = \{u \in \mathrm{H}^1(a,b) \mid u(a) = 0\},$$

which does not satisfy condition (V1).

6.2. System on an interval

In the previous example the second condition in (19) was trivially satisfied. In the following one the kernels of A_1 and A_1' are two-dimensional and both conditions in (19) are non-trivial.

For this example we take $L := \mathrm{L}^2(a,b) \times \mathrm{L}^2(a,b)$, $W_0 := \mathrm{H}_0^1(a,b) \times \mathrm{H}_0^1(a,b)$, and define $A_0, A_0' : W_0 \longrightarrow L$ by

$$A_0 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x} u_2 + \varepsilon u_1 \\ \frac{\mathrm{d}}{\mathrm{d}x} u_1 + \mu u_2 \end{bmatrix} \qquad \text{and} \qquad A_0' \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} := \begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} u_2 + \varepsilon u_1 \\ -\frac{\mathrm{d}}{\mathrm{d}x} u_1 + \mu u_2 \end{bmatrix}$$

for fixed $\varepsilon, \mu > 0$. As before, by $A_1 := (A_0')^*$ and $A_1' := A_0^*$, with domain $\operatorname{dom} A_1 = \operatorname{dom} A_1' = \mathrm{H}^1(a,b) \times \mathrm{H}^1(a,b) =: W$; as weak differential operators A_1 and A_1' acts formally as A_0 and A_0', respectively. Thus, $A_0 \subseteq A_1$ and $A_0' \subseteq A_1'$, whereas obviously $\overline{A_0 + A_0'}$ is a bounded self-adjoint operator in L with strictly positive bottom.

The boundary operator can easily be computed by integration by parts: for any $\mathsf{u}, \mathsf{v} \in W$, say, $\mathsf{u} = [u_1\ u_2]^\top$ and $\mathsf{v} = [v_1\ v_2]^\top$, one has

$$_{W'}\langle D\mathsf{u}, \mathsf{v} \rangle_W = u_2(b)\bar{v}_1(b) - u_2(a)\bar{v}_1(a) + u_1(b)\bar{v}_2(b) - u_1(a)\bar{v}_2(a)\,. \tag{24}$$

It is immediate to see that for $V = \widetilde{V} := \mathrm{H}_0^1(a,b) \times \mathrm{H}^1(a,b)$ conditions (V1)–(V2) are fulfilled (see [12, Section 3.3], [18, Section 5.3]). Thus, owing to Theorem 5, $A_\mathrm{r} := A_1|_V$ and A_r^* form an adjoint pair of bijective realisations with signed boundary map relative to (A_0, A_0'), and $\operatorname{dom} A_\mathrm{r} = \operatorname{dom} A_\mathrm{r}^* = V$. Theorem 18(ii) then controls *all* other adjoint pairs of bijective realisations with signed boundary map relative to (A_0, A_0').

The relevant kernels are the two-dimensional subspaces

$$\ker A_1 = \operatorname{span}\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2\} \qquad \text{and} \qquad \ker A_1' = \operatorname{span}\{\boldsymbol{\nu}_1', \boldsymbol{\nu}_2'\},$$

where

$$\boldsymbol{\nu}_1 := \begin{bmatrix} e^{\sqrt{\varepsilon\mu}x} \\ -\sqrt{\frac{\varepsilon}{\mu}}\, e^{\sqrt{\varepsilon\mu}x} \end{bmatrix}, \qquad \boldsymbol{\nu}_2 := \begin{bmatrix} e^{-\sqrt{\varepsilon\mu}x} \\ \sqrt{\frac{\varepsilon}{\mu}}\, e^{-\sqrt{\varepsilon\mu}x} \end{bmatrix}$$

and

$$\boldsymbol{\nu}_1' := \begin{bmatrix} e^{\sqrt{\varepsilon\mu}x} \\ \sqrt{\frac{\varepsilon}{\mu}} e^{\sqrt{\varepsilon\mu}x} \end{bmatrix}, \qquad \boldsymbol{\nu}_2' := \begin{bmatrix} e^{-\sqrt{\varepsilon\mu}x} \\ -\sqrt{\frac{\varepsilon}{\mu}} e^{-\sqrt{\varepsilon\mu}x} \end{bmatrix}.$$

We compute

$$
\begin{aligned}
\|\boldsymbol{\nu}_1\|_L^2 &= \|\boldsymbol{\nu}_1'\|_L^2 = \frac{\varepsilon+\mu}{2\mu\sqrt{\varepsilon\mu}}\left(e^{2\sqrt{\varepsilon\mu}b}-e^{2\sqrt{\varepsilon\mu}a}\right)\\
\|\boldsymbol{\nu}_2\|_L^2 &= \|\boldsymbol{\nu}_2'\|_L^2 = \frac{\varepsilon+\mu}{2\mu\sqrt{\varepsilon\mu}}\left(e^{-2\sqrt{\varepsilon\mu}a}-e^{-2\sqrt{\varepsilon\mu}b}\right).
\end{aligned}
$$

With respect to the above choice for A_{r}, the (non-orthogonal) projections defined in (13) act in this case as

$$
p_{\mathrm{K}}\mathsf{u} = C_1(\mathsf{u})\boldsymbol{\nu}_1 + C_2(\mathsf{u})\boldsymbol{\nu}_2 \qquad \text{and} \qquad p_{\mathrm{K}'}\mathsf{u} = C_1(\mathsf{u})\boldsymbol{\nu}_1' + C_2(\mathsf{u})\boldsymbol{\nu}_2',
$$

where

$$
\begin{aligned}
C_1(\mathsf{u}) &:= \frac{e^{-\sqrt{\varepsilon\mu}a}u_1(b)-e^{-\sqrt{\varepsilon\mu}b}u_1(a)}{e^{\sqrt{\varepsilon\mu}(b-a)}-e^{-\sqrt{\varepsilon\mu}(b-a)}}\\
C_2(\mathsf{u}) &:= -\frac{e^{\sqrt{\varepsilon\mu}a}u_1(b)-e^{\sqrt{\varepsilon\mu}b}u_1(a)}{e^{\sqrt{\varepsilon\mu}(b-a)}-e^{-\sqrt{\varepsilon\mu}(b-a)}}.
\end{aligned}
$$

The isometric realisations A_B, with $A_0 \subseteq A_B \subseteq A_1$, are indexed by bijections $B:\mathcal{V}\to\mathcal{W}$, which forces $\dim\mathcal{V}=\dim\mathcal{W}$. The trivial choice $\mathcal{V}=\mathcal{W}=\{0\}$ produces $A_B=A_{\mathrm{r}}$. When instead $\mathcal{V}=\ker A_1$ and $\mathcal{W}=\ker A_1'$, the second condition of (19) is trivially satisfied, as was the case in the previous example. In the following, let us study the case where $\mathcal{V}$ and $\mathcal{W}$ are one-dimensional.

In order to exploit conditions (19) and (20) conveniently, we observe that the second requirement (i.e., the identity) both in (19) and in (20) is formulated only in terms of the subspaces $\mathcal{V}$ and $\mathcal{W}$ and not of the extension parameters B and B^*. Thus, it is convenient to find first suitable $\mathcal{V}$ and $\mathcal{W}$ satisfying the identity requirement of (19), and then to seek for extension parameters B such that also the inequality requirement of (19) is satisfied.

To this aim, let us consider $\mathcal{V}:=\operatorname{span}\{a_1\boldsymbol{\nu}_1+a_2\boldsymbol{\nu}_2\}$ and $\mathcal{W}:=\operatorname{span}\{a_1'\boldsymbol{\nu}_1'+a_2'\boldsymbol{\nu}_2'\}$ for fixed $a_1,a_2,a_1',a_2'\in\mathbb{C}$ such that $|a_1|^2+|a_2|^2=1$, $|a_1'|^2+|a_2'|^2=1$ and $a_1,a_1'\geqslant 0$. Since $p_{\mathrm{K}'}\boldsymbol{\nu}_j=\boldsymbol{\nu}_j'$, $j=1,2$, the identity requirement of (19), namely,

$$
0 = \langle\, p_{\mathrm{K}'}(a_1\boldsymbol{\nu}_1+a_2\boldsymbol{\nu}_2)\,|\,\boldsymbol{\nu}_\perp'\,\rangle_L = \langle\, a_1\boldsymbol{\nu}_1'+a_2\boldsymbol{\nu}_2'\,|\,\boldsymbol{\nu}_\perp'\,\rangle_L \quad \forall\boldsymbol{\nu}_\perp'\in\ker,\, A_1'\ominus\mathcal{W}
$$

is satisfied if and only if $a_1\boldsymbol{\nu}_1'+a_2\boldsymbol{\nu}_2'\perp(\ker A_1'\ominus\mathcal{W})$, i.e., $a_1\boldsymbol{\nu}_1'+a_2\boldsymbol{\nu}_2'\in\mathcal{W}$. Owing to the normalisation of (a_1,a_2) and (a_1',a_2'), the latter condition is equivalent to

$$
a_1'=a_1 \qquad \text{and} \qquad a_2'=a_2\,. \tag{25}
$$

Arguing as in Remark 19, it is clear now that with respect to the extension parameter B defined by

$$
B(a_1\boldsymbol{\nu}_1+a_2\boldsymbol{\nu}_2) := (\alpha+\mathrm{i}\beta)a_1\boldsymbol{\nu}_1'+a_2\boldsymbol{\nu}_2',
$$

any choice

$$\alpha \geqslant \frac{\langle a_1\boldsymbol{\nu}_1 + a_2\boldsymbol{\nu}_2 \mid A_1'(a_1\boldsymbol{\nu}_1 + a_2\boldsymbol{\nu}_2)\rangle_L}{2\|a_1\boldsymbol{\nu}_1' + a_2\boldsymbol{\nu}_2'\|_L^2}, \qquad \beta \in \mathbb{R} \tag{26}$$

makes the assumptions of Theorem 18(i)–(b) fulfilled.

Let us focus on two special cases. The first is $a_1 = a_1' = 1$ and $a_2 = a_2' = 0$, in other words $\mathcal{V} = \operatorname{span}\{\boldsymbol{\nu}_1\}$, $\mathcal{W} = \operatorname{span}\{\boldsymbol{\nu}_1'\}$, and $B\boldsymbol{\nu}_1 = (\alpha + \mathrm{i}\beta)\boldsymbol{\nu}_1'$. Using the identity $\langle \boldsymbol{\nu}_1 \mid A_1'\boldsymbol{\nu}_1\rangle_L = \frac{4\varepsilon\mu}{\varepsilon+\mu}\|\boldsymbol{\nu}_1'\|_L^2$ we deduce from (26) that each choice $\beta \in \mathbb{R}$ and $\alpha \geqslant \frac{2\varepsilon\mu}{\varepsilon+\mu}$ identifies a pair (A_B, A_B^*) of bijective realisations with signed boundary map relative to (A_0, A_0'). Let us determine their domains.

We know from (15) that $\operatorname{dom} A_B$ consists of all $\mathsf{u} = [u_1\ u_2]^\top \in W$ such that $p_\mathrm{k}\mathsf{u} \in \mathcal{V}$ and $P_\mathcal{W}(A_1\mathsf{u}) = B(p_\mathrm{k}\mathsf{u})$. The condition $p_\mathrm{k}\mathsf{u} \in \mathcal{V}$ implies $C_2(\mathsf{u}) = 0$, hence

$$u_1(b) = e^{\sqrt{\varepsilon\mu}(b-a)}\,u_1(a)\,. \tag{27}$$

When this is the case,

$$\begin{aligned} B(p_\mathrm{k}\mathsf{u}) &= (\alpha + \mathrm{i}\beta)\frac{e^{-\sqrt{\varepsilon\mu}a}u_1(b) - e^{-\sqrt{\varepsilon\mu}b}u_1(a)}{e^{\sqrt{\varepsilon\mu}(b-a)} - e^{-\sqrt{\varepsilon\mu}(b-a)}}\,\boldsymbol{\nu}_1' \\ &= (\alpha + \mathrm{i}\beta)\,e^{-\sqrt{\varepsilon\mu}a}\,u_1(a)\,\boldsymbol{\nu}_1'\,. \end{aligned}$$

Acting on $A_1\mathsf{u}$ with the orthogonal projection onto $\mathcal{W}$ gives

$$P_\mathcal{W}(A_1\mathsf{u}) = \frac{1}{\|\boldsymbol{\nu}_1'\|_L^2}\langle A_1\mathsf{u} \mid \boldsymbol{\nu}_1'\rangle_L\boldsymbol{\nu}_1'\,,$$

and, by integration by parts,

$$\begin{aligned} \langle A_1\mathsf{u} \mid \boldsymbol{\nu}_1'\rangle_L &= \langle \frac{\mathrm{d}}{\mathrm{d}x}u_2 + \varepsilon u_1 \mid e^{\sqrt{\varepsilon\mu}x}\rangle_{\mathrm{L}^2(a,b)} + \sqrt{\frac{\varepsilon}{\mu}}\langle \frac{\mathrm{d}}{\mathrm{d}x}u_2 + \mu u_1 \mid e^{\sqrt{\varepsilon\mu}x}\rangle_{\mathrm{L}^2(a,b)} \\ &= -\sqrt{\varepsilon\mu}\langle u_2 \mid e^{\sqrt{\varepsilon\mu}x}\rangle_{\mathrm{L}^2(a,b)} + (e^{\sqrt{\varepsilon\mu}x}u_2)\Big|_a^b + \varepsilon\langle u_1 \mid e^{\sqrt{\varepsilon\mu}x}\rangle_{\mathrm{L}^2(a,b)} \\ &\quad - \sqrt{\frac{\varepsilon}{\mu}}\sqrt{\varepsilon\mu}\langle u_1 \mid e^{\sqrt{\varepsilon\mu}x}\rangle_{\mathrm{L}^2(a,b)} + \sqrt{\frac{\varepsilon}{\mu}}(e^{\sqrt{\varepsilon\mu}x}u_1)\Big|_a^b + \mu\sqrt{\frac{\varepsilon}{\mu}}\langle u_2 \mid e^{\sqrt{\varepsilon\mu}x}\rangle_{\mathrm{L}^2(a,b)} \\ &= (e^{\sqrt{\varepsilon\mu}x}u_2)\Big|_a^b + \sqrt{\frac{\varepsilon}{\mu}}(e^{\sqrt{\varepsilon\mu}x}u_1)\Big|_a^b\,. \end{aligned}$$

This, together with (27), yields

$$\begin{aligned} &P_\mathcal{W}(A_1\mathsf{u}) = \\ &= \Big(e^{\sqrt{\varepsilon\mu}b}u_2(b) - e^{\sqrt{\varepsilon\mu}a}u_2(a) + \sqrt{\frac{\varepsilon}{\mu}}e^{-\sqrt{\varepsilon\mu}a}(e^{2\sqrt{\varepsilon\mu}b} - e^{2\sqrt{\varepsilon\mu}a})u_1(a)\Big)\frac{\boldsymbol{\nu}_1'}{\|\boldsymbol{\nu}_1'\|_L^2} \\ &= \Big(\frac{e^{\sqrt{\varepsilon\mu}b}u_2(b) - e^{\sqrt{\varepsilon\mu}a}u_2(a)}{\|\boldsymbol{\nu}_1'\|_L^2} + \frac{2\varepsilon\mu}{\varepsilon+\mu}e^{-\sqrt{\varepsilon\mu}a}u_1(a)\Big)\boldsymbol{\nu}_1'\,, \end{aligned}$$

where in the second equality we used that $\|\boldsymbol{\nu}_1'\|_L^2 = \frac{\varepsilon+\mu}{2\mu\sqrt{\varepsilon\mu}}(e^{2\sqrt{\varepsilon\mu}b} - e^{2\sqrt{\varepsilon\mu}a})$. Finally, the identity $P_{\mathcal{W}}(A_1\mathsf{u}) = B(p_\mathrm{k}\mathsf{u})$ implies

$$u_2(b) \;=\; e^{-\sqrt{\varepsilon\mu}(b-a)}u_2(a) + \|\boldsymbol{\nu}_1'\|_L^2\Big(\alpha + \mathrm{i}\beta - \frac{2\varepsilon\mu}{\varepsilon+\mu}\Big)\, e^{-\sqrt{\varepsilon\mu}(a+b)}\, u_1(a)\,,$$

thus yielding

$$\operatorname{dom} A_B \;=\; \left\{ \mathsf{u}\in W \;\middle|\; \begin{aligned} & u_1(b) = e^{\sqrt{\varepsilon\mu}(b-a)}u_1(a) \\ & u_2(b) = e^{-\sqrt{\varepsilon\mu}(b-a)}u_2(a) \\ & \qquad\qquad + \|\boldsymbol{\nu}_1'\|_L^2(\alpha + \mathrm{i}\beta - \tfrac{2\varepsilon\mu}{\varepsilon+\mu})e^{-\sqrt{\varepsilon\mu}(a+b)}u_1(a) \end{aligned} \right\}.$$

Analogous arguments yield

$$\operatorname{dom} A_B^* \;=\; \left\{ \mathsf{u}\in W \;\middle|\; \begin{aligned} & u_1(b) = e^{\sqrt{\varepsilon\mu}(b-a)}u_1(a) \\ & u_2(b) = e^{-\sqrt{\varepsilon\mu}(b-a)}u_2(a) \\ & \qquad\qquad - \|\boldsymbol{\nu}_1'\|_L^2(\alpha - \mathrm{i}\beta - \tfrac{2\varepsilon\mu}{\varepsilon+\mu})e^{-\sqrt{\varepsilon\mu}(a+b)}u_1(a) \end{aligned} \right\}.$$

For all $\beta\in\mathbb{R}$ and $\alpha \geqslant \frac{2\varepsilon\mu}{\varepsilon+\mu}$, $\operatorname{dom} A_B + \operatorname{dom} A_B^*$ is a proper subspace of W, while for $\alpha = \frac{2\varepsilon\mu}{\varepsilon+\mu}$ we have $\operatorname{dom} A_B = \operatorname{dom} A_B^*$.

The second special case we want to consider is $a_1 = a_2' = 1$, $a_2 = a_1' = 0$, that is, a choice that is *not* covered by formula (25): indeed, we want to highlight the mechanism why, out of the regime (25) determined by our Theorem 18, one fails in having adjoint pairs of bijective realisations relative to (A_0, A_0').

In this case $\mathcal{V} = \operatorname{span}\{\boldsymbol{\nu}_1\}$, $\mathcal{W} = \operatorname{span}\{\boldsymbol{\nu}_2'\}$, and $B\boldsymbol{\nu}_1 := (\alpha+\mathrm{i}\beta)\boldsymbol{\nu}_2'$. Let us compute the domains of A_B and A_B^*, and investigate condition (V1) directly. Since $\mathcal{V}$ is the same as in the previous case, formula (27) still holds, and hence

$$B(p_\mathrm{k}\mathsf{u}) \;=\; (\alpha+\mathrm{i}\beta)\, e^{-\sqrt{\varepsilon\mu}a}u_1(a)\,\boldsymbol{\nu}_2'\,.$$

Proceeding analogously to what we did in the first part of this Subsection, we obtain

$$\begin{aligned} P_{\mathcal{W}}(A_1\mathsf{u}) \;&=\; \frac{1}{\|\boldsymbol{\nu}_2'\|_L^2}\Big((e^{-\sqrt{\varepsilon\mu}x}u_2)\big|_a^b - \sqrt{\frac{\varepsilon}{\mu}}(e^{-\sqrt{\varepsilon\mu}x}u_1)\big|_a^b\Big)\,\boldsymbol{\nu}_2' \\ &=\; \frac{1}{\|\boldsymbol{\nu}_2'\|_L^2}\,(e^{-\sqrt{\varepsilon\mu}b}u_2(b) - e^{-\sqrt{\varepsilon\mu}a}u_2(a))\boldsymbol{\nu}_2'\,, \end{aligned}$$

having used (27) in the second identity. Therefore,

$$\operatorname{dom} A_B \;=\; \left\{ \mathsf{u}\in W \;\middle|\; \begin{aligned} & u_1(b) = e^{\sqrt{\varepsilon\mu}(b-a)}u_1(a) \\ & u_2(b) = e^{\sqrt{\varepsilon\mu}(b-a)}u_2(a) \\ & \qquad\qquad + \|\boldsymbol{\nu}_2'\|_L^2(\alpha + \mathrm{i}\beta)e^{\sqrt{\varepsilon\mu}(b-a)}u_1(a) \end{aligned} \right\}$$

and

$$\operatorname{dom} A_B^* = \left\{ \mathsf{u} \in W \;\middle|\; \begin{array}{l} u_1(b) = e^{-\sqrt{\varepsilon\mu}(b-a)} u_1(a) \\ u_2(b) = e^{-\sqrt{\varepsilon\mu}(b-a)} u_2(a) \\ \qquad\qquad - \|\boldsymbol{\nu}_2'\|_L^2 (\alpha - \mathrm{i}\beta) u_1(a) \end{array} \right\}.$$

However, we now see that in this case condition (V1) fails to hold irrespectively of what α and β are. Indeed, for $\mathsf{u} \in \operatorname{dom} A_B$ by (24) we have

$$\begin{aligned} {}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\, \rangle_W &= 2\Big(\|\boldsymbol{\nu}_2'\|_L^2 \alpha e^{2\sqrt{\varepsilon\mu}(b-a)} |u_1(a)|^2 \\ &\quad + (e^{2\sqrt{\varepsilon\mu}(b-a)} - 1)\, \mathfrak{Re}(u_1(a)\bar{u}_2(a))\Big). \end{aligned}$$

Thus, for any α, β and $u_1(a) \neq 0$ one can always choose $u_2(a)$ such that the expression above is negative or positive at will. Hence, (V1) is not satisfied because the above expression does not have a definite sign.

References

[1] N. Antonić, K. Burazin, On equivalent descriptions of boundary conditions for Friedrichs systems, Math. Montisnigri 22–23 (2009–2010) 5–13.

[2] N. Antonić, K. Burazin, Intrinsic boundary conditions for Friedrichs systems, Comm. Partial Differential Equations 35 (2010) 1690–1715.

[3] N. Antonić, K. Burazin, Boundary operator from matrix field formulation of boundary conditions for Friedrichs systems, J. Differential Equations 250 (2011) 3630–3651.

[4] N. Antonić, K. Burazin, I. Crnjac, M. Erceg, Complex Friedrichs systems and applications, submitted for publication, 22 pp.

[5] N. Antonić, K. Burazin, M. Vrdoljak, Second-order equations as Friedrichs systems, Nonlinear Anal. Real World Appl. 15 (2014) 290–305.

[6] N. Antonić, K. Burazin, M. Vrdoljak, Connecting classical and abstract theory of Friedrichs systems via trace operator, ISRN Math. Anal. 2011 (2011) 469795, http://dx.doi.org/10.5402/2011/469795.

[7] N. Antonić, K. Burazin, M. Vrdoljak, Heat equation as a Friedrichs system, J. Math. Anal. Appl. 404 (2013) 537–553.

[8] M.v. Birman, On the theory of self-adjoint extensions of positive definite operators, Mat. Sb. 38 (80) (1956) 431–450.

[9] J. Bognár, Indefinite Inner Product Spaces, Springer, 1974.

[10] T. Bui-Thanh, L. Demkowicz, O. Ghattas, A unified discontinuous Petrov–Galerkin method and its analysis for Friedrichs' systems, SIAM J. Numer. Anal. 51 (2013) 1933–1958.

[11] T. Bui-Thanh, From Godunov to a unified hybridized discontinuous Galerkin framework for partial differential equations, J. Comput. Phys. 295 (2015) 114–146.

[12] K. Burazin, M. Erceg, Non-stationary abstract Friedrichs systems, Mediterr. J. Math. 13 (2016) 3777–3796.

[13] E. Burman, A. Ern, M.A. Fernandez, Explicit Runge–Kutta schemes and finite elements with symmetric stabilization for first-order linear PDE systems, SIAM J. Numer. Anal. 48 (2010) 2019–2042.

[14] B. Després, F. Lagoutière, N. Seguin, Weak solutions to Friedrichs systems with convex constraints, Nonlinearity 24 (2011) 3055–3081.

[15] A. Ern, J.-L. Guermond, Discontinuous Galerkin methods for Friedrichs' systems. I. General theory, SIAM J. Numer. Anal. 44 (2006) 753–778.

[16] A. Ern, J.-L. Guermond, Discontinuous Galerkin methods for Friedrichs' systems. II. Second-order elliptic PDEs, SIAM J. Numer. Anal. 44 (2006) 2363–2388.

[17] A. Ern, J.-L. Guermond, Discontinuous Galerkin methods for Friedrichs' systems. III. Multifield theories with partial coercivity, SIAM J. Numer. Anal. 46 (2008) 776–804.

[18] A. Ern, J.-L. Guermond, G. Caplain, An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems, Comm. Partial Differential Equations 32 (2007) 317–341.

[19] K.O. Friedrichs, Symmetric hyperbolic linear differential equations, Comm. Pure Appl. Math. 7 (1954) 345–392.

[20] K.O. Friedrichs, Symmetric positive linear differential equations, Comm. Pure Appl. Math. 11 (1958) 333–418.

[21] K.O. Friedrichs, P.D. Lax, Boundary value problems for first order operators, Comm. Pure Appl. Math. 18 (1965) 355–388.

[22] G. Grubb, A characterization of the non-local boundary value problems associated with an elliptic operator, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 22 (1968) 425–513.

[23] G. Grubb, Distributions and Operators, Springer, 2009.

[24] P. Houston, J.A. Mackenzie, E. Süli, G. Warnecke, A posteriori error analysis for numerical approximation of Friedrichs systems, Numer. Math. 82 (1999) 433–470.

[25] M. Jensen, Discontinuous Galerkin Methods for Friedrichs Systems with Irregular Solutions, Ph.D. thesis, University of Oxford, 2004, http://sro.sussex.ac.uk/45497/1/thesisjensen.pdf.

[26] M.G. Kreĭn, The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I, Mat. Sb. 20 (62) (1947) 431–495.

[27] A. Michelangeli, Kreĭn–Višik–Birman self-adjoint extension theory revisited, SISSA preprint 59/2015/MAT, http://preprints.sissa.it/xmlui/handle/1963/35174, 2015.

[28] C. Mifsud, B. Després, N. Seguin, Dissipative formulation of initial boundary value problems for Friedrichs' systems, Comm. Partial Differential Equations 41 (2016) 51–78.

[29] R.S. Phillips, L. Sarason, Singular symmetric positive first order differential operators, J. Math. Mech. 15 (1966) 235–271.

[30] K. Schmüdgen, Unbounded Self-Adjoint Operators on Hilbert Space, Graduate Texts in Mathematics, vol. 265, Springer, Dordrecht, 2012.

[31] M.I. Višik, On general boundary problems for elliptic differential equations, Tr. Mosk. Mat. Obs. 1 (1952) 187–246.