

# A QUASISTATIC EVOLUTION MODEL FOR PERFECTLY PLASTIC PLATES DERIVED BY $\Gamma$ -CONVERGENCE

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ABSTRACT. The subject of this paper is the rigorous derivation of a quasistatic evolution model for a linearly elastic - perfectly plastic thin plate. As the thickness of the plate tends to zero, we prove via  $\Gamma$ -convergence techniques that solutions to the three-dimensional quasistatic evolution problem of Prandtl-Reuss elastoplasticity converge to a quasistatic evolution of a suitable reduced model. In this limiting model the admissible displacements are of Kirchhoff-Love type and the stretching and bending components of the stress are coupled through a plastic flow rule. Some equivalent formulations of the limiting problem in rate form are derived, together with some two-dimensional characterizations for suitable choices of the data.

## 1. INTRODUCTION

The rigorous derivation of lower dimensional models for thin structures is a question of great interest in materials science and its applications. In the early 90's a rigorous approach to dimension reduction has emerged in the stationary framework and in the context of nonlinear elasticity [2, 22]. This approach is based on  $\Gamma$ -convergence and, starting from the seminal paper [18], has led to establish a hierarchy of limit models for plates [18, 19], rods [30, 31, 33, 34], and shells [17, 23, 24]. More recently, the  $\Gamma$ -convergence approach to dimension reduction has gained attention also in the evolutionary framework: in nonlinear elasticity [1], crack propagation [6, 16], elastoplasticity with hardening [25, 26], and delamination problems [29].

In this paper we focus on the rigorous justification of a quasistatic evolution model for a thin plate in perfect plasticity. More precisely, we shall consider a three-dimensional plate of small thickness, whose elastic behaviour is linear and isotropic and whose plastic response is governed by the Prandtl-Reuss flow rule without hardening, and derive via  $\Gamma$ -convergence techniques a reduced quasistatic evolution model, by sending the thickness parameter to zero.

Let  $\omega$  be a domain in  $\mathbb{R}^2$  with a  $C^2$  boundary and let  $\varepsilon > 0$ . For a plate of reference configuration

$$\Omega_\varepsilon := \omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$$

the quasistatic evolution problem in perfect plasticity can be formulated as follows. Let  $u^\varepsilon(t)$  denote the displacement field at time  $t$  and let  $Eu^\varepsilon(t)$  denote the infinitesimal strain tensor at  $t$ , that is, the symmetric part of  $Du^\varepsilon(t)$ . Let  $\sigma^\varepsilon(t)$  be the stress tensor at  $t$  and let  $e^\varepsilon(t)$  and  $p^\varepsilon(t)$  (a deviatoric symmetric matrix) be the elastic and plastic strain tensors at  $t$ . Assume that the plate is subjected to a time-dependent boundary condition  $w^\varepsilon(t)$  prescribed on a subset  $\Gamma_\varepsilon := \gamma_d \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$  of the lateral boundary of  $\Omega_\varepsilon$  and that for simplicity there are no applied loads. The classical formulation of the quasistatic evolution problem on a time interval  $[0, T]$  consists in finding  $u^\varepsilon(t)$ ,  $e^\varepsilon(t)$ ,  $p^\varepsilon(t)$ , and  $\sigma^\varepsilon(t)$  such that the following conditions are satisfied for every  $t \in [0, T]$ :

- (cf1) *kinematic admissibility*:  $Eu^\varepsilon(t) = e^\varepsilon(t) + p^\varepsilon(t)$  in  $\Omega_\varepsilon$  and  $u^\varepsilon(t) = w^\varepsilon(t)$  on  $\Gamma_\varepsilon$ ;
- (cf2) *constitutive law*:  $\sigma^\varepsilon(t) = \mathbb{C}e^\varepsilon(t)$  in  $\Omega_\varepsilon$ , where  $\mathbb{C}$  is the elasticity tensor;

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- (cf3) *equilibrium*:  $\operatorname{div} \sigma^\varepsilon(t) = 0$  in  $\Omega_\varepsilon$ ;
- (cf4) *stress constraint*:  $\sigma_D^\varepsilon(t) \in K$ , where  $\sigma_D^\varepsilon$  is the deviatoric part of  $\sigma^\varepsilon$  and  $K$  is a given convex and compact subset of deviatoric  $3 \times 3$  matrices, representing the set of admissible stresses;
- (cf5) *flow rule*:  $\dot{p}^\varepsilon(t) = 0$  if  $\sigma_D^\varepsilon(t) \in \operatorname{int} K$ , while  $\dot{p}^\varepsilon(t)$  belongs to the normal cone to  $K$  at  $\sigma_D^\varepsilon(t)$  if  $\sigma_D^\varepsilon(t) \in \partial K$ .

The first existence result of a quasistatic evolution in perfect plasticity has been proved in [35] by means of viscoplastic approximations. More recently, in [9] the problem has been reformulated within the framework of the variational theory for rate-independent processes, developed in [27]. The variational formulation reads as follows: to find a triple  $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$  such that for every  $t \in [0, T]$  we have

- (qs1) *global stability*:  $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$  satisfies  $Eu^\varepsilon(t) = e^\varepsilon(t) + p^\varepsilon(t)$  in  $\Omega_\varepsilon$ ,  $u^\varepsilon(t) = w^\varepsilon(t)$  on  $\Gamma_\varepsilon$ , and minimizes

$$\frac{1}{2} \int_{\Omega_\varepsilon} \mathbb{C}f : f \, dx + \int_{\Omega_\varepsilon} H(q - p^\varepsilon(t)) \, dx$$

among all kinematically admissible triples  $(v, f, q)$ , where  $H$  is the support function of  $K$ , i.e.,  $H(p) := \sup\{\sigma : p : \sigma \in K\}$ ;

- (qs2) *energy balance*:

$$\begin{aligned} \frac{1}{2} \int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(t) : e^\varepsilon(t) \, dx + \int_0^t \int_{\Omega_\varepsilon} H(\dot{p}^\varepsilon(s)) \, dx ds \\ = \frac{1}{2} \int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(0) : e^\varepsilon(0) \, dx + \int_0^t \int_{\Omega_\varepsilon} \mathbb{C}e^\varepsilon(s) : E\dot{w}^\varepsilon(s) \, dx ds. \end{aligned}$$

The existence of a quasistatic evolution according to the previous formulation and the extent to which this is equivalent to the original formulation is the main focus of [9].

The main purpose of this paper is to characterize the limiting behaviour of a sequence of solutions  $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$ , as  $\varepsilon \rightarrow 0$ . We observe that the abstract theory of evolutionary  $\Gamma$ -convergence for rate-independent systems developed in [28] cannot be directly applied here. Indeed, it consists in studying separately the  $\Gamma$ -limit of the stored-energy functionals and that of the dissipation distances and in coupling them through the construction of a joint recovery sequence. This technique has been applied, e.g., in [25, 26], where the presence of hardening gives rise to a stored-energy functional that is coercive in the  $L^2$  norm both with respect to  $e$  and  $p$ . This approach is not suited to our case, since the elastic energy is coercive only with respect to the elastic strain  $e$ , while the plastic strain  $p$  can be controlled only through the dissipation. For this reason, to identify the correct limiting energy we study the  $\Gamma$ -convergence of the total energy functional, given by the sum of the stored energy with the dissipation distance. More precisely, we first focus on the static case, that is, we consider a boundary displacement  $w^\varepsilon$  independent of time, we introduce the functional

$$\mathcal{E}_\varepsilon(u, e, p) := \frac{1}{2} \int_{\Omega_\varepsilon} \mathbb{C}e : e \, dx + \int_{\Omega_\varepsilon} H(p) \, dx$$

defined on the class  $\mathcal{A}_\varepsilon(w^\varepsilon, \Omega_\varepsilon)$  of all triples  $(u, e, p)$  satisfying  $Eu = e + p$  in  $\Omega_\varepsilon$  and  $u = w^\varepsilon$  on  $\Gamma_\varepsilon$ , and we study its limit, as  $\varepsilon \rightarrow 0$ , in the sense of  $\Gamma$ -convergence.

As pointed out in [9], because of the linear growth of  $H$ , the functional  $\mathcal{E}_\varepsilon$  is not coercive in any Sobolev norm. The natural setting for a weak formulation is the space  $BD(\Omega_\varepsilon)$  of functions with bounded deformation for the displacement  $u$  and the space  $M_b(\Omega_\varepsilon \cup \Gamma_\varepsilon; \mathbb{M}_D^{3 \times 3})$  of  $\mathbb{M}_D^{3 \times 3}$ -valued bounded Borel measures on  $\Omega_\varepsilon \cup \Gamma_\varepsilon$  for the plastic strain  $p$ . This is also natural from a mechanical point of view, because in absence of hardening it is well known that displacements may develop jump discontinuities along so-called slip surfaces, on which plastic strain concentrates.

Since  $p \in M_b(\Omega_\varepsilon \cup \Gamma_\varepsilon; \mathbb{M}_D^{3 \times 3})$ , the functional

$$\int_{\Omega_\varepsilon} H(p) dx$$

has to be interpreted according to the theory of convex functions of measures, developed in [20, 36] (see also Section 2), as

$$\int_{\Omega_\varepsilon \cup \Gamma_\varepsilon} H\left(\frac{dp}{d|p|}\right) d|p|,$$

where  $dp/d|p|$  is the Radon-Nicodym derivative of  $p$  with respect to its total variation  $|p|$ . Moreover, the boundary condition is relaxed by requiring that

$$p = (w^\varepsilon - u) \odot \nu_{\partial\Omega_\varepsilon} \mathcal{H}^2 \quad \text{on } \Gamma_\varepsilon, \quad (1.1)$$

where  $\odot$  denotes the symmetric tensor product and  $\nu_{\partial\Omega_\varepsilon}$  is the outer unit normal to  $\partial\Omega_\varepsilon$ . The mechanical interpretation of (1.1) is that  $u$  may not attain the boundary condition: in this case a plastic slip is developed along  $\Gamma_\varepsilon$ , whose amount is proportional to the difference between the prescribed boundary value and the actual value.

The  $\Gamma$ -convergence of  $\mathcal{E}_\varepsilon$  (rescaled to the domain  $\Omega := \omega \times (-\frac{1}{2}, \frac{1}{2})$  independent of  $\varepsilon$ ) is the subject of Section 4. For simplicity we assume that the prescribed boundary datum  $w^\varepsilon$  is of the form

$$w^\varepsilon(x) = \left(w_1(x', \frac{x_3}{\varepsilon}), w_2(x', \frac{x_3}{\varepsilon}), \frac{1}{\varepsilon} w_3(x')\right) \quad \text{for a.e. } x = (x', x_3) \in \Omega_\varepsilon, \quad (1.2)$$

where  $w \in W^{1,2}(\Omega; \mathbb{R}^3)$  is a Kirchhoff-Love displacement, that is,  $w_3 \in W^{2,2}(\omega)$  and there exists  $\bar{w} \in W^{1,2}(\omega; \mathbb{R}^2)$  such that

$$w(x) = (\bar{w}_1(x') - x_3 \partial_1 w_3(x'), \bar{w}_2(x') - x_3 \partial_2 w_3(x'), w_3(x')) \quad \text{for a.e. } x = (x', x_3) \in \Omega. \quad (1.3)$$

More general boundary conditions can be considered (see Remark 4.3).

Setting  $\Gamma_d := \gamma_d \times (-\frac{1}{2}, \frac{1}{2})$ , we show that the  $\Gamma$ -limit of  $\mathcal{E}_\varepsilon$  is finite only on the class  $\mathcal{A}_{KL}(w)$  of triples  $(u, e, p)$  such that  $u \in BD(\Omega)$ ,  $e \in L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ ,  $p \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ , and

$$Eu = e + p \quad \text{in } \Omega, \quad p = (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_d, \quad (1.4)$$

$$e_{i3} = 0 \quad \text{in } \Omega, \quad p_{i3} = 0 \quad \text{in } \Omega \cup \Gamma_d, \quad i = 1, 2, 3, \quad (1.5)$$

where  $\nu_{\partial\Omega}$  is the outer unit normal to  $\partial\Omega$ . On this class the  $\Gamma$ -limit is given by the functional

$$\mathcal{J}(u, e, p) := \frac{1}{2} \int_{\Omega} \mathbb{C}_r e : e dx + \mathcal{H}_r(p)$$

where

$$\mathcal{H}_r(p) := \int_{\Omega \cup \Gamma_d} H_r\left(\frac{dp}{d|p|}\right) d|p|$$

and the tensor  $\mathbb{C}_r$  and the function  $H_r$  are defined through pointwise minimization formulas (see (4.1), (4.4), and (4.6)).

Conditions (1.4)–(1.5) imply that  $u$  is a Kirchhoff-Love displacement in  $BD(\Omega)$ , that is,  $u_3$  belongs to the space  $BH(\omega)$  of functions with bounded Hessian and there exists  $\bar{u} \in BD(\omega)$  such that

$$u(x) = (\bar{u}_1(x') - x_3 \partial_1 u_3(x'), \bar{u}_2(x') - x_3 \partial_2 u_3(x'), u_3(x')) \quad \text{for a.e. } x = (x', x_3) \in \Omega.$$

Moreover,

$$(Eu)_{\alpha\beta} = (E\bar{u})_{\alpha\beta} - x_3 \partial_{\alpha\beta}^2 u_3 \quad \text{for } \alpha, \beta = 1, 2.$$

We note that the averaged tangential displacement  $\bar{u}$  may exhibit jump discontinuities, while, because of the embedding of  $BH(\omega)$  into  $C(\bar{\omega})$ , the normal displacement  $u_3$  is continuous, but its gradient may have jump discontinuities. Moreover, the second equality in (1.4), together with the second condition in (1.5), implies that  $u_3$  satisfies the boundary condition  $u_3 = w_3$  on  $\gamma_d$ . In particular, in the limit model slip surfaces are vertical surfaces whose projection on  $\omega$  is the union of the jump set of  $\bar{u}$  and the jump set of  $\nabla u_3$ .

We also remark that conditions (1.4)–(1.5) do not imply that  $e$  and  $p$  are affine with respect to the  $x_3$  variable. Therefore, in contrast with the case of linearized elasticity [5, 7], the limit functional  $\mathcal{J}$  cannot be in general expressed in terms of two-dimensional quantities only. A precise characterization of conditions (1.4)–(1.5) in terms of the moments of  $e$  and  $p$  is given in Proposition 3.5.

The main technical ingredient in the  $\Gamma$ -convergence result is Theorem 3.9, which ensures the density of smooth enough triples in the class  $\mathcal{A}_{KL}(w)$  and is the key argument in the construction of a recovery sequence (Theorem 4.5). A first difficulty in the proof of Theorem 3.9 is due to the fact that one has to preserve the Kirchhoff-Love structure. This can be done by mollifying separately the Kirchhoff-Love components  $\bar{u}$  and  $u_3$  of  $u$  and the zeroth and first order moments of  $e$  and  $p$ . A more delicate issue comes from the fact that the boundary conditions are imposed on a portion of the lateral boundary of  $\Omega$  and not on the whole  $\partial\omega \times (-\frac{1}{2}, \frac{1}{2})$ . In particular, to have that mollifications satisfy the boundary datum, we need first to approximate  $u_3$  in such a way that the equality  $u_3 = w_3$  holds on an open subset of  $\partial\omega$  strictly containing  $\gamma_d$ . To do that we use in a crucial way that the boundary of  $\omega$  is of class  $C^2$ , as well as the additional assumption that the relative boundary of  $\gamma_d$  in  $\partial\omega$  is made of two points. A similar approximation has to be applied to  $\bar{u}$  and to the moments of  $e$  and  $p$  to guarantee the strict convergence of the full plastic strains in the sense of measures, which is needed to ensure convergence of the energies.

A related dimension reduction problem for perfectly plastic plates in the stationary case has been studied in [32]. In that paper the elastic strain  $e$  is assumed a priori to coincide with  $\text{tr} Eu = \text{div} u$  and the plastic strain with the deviatoric part of  $Eu$ . Moreover, the set  $K$  is supposed to be symmetric with respect to the origin. This last assumption allows one, via minimization arguments, to express the  $\Gamma$ -limit in terms of the normal displacement  $u_3$  and the first order moments of  $e$  and  $p$ , only, so that the limit model turns out to be two-dimensional. Another crucial difference is that in [32] the boundary conditions are prescribed on the whole lateral boundary of  $\Omega_\varepsilon$  and, as explained above, this makes the approximation arguments much easier.

In Section 5 we introduce time and study the convergence of quasistatic evolutions. We prescribe on  $\Omega_\varepsilon$  a boundary datum  $w^\varepsilon(t)$  of the form (1.2)–(1.3) and we consider a sequence of initial data  $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon)$  converging in a suitable sense to a limit triple  $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$ . In Theorem 5.4 we show that, if for every  $\varepsilon > 0$  the triple  $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$  is a quasistatic evolution in the sense of (qs1)–(qs2) for the boundary datum  $w^\varepsilon(t)$  and the initial datum  $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon)$ , then, up to a suitable scaling,  $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$  converges, as  $\varepsilon \rightarrow 0$ , to a limit triple  $(u(t), e(t), p(t))$  that satisfies:

(qs0)<sub>r</sub> *initial condition*:  $(u(0), e(0), p(0)) = (u_0, e_0, p_0)$ ;

(qs1)<sub>r</sub> *global stability*: for every  $t \in [0, T]$   $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$  and minimizes

$$\frac{1}{2} \int_{\Omega} \mathbb{C}_r f : f \, dx + \mathcal{H}_r(q - p(t))$$

among all triples  $(v, f, q)$  in  $\mathcal{A}_{KL}(w(t))$ ;

(qs2)<sub>r</sub> *energy balance*: for every  $t \in [0, T]$

$$\frac{1}{2} \int_{\Omega} \mathbb{C}_r e(t) : e(t) \, dx + \int_0^t \mathcal{H}_r(\dot{p}(s)) \, ds = \frac{1}{2} \int_{\Omega} \mathbb{C}_r e(0) : e(0) \, dx + \int_0^t \int_{\Omega} \mathbb{C}_r e(s) : E\dot{w}(s) \, dx \, ds.$$

We call a triple satisfying (qs1)<sub>r</sub>–(qs2)<sub>r</sub> a reduced quasistatic evolution.

The proof of Theorem 5.4 mainly relies on the  $\Gamma$ -convergence result of Section 4. Even if the abstract theory of [28] cannot be directly applied, we follow the general scheme proposed in that paper. In particular, the role of the so-called joint recovery sequence is played in our case by the recovery sequence provided by Theorem 4.5.

In the last part of the paper we discuss some properties of reduced quasistatic evolutions. We show three equivalent formulations in rate form (Theorem 6.13). In all of them the

global stability condition is replaced by a system of two equilibrium conditions, one for the stretching component of the stress and the other for the bending component. These two components are coupled in the energy balance, which is rephrased in the three different formulations in terms of a maximal dissipation principle, of a flow rule in a weak form, and of a variational inequality for the stress (analogous to the formulation considered in [35] in the case of three-dimensional perfect plasticity), respectively. To prove these results we define a suitable notion of duality between stresses and plastic strains in the footsteps of [21] and [10].

In the last subsection we focus on two examples, where a reduced quasistatic evolution can be characterized in terms of two-dimensional quantities only. In particular, in Proposition 6.16 we show that, if the set  $K$  is symmetric with respect to the origin and the boundary datum and the initial data are properly chosen, our notion of reduced quasistatic evolution coincides with that studied in [8, 12, 13].

The paper is organised as follows: in Section 2 we recall some preliminary results and describe the formulation of the problem. In Section 3 we discuss the properties of Kirchhoff-Love admissible triples and prove some approximation results. Section 4 is devoted to the  $\Gamma$ -convergence result in the stationary case, while Section 5 concerns the convergence of quasistatic evolutions. Finally, in Section 6 we show some equivalent formulations of the reduced quasistatic evolution problem and discuss some examples.

## 2. PRELIMINARIES AND SETTING OF THE PROBLEM

**2.1. Mathematical preliminaries.** In this subsection we recall some notions from measure theory and from the theory of functions with bounded deformation and with bounded Hessian.

**Measures.** Given a Borel set  $B \subset \mathbb{R}^N$  and a finite dimensional Hilbert space  $X$ ,  $M_b(B; X)$  denotes the space of all bounded Borel measures on  $B$  with values in  $X$ , endowed with the norm  $\|\mu\|_{M_b} := |\mu|(B)$ , where  $|\mu| \in M_b(B; \mathbb{R})$  is the variation of the measure  $\mu$ . For every  $\mu \in M_b(B; X)$  we consider the Lebesgue decomposition  $\mu = \mu^a + \mu^s$ , where  $\mu^a$  is absolutely continuous with respect to the Lebesgue measure  $\mathcal{L}^N$  and  $\mu^s$  is singular with respect to  $\mathcal{L}^N$ . If  $\mu^s = 0$ , we always identify  $\mu$  with its density with respect to  $\mathcal{L}^N$ , which is a function in  $L^1(B; X)$ .

If the relative topology of  $B$  is locally compact, by Riesz representation Theorem the space  $M_b(B; X)$  can be identified with the dual of  $C_0(B; X)$ , which is the space of all continuous functions  $\varphi : B \rightarrow X$  such that the set  $\{|\varphi| \geq \delta\}$  is compact for every  $\delta > 0$ . The weak\* topology on  $M_b(B; X)$  is defined using this duality.

**Convex functions of measures.** For every  $\mu \in M_b(B; X)$  let  $d\mu/d|\mu|$  be the Radon-Nicodym derivative of  $\mu$  with respect to its variation  $|\mu|$ . Let  $H_0 : X \rightarrow [0, +\infty)$  be a convex and positively one-homogeneous function such that

$$r_0|\xi| \leq H_0(\xi) \leq R_0|\xi| \quad \text{for every } \xi \in X,$$

where  $r_0$  and  $R_0$  are two constants, with  $0 < r_0 \leq R_0$ . According to the theory of convex functions of measures, developed in [20], we introduce the nonnegative Radon measure  $H_0(\mu) \in M_b(B)$  defined by

$$H_0(\mu)(A) := \int_A H_0\left(\frac{d\mu}{d|\mu|}\right) d|\mu|$$

for every Borel set  $A \subset B$ . We also consider the functional  $\mathcal{H}_0 : M_b(B; X) \rightarrow [0, +\infty)$  defined by

$$\mathcal{H}_0(\mu) := H_0(\mu)(B) = \int_B H_0\left(\frac{d\mu}{d|\mu|}\right) d|\mu|$$

for every  $\mu \in M_b(B; X)$ . One can prove that  $H_0(\mu)$  coincides with the measure studied in [36, Chapter II, Section 4]. Hence,

$$\mathcal{H}_0(\mu) = \sup \left\{ \int_B \varphi : d\mu : \varphi \in C_0(B; X), \varphi(x) \in K_0 \text{ for every } x \in B \right\}, \quad (2.1)$$

where  $K_0 := \partial H_0(0)$  is the subdifferential of  $H_0$  at 0. Moreover,  $\mathcal{H}_0$  is lower semicontinuous on  $M_b(B; X)$  with respect to weak\* convergence.

**Functions with bounded deformation.** Let  $U$  be an open set of  $\mathbb{R}^N$ . The space  $BD(U)$  of functions with *bounded deformation* is the space of all functions  $u \in L^1(U; \mathbb{R}^N)$  whose symmetric gradient  $Eu := \text{sym } Du$  (in the sense of distributions) belongs to  $M_b(U; \mathbb{M}_{sym}^{N \times N})$ . It is easy to see that  $BD(U)$  is a Banach space endowed with the norm

$$\|u\|_{L^1} + \|Eu\|_{M_b}.$$

We say that a sequence  $(u^k)$  converges to  $u$  weakly\* in  $BD(U)$  if  $u^k \rightharpoonup u$  weakly in  $L^1(U; \mathbb{R}^N)$  and  $Eu^k \rightharpoonup Eu$  weakly\* in  $M_b(U; \mathbb{M}_{sym}^{N \times N})$ . Every bounded sequence in  $BD(U)$  has a weakly\* converging subsequence. If  $U$  is bounded and has Lipschitz boundary,  $BD(U)$  can be embedded into  $L^{N/(N-1)}(U; \mathbb{R}^N)$  and every function  $u \in BD(U)$  has a trace, still denoted by  $u$ , which belongs to  $L^1(\partial U; \mathbb{R}^N)$ . Moreover, if  $\Gamma$  is a nonempty open subset of  $\partial U$ , there exists a constant  $C > 0$ , depending on  $U$  and  $\Gamma$ , such that

$$\|u\|_{L^1(\Omega)} \leq C\|u\|_{L^1(\Gamma)} + C\|Eu\|_{M_b}. \quad (2.2)$$

(see [36, Chapter II, Proposition 2.4 and Remark 2.5]). For the general properties of the space  $BD(U)$  we refer to [36].

**Functions with bounded Hessian.** Let  $U$  be an open set of  $\mathbb{R}^N$ . The space  $BH(U)$  of functions with *bounded Hessian* is the space of all functions  $u \in W^{1,1}(U)$  whose Hessian  $D^2u$  belongs to  $M_b(U; \mathbb{M}_{sym}^{N \times N})$ . It is easy to see that  $BH(U)$  is a Banach space endowed with the norm

$$\|u\|_{L^1} + \|\nabla u\|_{L^1} + \|D^2u\|_{M_b}.$$

If  $U$  has the cone property, then  $BH(U)$  coincides with the space of functions in  $L^1(U)$  whose Hessian belongs to  $M_b(U; \mathbb{M}_{sym}^{N \times N})$ . If  $U$  is bounded and has Lipschitz boundary,  $BH(U)$  can be embedded into  $W^{1,N/(N-1)}(U)$ . If, in addition, the boundary of  $U$  is  $C^2$ , then  $BH(U)$  is embedded into  $C(\bar{U})$ , which is the space of all continuous functions on  $\bar{U}$ . Moreover, if  $U$  is bounded and has a  $C^2$  boundary, for every function  $u \in BH(U)$  one can define the traces of  $u$  and of  $\nabla u$ , still denoted by  $u$  and  $\nabla u$ ; they satisfy  $u \in W^{1,1}(\partial U)$ ,  $\nabla u \in L^1(\partial U; \mathbb{R}^N)$ , and  $\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau$  in  $L^1(\partial U)$ , where  $\tau$  is any tangent vector to  $\partial U$ . For the general properties of the space  $BH(U)$  we refer to [11].

**2.2. Setting of the problem.** Throughout the paper  $\omega$  is a bounded and connected open set of  $\mathbb{R}^2$  with a  $C^2$  boundary. We suppose that the boundary  $\partial\omega$  is partitioned into two disjoint open subsets  $\gamma_d, \gamma_n$  and their common boundary  $\partial|_{\partial\omega}\gamma_d = \partial|_{\partial\omega}\gamma_n$  (topological notions refer here to the relative topology of  $\partial\omega$ ). We assume that  $\gamma_d \neq \emptyset$  and that  $\partial|_{\partial\omega}\gamma_d = \{P_1, P_2\}$ , where  $P_1, P_2$  are two points in  $\partial\omega$ .

The reference configuration of the plate is given by the set

$$\Omega_\varepsilon := \omega \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right),$$

where  $\varepsilon > 0$ . We denote by  $\Gamma_\varepsilon$  the Dirichlet part of the boundary, given by  $\Gamma_\varepsilon := \gamma_d \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ , and by  $\nu_{\partial\Omega_\varepsilon}$  the outer unit normal to  $\partial\Omega_\varepsilon$ .

**The elasticity tensor.** Let  $\mathbb{C}$  be the *elasticity tensor*, considered as a symmetric positive definite linear operator  $\mathbb{C} : \mathbb{M}_{sym}^{3 \times 3} \rightarrow \mathbb{M}_{sym}^{3 \times 3}$  and let  $Q : \mathbb{M}_{sym}^{3 \times 3} \rightarrow [0, +\infty)$  be the quadratic form associated with  $\mathbb{C}$ , given by

$$Q(\xi) := \frac{1}{2}\mathbb{C}\xi : \xi \quad \text{for every } \xi \in \mathbb{M}_{sym}^{3 \times 3}.$$

It follows that there exist two constants  $r_{\mathbb{C}}$  and  $R_{\mathbb{C}}$ , with  $0 < r_{\mathbb{C}} \leq R_{\mathbb{C}}$ , such that

$$r_{\mathbb{C}}|\xi|^2 \leq Q(\xi) \leq R_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}_{sym}^{3 \times 3}. \quad (2.3)$$

These inequalities imply

$$|\mathbb{C}\xi| \leq 2R_{\mathbb{C}}|\xi| \quad \text{for every } \xi \in \mathbb{M}_{sym}^{3 \times 3}. \quad (2.4)$$

**The dissipation potential.** Let  $\mathbb{M}_D^{3 \times 3}$  be the space of all matrices in  $\mathbb{M}_{sym}^{3 \times 3}$  with zero trace. Let  $K$  be a closed convex set of  $\mathbb{M}_D^{3 \times 3}$  such that there exist two constants  $r_K$  and  $R_K$ , with  $0 < r_K \leq R_K$ , such that

$$\{\xi \in \mathbb{M}_D^{3 \times 3} : |\xi| \leq r_K\} \subset K \subset \{\xi \in \mathbb{M}_D^{3 \times 3} : |\xi| \leq R_K\}.$$

The boundary of  $K$  is interpreted as the *yield surface*. The *plastic dissipation potential* is given by the support function  $H : \mathbb{M}_D^{3 \times 3} \rightarrow [0, +\infty)$  of  $K$ , defined as

$$H(\xi) := \sup_{\sigma \in K} \sigma : \xi.$$

It follows that  $H$  is a convex and positively one-homogeneous function such that

$$r_K|\xi| \leq H(\xi) \leq R_K|\xi| \quad \text{for every } \xi \in \mathbb{M}_D^{3 \times 3}. \quad (2.5)$$

In particular,  $H$  satisfies the triangle inequality

$$H(\xi + \zeta) \leq H(\xi) + H(\zeta) \quad \text{for every } \xi, \zeta \in \mathbb{M}_D^{3 \times 3}. \quad (2.6)$$

**Admissible triples and energy.** On  $\Gamma_\varepsilon$  we prescribe a boundary datum  $w^\varepsilon \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3)$  of the following form:

$$w^\varepsilon(z) := (\bar{w}_1(z') - \frac{z_3}{\varepsilon} \partial_1 w_3(z'), \bar{w}_2(z') - \frac{z_3}{\varepsilon} \partial_2 w_3(z'), \frac{1}{\varepsilon} w_3(z')) \quad \text{for a.e. } z = (z', z_3) \in \Omega_\varepsilon,$$

where  $\bar{w}_\alpha \in W^{1,2}(\omega)$ ,  $\alpha = 1, 2$ , and  $w_3 \in W^{2,2}(\omega)$ . The *set of admissible displacements and strains* for the boundary datum  $w^\varepsilon$  is denoted by  $\mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$  and is defined as the class of all triples  $(v, f, q) \in BD(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon; \mathbb{M}_{sym}^{3 \times 3}) \times M_b(\Omega_\varepsilon; \mathbb{M}_D^{3 \times 3})$  satisfying

$$\begin{aligned} Ev &= f + q \quad \text{in } \Omega_\varepsilon, \\ q &= (w^\varepsilon - v) \odot \nu_{\partial\Omega_\varepsilon} \mathcal{H}^2 \quad \text{on } \Gamma_\varepsilon, \end{aligned}$$

where  $\odot$  stands for the symmetrized tensor product and  $\mathcal{H}^2$  is the two-dimensional Hausdorff measure. The function  $v$  represents the *displacement* of the plate, while  $f$  and  $q$  are called the *elastic* and *plastic strain*, respectively.

For every admissible triple  $(v, f, q) \in \mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$  we define the associated *energy* as

$$\mathcal{E}_\varepsilon(v, f, q) := \int_{\Omega_\varepsilon} Q(f(z)) dz + \int_{\Omega_\varepsilon \cup \Gamma_\varepsilon} H\left(\frac{dq}{d|q|}\right) d|q|. \quad (2.7)$$

The first term represents the elastic energy, while the second term accounts for plastic dissipation.

**2.3. The rescaled problem.** As usual in dimension reduction problems, it is convenient to perform a change of variable in such a way to rewrite the system on a fixed domain independent of  $\varepsilon$ . To this purpose, we set

$$\Omega := \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \Gamma_d := \gamma_d \times \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \Gamma_n := \gamma_n \times \left(-\frac{1}{2}, \frac{1}{2}\right),$$

and we denote by  $\nu_{\partial\Omega}$  the outer unit normal to  $\partial\Omega$ . We consider the change of variable  $\psi_\varepsilon : \bar{\Omega} \rightarrow \bar{\Omega}_\varepsilon$  given by

$$\psi_\varepsilon(x) := (x', \varepsilon x_3) \quad \text{for every } x = (x', x_3) \in \bar{\Omega}$$

and the linear operator  $\Lambda_\varepsilon : \mathbb{M}_{sym}^{3 \times 3} \rightarrow \mathbb{M}_{sym}^{3 \times 3}$  given by

$$\Lambda_\varepsilon \xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \frac{1}{\varepsilon} \xi_{13} \\ \xi_{21} & \xi_{22} & \frac{1}{\varepsilon} \xi_{23} \\ \frac{1}{\varepsilon} \xi_{31} & \frac{1}{\varepsilon} \xi_{32} & \frac{1}{\varepsilon^2} \xi_{33} \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}_{sym}^{3 \times 3}.$$

To any triple  $(v, f, q) \in \mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$  we associate a triple  $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  defined as follows:

$$u := (v_1 \circ \psi_\varepsilon, v_2 \circ \psi_\varepsilon, \varepsilon v_3 \circ \psi_\varepsilon), \quad e := \Lambda_\varepsilon^{-1} f \circ \psi_\varepsilon, \quad p := \frac{1}{\varepsilon} \Lambda_\varepsilon^{-1} \psi_\varepsilon^\#(q).$$

Here the measure  $\psi_\varepsilon^\#(q) \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3})$  is the pull-back measure of  $q$ , satisfying

$$\int_{\Omega \cup \Gamma_d} \varphi : d\psi_\varepsilon^\#(q) = \int_{\Omega_\varepsilon \cup \Gamma_\varepsilon} \varphi \circ \psi_\varepsilon^{-1} : dq \quad \text{for every } \varphi \in C_0(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3}).$$

According to this change of variable we have

$$\mathcal{E}_\varepsilon(v, f, q) = \varepsilon \mathcal{Q}(\Lambda_\varepsilon e) + \varepsilon \mathcal{H}(\Lambda_\varepsilon p),$$

where

$$\mathcal{Q}(\Lambda_\varepsilon e) := \int_{\Omega} \mathcal{Q}(\Lambda_\varepsilon e(x)) dx, \quad \mathcal{H}(\Lambda_\varepsilon p) := \int_{\Omega \cup \Gamma_d} H\left(\frac{d\Lambda_\varepsilon p}{|d\Lambda_\varepsilon p|}\right) d|\Lambda_\varepsilon p|.$$

We also introduce the scaled Dirichlet boundary datum  $w \in W^{1,2}(\Omega; \mathbb{R}^3)$ , given by

$$w(x) := (\bar{w}_1(x') - x_3 \partial_1 w_3(x'), \bar{w}_2(x') - x_3 \partial_2 w_3(x'), w_3(x')) \quad \text{for every } x \in \Omega.$$

From the definition of the class  $\mathcal{A}(\Omega_\varepsilon, w^\varepsilon)$  it immediately follows that the scaled triple  $(u, e, p)$  satisfies the equalities

$$Eu = e + p \quad \text{in } \Omega, \tag{2.8}$$

$$p = (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_d, \tag{2.9}$$

$$p_{11} + p_{22} + \frac{1}{\varepsilon^2} p_{33} = 0 \quad \text{in } \Omega \cup \Gamma_d. \tag{2.10}$$

We are thus led to introduce the class  $\mathcal{A}_\varepsilon(w)$  of all triples  $(u, e, p) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  satisfying (2.8)–(2.10), and to define the functional

$$\mathcal{J}_\varepsilon(u, e, p) := \mathcal{Q}(\Lambda_\varepsilon e) + \mathcal{H}(\Lambda_\varepsilon p) \tag{2.11}$$

for every  $(u, e, p) \in \mathcal{A}_\varepsilon(w)$ . In the following we shall study the asymptotic behaviour of the minimizers of  $\mathcal{J}_\varepsilon$  and of the quasistatic evolution associated with  $\mathcal{J}_\varepsilon$ , as  $\varepsilon \rightarrow 0$ .

### 3. THE CLASS OF KIRCHHOFF-LOVE ADMISSIBLE TRIPLES

In this section we introduce the class of Kirchhoff-Love admissible triples, which will be the domain of the minimum problem describing the asymptotic behaviour of minimizers of  $\mathcal{J}_\varepsilon$ , as  $\varepsilon \rightarrow 0$ , and the space where the limiting quasistatic evolution takes place. We prove some approximation results, which will be crucial in the proofs of both convergence results. To this purpose we first define the set of Kirchhoff-Love displacements as

$$KL(\Omega) := \{u \in BD(\Omega) : (Eu)_{i3} = 0 \quad \text{for } i = 1, 2, 3\}.$$

**Remark 3.1.** Note that  $u \in KL(\Omega)$  if and only if  $u_3 \in BH(\omega)$  and there exists  $\bar{u} \in BD(\omega)$  such that

$$u_\alpha = \bar{u}_\alpha - x_3 \partial_\alpha u_3, \quad \alpha = 1, 2.$$

In particular, if  $u \in KL(\Omega)$ , then  $(Eu)_{\alpha\beta} = (E\bar{u})_{\alpha\beta} - x_3 \partial_\alpha^2 u_3$  for  $\alpha, \beta = 1, 2$ . If, in addition,  $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ , then  $\bar{u} \in W^{1,p}(\omega; \mathbb{R}^2)$  and  $u_3 \in W^{2,p}(\omega)$ . We call  $\bar{u}, u_3$  the *Kirchhoff-Love components* of  $u$ .

For every  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  we define the class  $\mathcal{A}_{KL}(w)$  of Kirchhoff-Love admissible triples for the boundary datum  $w$  as the set of all triples  $(u, e, p) \in KL(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  satisfying

$$\begin{aligned} Eu &= e + p \quad \text{in } \Omega, & p &= (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_d, \\ e_{i3} &= 0 \quad \text{in } \Omega, & p_{i3} &= 0 \quad \text{in } \Omega \cup \Gamma_d, \quad i = 1, 2, 3. \end{aligned}$$



**Remark 3.2.** The space

$$\{\xi \in \mathbb{M}_{sym}^{3 \times 3} : \xi_{i3} = 0 \text{ for } i = 1, 2, 3\}$$

is canonically isomorphic to  $\mathbb{M}_{sym}^{2 \times 2}$ . Therefore, in the following, given a triple  $(u, e, p) \in \mathcal{A}_{KL}(w)$  we will usually identify  $e$  with a function in  $L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  and  $p$  with a measure in  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$ .

We notice that the set  $\mathcal{A}_{KL}(w)$  is always nonempty as it contains the triple  $(w, Ew, 0)$ . We also point out that if  $(u, e, p) \in \mathcal{A}_{KL}(w)$ , then in general one cannot conclude that  $e$  and  $p$  are affine in the  $x_3$  variable. However, some conditions on the structure of  $e$  and  $p$  can be deduced. To this purpose, we introduce the following definitions.

**Definition 3.3.** Let  $f \in L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ . We denote by  $\bar{f}, \hat{f} \in L^2(\omega; \mathbb{M}_{sym}^{3 \times 3})$  and by  $f_\perp \in L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  the following orthogonal components (in the sense of  $L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ ) of  $f$ :

$$\bar{f}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x', x_3) dx_3, \quad \hat{f}(x') := 12 \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 f(x', x_3) dx_3$$

for a.e.  $x' \in \omega$ , and

$$f_\perp(x) := f(x) - \bar{f}(x') - x_3 \hat{f}(x')$$

for a.e.  $x \in \Omega$ . The component  $\bar{f}$  is called the *zero-th order moment* of  $f$ , while  $\hat{f}$  is called the *first order moment* of  $f$ .

**Definition 3.4.** Let  $q \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ . The *zero-th order moment* of  $q$  is the measure  $\bar{q} \in M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  defined by

$$\int_{\omega \cup \gamma_d} \varphi : d\bar{q} := \int_{\Omega \cup \Gamma_d} \varphi : dq$$

for every  $\varphi \in C_0(\omega \cup \gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ , while the *first order moment* of  $q$  is the measure  $\hat{q} \in M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  defined by

$$\int_{\omega \cup \gamma_d} \varphi : d\hat{q} := 12 \int_{\Omega \cup \Gamma_d} x_3 \varphi : dq$$

for every  $\varphi \in C_0(\omega \cup \gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ . We also define  $q_\perp \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  as the measure given by

$$q_\perp := q - \bar{q} \otimes \mathcal{L}^1 - x_3 \hat{q} \otimes \mathcal{L}^1,$$

where the symbol  $\otimes$  denotes the generalized product of measures (see, e.g., [4, Definition 2.27]).

With these definitions at hand one can easily prove the following characterization of the class  $\mathcal{A}_{KL}(\Omega)$ .

**Proposition 3.5.** *Let  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and  $(u, e, p) \in KL(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  with  $e_{i3} = 0$  in  $\Omega$  and  $p_{i3} = 0$  in  $\Omega \cup \Gamma_d$  for  $i = 1, 2, 3$ . Let  $\bar{u} \in BD(\omega)$ ,  $u_3 \in BH(\omega)$ , and  $\bar{w} \in W^{1,2}(\omega; \mathbb{R}^2)$ ,  $w_3 \in W^{2,2}(\omega)$  be the Kirchhoff-Love components of  $u$  and  $w$ , respectively. Finally, let  $\bar{e}, \hat{e} \in L^2(\omega; \mathbb{M}_{sym}^{3 \times 3})$ ,  $e_\perp \in L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ ,  $\bar{p}, \hat{p} \in M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ , and  $p_\perp \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  be the moments of  $e$  and  $p$ , according to Definitions 3.3 and 3.4. Then  $(u, e, p) \in \mathcal{A}_{KL}(\Omega)$  if and only if the following three conditions are satisfied:*

- (i)  $E\bar{u} = \bar{e} + \bar{p}$  in  $\omega$  and  $\bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega} \mathcal{H}^1$  on  $\gamma_d$ ;
- (ii)  $D^2 u_3 = -(\hat{e} + \hat{p})$  in  $\omega$ ,  $u_3 = w_3$  on  $\gamma_d$ , and  $\hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega} \mathcal{H}^1$  on  $\gamma_d$ ;
- (iii)  $p_\perp = -e_\perp$  in  $\Omega$  and  $p_\perp = 0$  on  $\Gamma_d$ ,

where we have identified  $\bar{e}, \hat{e}$  with functions in  $L^2(\omega; \mathbb{M}_{sym}^{2 \times 2})$  and  $\bar{p}, \hat{p}$  with measures in  $M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{2 \times 2})$ . Here  $\nu_{\partial\omega}$  denotes the outer unit normal to  $\partial\omega$  and  $\mathcal{H}^1$  is the one-dimensional Hausdorff measure.

We now prove some approximation results for Kirchhoff-Love admissible triples. We first need a technical lemma.

**Lemma 3.6.** *Let  $H_0 : \mathbb{M}_{sym}^{2 \times 2} \rightarrow [0, +\infty)$  be a convex and positively one-homogeneous function such that*

$$r_0|\xi| \leq H_0(\xi) \leq R_0|\xi| \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2},$$

where  $r_0$  and  $R_0$  are two constants, with  $0 < r_0 \leq R_0$ . Let  $\mu \in M_b(\bar{\omega} \times (-\frac{1}{2}, \frac{1}{2}); \mathbb{M}_{sym}^{2 \times 2})$  be such that  $\mu = \mu_{x_3} \otimes \mathcal{L}^1$ , where the measure-valued map

$$x_3 \in (-\frac{1}{2}, \frac{1}{2}) \mapsto \mu_{x_3} \in M_b(\bar{\omega}; \mathbb{M}_{sym}^{2 \times 2})$$

is  $\mathcal{L}^1$ -measurable in the sense of [4, Definition 2.25]. Assume that  $|\mu_{x_3}|(\partial\omega) = 0$  for  $\mathcal{L}^1$ -a.e.  $x_3 \in (-\frac{1}{2}, \frac{1}{2})$ . Let  $(\rho_\delta) \subset C_c^\infty(\mathbb{R}^2)$  be a sequence of mollifiers, with  $\text{supp } \rho_\delta \subset B_\delta(0)$ . Then

$$\lim_{\delta \rightarrow 0} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \int_{\omega} H_0(\rho_\delta * \mu_{x_3}) dx' \right) dx_3 = H_0(\mu)(\Omega).$$

*Proof.* We first observe that, since

$$\frac{d\mu}{d|\mu|}(x) = \frac{d\mu_{x_3}}{d|\mu_{x_3}|}(x')$$

for  $|\mu|$ -a.e.  $x \in \Omega$ , the following equality holds:

$$H_0(\mu)(\Omega) = \int_{-\frac{1}{2}}^{\frac{1}{2}} H_0(\mu_{x_3})(\omega) dx_3.$$

We extend  $\mu_{x_3}$  to 0 outside  $\bar{\omega}$ , so that the convolutions  $\rho_\delta * \mu_{x_3}$  are well defined on  $\mathbb{R}^2$ . By Jensen inequality (see, e.g., [20, Theorem 1]) and Fubini-Tonelli Theorem we obtain

$$\begin{aligned} \int_{\omega} H_0(\rho_\delta * \mu_{x_3}) dx' &= \int_{\omega} H_0 \left( \int_{\mathbb{R}^2} \rho_\delta(x' - y') d\mu_{x_3}(y') \right) dx' \\ &\leq \int_{\omega} \int_{\mathbb{R}^2} \rho_\delta(x' - y') dH_0(\mu_{x_3})(y') dx' \leq H_0(\mu_{x_3})(\omega) \end{aligned}$$

for  $\mathcal{L}^1$ -a.e.  $x_3 \in (-\frac{1}{2}, \frac{1}{2})$ . By integrating with respect to  $x_3$  we deduce

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \int_{\omega} H_0(\rho_\delta * \mu_{x_3}) dx' \right) dx_3 \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} H_0(\mu_{x_3})(\omega) dx_3 = H_0(\mu)(\Omega).$$

On the other hand, we have that  $\rho_\delta * \mu_{x_3} \rightharpoonup \mu_{x_3}$  weakly\* in  $M_b(\omega; \mathbb{M}_{sym}^{2 \times 2})$  for  $\mathcal{L}^1$ -a.e.  $x_3 \in (-\frac{1}{2}, \frac{1}{2})$ . Hence, by lower semicontinuity

$$H_0(\mu_{x_3})(\omega) \leq \liminf_{\delta \rightarrow 0} \int_{\omega} H_0(\rho_\delta * \mu_{x_3}) dx'$$

for  $\mathcal{L}^1$ -a.e.  $x_3 \in (-\frac{1}{2}, \frac{1}{2})$ . Integration with respect to  $x_3$  and Fatou's Lemma yield the thesis.  $\square$

The next lemma allows one to approximate in energy any Kirchhoff-Love admissible triple by means of triples  $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(\Omega)$  with  $u^\varepsilon$  smooth. The proof of this result is based on an adaptation of [11, Proposition 1.4].

**Lemma 3.7.** *Let  $H_0$  be as in Lemma 3.6. Let  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and let  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . Then, there exists a sequence of triples  $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(w)$  such that*

$$u^\varepsilon \in C^\infty(\Omega; \mathbb{R}^3) \cap W^{1,1}(\Omega; \mathbb{R}^3)$$

and the following properties hold:

$$u^\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \tag{3.1}$$

$$e^\varepsilon \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \tag{3.2}$$

$$p^\varepsilon \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}), \tag{3.3}$$

$$\mathcal{H}_0(p^\varepsilon) \rightarrow \mathcal{H}_0(p). \tag{3.4}$$

*Proof. Step 1.* We first show that any triple  $(u, e, p) \in \mathcal{A}_{KL}(w)$  can be approximated in the sense of (3.1)–(3.4) by a sequence of triples  $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(w)$  with  $u^\varepsilon \in C^\infty(\Omega; \mathbb{R}^3) \cap BD(\Omega)$ .

Let  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and let  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . By Proposition 3.5 the Kirchhoff-Love components  $\bar{u} \in BD(\omega)$  and  $u_3 \in BH(\omega)$  of  $u$  satisfy

$$\begin{aligned} E\bar{u} &= \bar{e} + \bar{p} \text{ in } \omega, \quad \bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega} \mathcal{H}^1 \text{ on } \gamma_d, \\ D^2 u_3 &= -(\hat{e} + \hat{p}) \text{ in } \omega, \quad u_3 = w_3 \text{ on } \gamma_d, \quad \hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega} \mathcal{H}^1 \text{ on } \gamma_d, \end{aligned}$$

where  $\bar{e}, \hat{e}$  have been identified with functions in  $L^2(\omega; \mathbb{M}_{sym}^{2 \times 2})$  and  $\bar{p}, \hat{p}$  with measures in  $M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{2 \times 2})$ . Moreover,

$$p_\perp = -e_\perp \text{ in } \Omega, \quad p_\perp = 0 \text{ on } \Gamma_d.$$

Fix  $\varepsilon > 0$ . Let  $r > 0$  be such that the set

$$\omega_0 := \{x' \in \omega : \text{dist}(x', \partial\omega) > \frac{1}{r}\}$$

satisfies

$$|\bar{p}|(\omega \setminus \omega_0) + |\hat{p}|(\omega \setminus \omega_0) \leq \varepsilon. \quad (3.5)$$

We set

$$\begin{aligned} \omega_j &:= \{x' \in \omega : \text{dist}(x', \partial\omega) > \frac{1}{j+r}\} \quad \text{for every } j \in \mathbb{N}, \\ A_j &:= \omega_{j+1} \setminus \bar{\omega}_{j-1} \quad \text{for } j \geq 2, \quad A_1 := \omega_2. \end{aligned}$$

Let  $\{\varphi_j\}$  be a  $C^\infty$  partition of unity for  $\omega$  subordinate to the covering  $\{A_j\}$ , that is,  $\varphi_j \in C_c^\infty(A_j)$ ,  $0 \leq \varphi_j \leq 1$  for every  $j \in \mathbb{N}$ , and

$$\sum_{j=1}^{\infty} \varphi_j = 1 \quad \text{in } \omega. \quad (3.6)$$

Let  $(\rho_\delta)$  be a sequence of convolution kernels with  $\rho_\delta \in C_0^\infty(B_\delta(0))$  for every  $\delta > 0$ . For every  $j \in \mathbb{N}$  we choose  $\delta_j$  such that

$$\{x' \in \omega : \text{dist}(x', \text{supp } \varphi_j) < \delta_j\} \subset\subset A_j, \quad (3.7)$$

$$\|(\varphi_j u_3) * \rho_{\delta_j} - \varphi_j u_3\|_{W^{1,2}} + \|(\varphi_j \bar{u}) * \rho_{\delta_j} - \varphi_j \bar{u}\|_{L^2} \leq \varepsilon 2^{-j}, \quad (3.8)$$

$$\|(\varphi_j \bar{e}) * \rho_{\delta_j} - \varphi_j \bar{e}\|_{L^2} + \|(\varphi_j \hat{e}) * \rho_{\delta_j} - \varphi_j \hat{e}\|_{L^2} \leq \varepsilon 2^{-j}, \quad (3.9)$$

$$\|(u_3 D^2 \varphi_j) * \rho_{\delta_j} - u_3 D^2 \varphi_j\|_{L^2} + \|(\nabla u_3 \odot \nabla \varphi_j) * \rho_{\delta_j} - \nabla u_3 \odot \nabla \varphi_j\|_{L^2} \leq \varepsilon 2^{-j}, \quad (3.10)$$

$$\|(\bar{u} \odot \nabla \varphi_j) * \rho_{\delta_j} - \bar{u} \odot \nabla \varphi_j\|_{L^2} \leq \varepsilon 2^{-j}. \quad (3.11)$$

Moreover, we extend the function  $\varphi_j e_\perp$  to 0 outside  $A_j \times (-\frac{1}{2}, \frac{1}{2})$  and consider the convolution

$$(\varphi_j e_\perp) * \rho_{\delta_j}(x) := \int_{\mathbb{R}^2} \rho_{\delta_j}(x' - y') \varphi_j(y') e_\perp(y', x_3) dy'$$

defined for every  $x \in \Omega$ . Since  $\varphi_j p = \varphi_j(\bar{p} + x_3 \hat{p} - e_\perp(\cdot, x_3)) \otimes \mathcal{L}^1$ , by Lemma 3.6 we can assume  $\delta_j$  to be so small that

$$\|(\varphi_j e_\perp) * \rho_{\delta_j} - \varphi_j e_\perp\|_{L^2(\Omega)} \leq \varepsilon 2^{-j}, \quad (3.12)$$

$$\left| \int_{\Omega} H_0((\varphi_j \bar{p}) * \rho_{\delta_j} + x_3(\varphi_j \hat{p}) * \rho_{\delta_j} - (\varphi_j e_\perp) * \rho_{\delta_j}) dx - H_0(\varphi_j p)(\Omega) \right| \leq \varepsilon 2^{-j}. \quad (3.13)$$

Finally, we define

$$\begin{aligned} \bar{u}^\varepsilon &:= \sum_{j=1}^{\infty} (\varphi_j \bar{u}) * \rho_{\delta_j}, \quad u_3^\varepsilon := \sum_{j=1}^{\infty} (\varphi_j u_3) * \rho_{\delta_j}, \quad u_\alpha^\varepsilon := \bar{u}_\alpha^\varepsilon - x_3 \partial_\alpha u_3^\varepsilon \quad (\alpha = 1, 2), \\ e^\varepsilon &:= \bar{e}^\varepsilon + x_3 \hat{e}^\varepsilon + e_\perp^\varepsilon, \end{aligned}$$

where

$$\bar{e}^\varepsilon := \sum_{j=1}^{\infty} [(\varphi_j \bar{e}) * \rho_{\delta_j} + (\bar{u} \odot \nabla \varphi_j) * \rho_{\delta_j}],$$

$$\hat{e}^\varepsilon := \sum_{j=1}^{\infty} [(\varphi_j \hat{e}) * \rho_{\delta_j} - (u_3 D^2 \varphi_j) * \rho_{\delta_j} - 2(\nabla u_3 \odot \nabla \varphi_j) * \rho_{\delta_j}], \quad e_\perp^\varepsilon := \sum_{j=1}^{\infty} (\varphi_j e_\perp) * \rho_{\delta_j},$$

and

$$p^\varepsilon := \begin{cases} \sum_{j=1}^{\infty} [(\varphi_j \bar{p}) * \rho_{\delta_j} + x_3(\varphi_j \hat{p}) * \rho_{\delta_j} - (\varphi_j e_\perp) * \rho_{\delta_j}] & \text{in } \Omega, \\ (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 & \text{on } \Gamma_d. \end{cases}$$

It is easy to see that  $\bar{u}^\varepsilon \in C^\infty(\omega; \mathbb{R}^2) \cap BD(\omega)$ ,  $u_3^\varepsilon \in C^\infty(\omega) \cap W^{2,1}(\omega)$ , hence  $u^\varepsilon \in C^\infty(\Omega; \mathbb{R}^3) \cap BD(\Omega)$ . Moreover,

$$E\bar{u}^\varepsilon = \bar{e}^\varepsilon + \bar{p}^\varepsilon \quad \text{and} \quad D^2 u_3^\varepsilon = -(\hat{e}^\varepsilon + \hat{p}^\varepsilon) \quad \text{in } \Omega \quad (3.14)$$

for every  $\varepsilon$ . Arguing as in [11, Proof of Proposition 1.4], one can also show that  $u_3^\varepsilon = u_3$ ,  $\nabla u_3^\varepsilon = \nabla u_3$ , and  $\bar{u}^\varepsilon = \bar{u}$  on  $\partial\omega$ . By Proposition 3.5 this implies that  $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(w)$ .

By (3.6) and (3.8) we deduce that

$$u^\varepsilon \rightarrow u \quad \text{strongly in } L^2(\Omega; \mathbb{R}^3), \quad (3.15)$$

while by (3.9)–(3.12) we obtain (3.2).

To prove (3.3) it is enough to show that

$$\bar{p}^\varepsilon \rightharpoonup \bar{p} \quad \text{and} \quad \hat{p}^\varepsilon \rightharpoonup \hat{p} \quad \text{weakly* in } (C_b(\omega; \mathbb{M}_{sym}^{2 \times 2}))', \quad (3.16)$$

where  $C_b(\omega; \mathbb{M}_{sym}^{2 \times 2})$  is the space of all bounded functions in  $C(\omega; \mathbb{M}_{sym}^{2 \times 2})$ . Indeed, if (3.16) holds, for every  $\phi \in C_0(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  we have

$$\int_{\Omega \cup \Gamma_d} \phi : dp^\varepsilon = \int_{\omega} \bar{\phi} : d\bar{p}^\varepsilon + \frac{1}{12} \int_{\omega} \hat{\phi} : d\hat{p}^\varepsilon - \int_{\Omega} \phi_\perp : e_\perp^\varepsilon dx + \int_{\Gamma_d} \phi : ((w - u) \odot \nu_{\partial\Omega}) d\mathcal{H}^2,$$

where  $\bar{\phi}$ ,  $\hat{\phi}$ ,  $\phi_\perp$  are defined according to Definition 3.3. Convergence (3.3) follows now by (3.2) and (3.16).

We prove (3.16) for the sequence  $(\bar{p}^\varepsilon)$ , the same argument applies to  $(\hat{p}^\varepsilon)$ . By (3.2), (3.14), and (3.15) it is enough to check that

$$\limsup_{\varepsilon \rightarrow 0} \|\bar{p}^\varepsilon\|_{M_b(\omega)} \leq \|\bar{p}\|_{M_b(\omega)}. \quad (3.17)$$

Now, let  $\phi \in C_c^\infty(\omega; \mathbb{M}_{sym}^{2 \times 2})$  with  $\|\phi\|_\infty \leq 1$ . Denoting by  $\check{\rho}_{\delta_j}$  the function  $\check{\rho}_{\delta_j}(z') = \rho_{\delta_j}(-z')$  for every  $z' \in \mathbb{R}^2$  and for every  $j$ , we have

$$\begin{aligned} \left| \int_{\omega} \phi : d\bar{p}^\varepsilon \right| &\leq \left| \sum_{j=1}^{\infty} \int_{\omega} \phi(x') : \left( \int_{\omega} \varphi_j(y') \rho_{\delta_j}(x' - y') d\bar{p}(y') \right) dx' \right| \\ &= \left| \sum_{j=1}^{\infty} \int_{\omega} \varphi_j(\phi * \check{\rho}_{\delta_j}) : d\bar{p} \right| \\ &\leq \sum_{j=2}^{\infty} \int_{\omega} \varphi_j |\phi * \check{\rho}_{\delta_j}| |d\bar{p}| + \int_{\omega} \varphi_1 |\phi * \check{\rho}_{\delta_1}| |d\bar{p}| \\ &\leq |\bar{p}|(\omega \setminus \omega_0) + |\bar{p}|(\omega). \end{aligned}$$

Hence, (3.17) follows from (3.5). Therefore, we deduce (3.16), which in turn yields (3.3). Combining (3.2), (3.3), and (3.15), we also have (3.1).

It remains to prove (3.4). By (3.3) and by the lower semicontinuity of  $\mathcal{H}_0$  with respect to weak\* convergence in  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$ , we deduce that

$$\mathcal{H}_0(p) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{H}_0(p^\varepsilon).$$

To prove the opposite inequality, we notice that

$$\begin{aligned} \mathcal{H}_0(p^\varepsilon) &= H_0(p^\varepsilon)(\Omega) + H_0(p)(\Gamma_d) \\ &\leq \sum_{j=1}^{\infty} \int_{\Omega} H_0((\varphi_j \bar{p}) * \rho_{\delta_j} + x_3(\varphi_j \hat{p}) * \rho_{\delta_j} - (\varphi_j e_{\perp}) * \rho_{\delta_j}) dx + H_0(p)(\Gamma_d) \\ &\leq \sum_{j=1}^{\infty} H_0(\varphi_j p)(\Omega) + H_0(p)(\Gamma_d) + \varepsilon, \end{aligned}$$

by (3.13). Therefore,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \mathcal{H}_0(p^\varepsilon) &\leq \sum_{j=1}^{\infty} H_0(\varphi_j p)(\Omega) + H_0(p)(\Gamma_d) \\ &= \sum_{j=1}^{\infty} \int_{\Omega} \varphi_j(x') H_0\left(\frac{dp}{d|p|}(x)\right) d|p|(x) + H_0(p)(\Gamma_d) = \mathcal{H}_0(p), \end{aligned}$$

which completes the proof of (3.4) and of Step 1.

**Step 2.** To conclude the proof of the lemma we shall prove that any triple  $(u, e, p) \in \mathcal{A}_{KL}(w)$  with  $u \in C^\infty(\Omega; \mathbb{R}^3) \cap BD(\Omega)$  can be approximated in the sense of (3.1)–(3.4) by a sequence of triples  $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in \mathcal{A}_{KL}(w)$  with  $u^\varepsilon \in C^\infty(\Omega; \mathbb{R}^3) \cap W^{1,1}(\Omega; \mathbb{R}^3)$ .

Let  $(u, e, p) \in \mathcal{A}_{KL}(w)$  with  $u \in C^\infty(\Omega; \mathbb{R}^3) \cap BD(\Omega)$ . The Kirchhoff-Love components of  $u$  satisfy  $\bar{u} \in C^\infty(\omega; \mathbb{R}^2) \cap BD(\omega)$  and  $u_3 \in C^\infty(\omega) \cap W^{2,1}(\omega)$ . By [36, Chapter I, Proposition 1.3] and the regularity of  $\partial\omega$  we can construct a sequence  $(\bar{u}^\varepsilon) \subset C^\infty(\bar{\omega}; \mathbb{R}^2)$  such that

$$\bar{u}^\varepsilon \rightarrow \bar{u} \quad \text{strongly in } L^1(\omega; \mathbb{R}^2) \quad \text{and} \quad E\bar{u}^\varepsilon \rightarrow E\bar{u} \quad \text{strongly in } L^1(\omega; \mathbb{M}_{sym}^{2 \times 2}). \quad (3.18)$$

This implies, in particular, that  $\bar{u}^\varepsilon \rightarrow \bar{u}$  strongly in  $L^1(\Gamma_d; \mathbb{R}^2)$ . The sequence of triples  $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$  defined by

$$u_\alpha^\varepsilon := \bar{u}_\alpha^\varepsilon - x_3 \partial_\alpha u_3 \quad (\alpha = 1, 2), \quad u_3^\varepsilon := u_3, \quad e^\varepsilon := e,$$

and

$$p^\varepsilon := \begin{cases} E\bar{u}^\varepsilon - e - x_3 D^2 u_3 & \text{in } \Omega, \\ (w - u^\varepsilon) \odot \nu_{\partial\Omega} \mathcal{H}^2 & \text{on } \Gamma_d, \end{cases}$$

satisfies all the required properties.  $\square$

**Remark 3.8.** We observe that by (3.15) and (3.18) and the continuous embedding of  $BD(\omega)$  into  $L^2(\omega; \mathbb{R}^2)$  the approximating sequence  $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$  in Lemma 3.7 satisfies also

$$\bar{u}^\varepsilon \rightarrow \bar{u} \quad \text{strongly in } L^2(\omega; \mathbb{R}^2). \quad (3.19)$$

Moreover, the construction of  $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$  can be modified in such a way to satisfy also the following convergence properties:

$$\|E\bar{u}^\varepsilon\|_{L^1} \rightarrow \|E\bar{u}\|_{M_b}, \quad (3.20)$$

$$\|D^2 u_3^\varepsilon\|_{L^1} \rightarrow \|D^2 u_3\|_{M_b}, \quad (3.21)$$

$$u_3^\varepsilon \rightarrow u_3 \quad \text{in } C(\bar{\omega}). \quad (3.22)$$

Indeed, let us denote by  $\bar{p}^a$ ,  $\hat{p}^a$  and  $\bar{p}^s$ ,  $\hat{p}^s$  the absolutely continuous parts and the singular parts of  $\bar{p}$  and  $\hat{p}$ , respectively. In Step 1 we can choose  $\delta_j$  in such a way to satisfy also the following estimates:

$$\|(\varphi_j \bar{p}^a) * \rho_{\delta_j} - \varphi_j \bar{p}^a\|_{L^1} + \|(\varphi_j \hat{p}^a) * \rho_{\delta_j} - \varphi_j \hat{p}^a\|_{L^1} \leq \varepsilon 2^{-j} \quad (3.23)$$

$$\|(\varphi_j \bar{p}^s) * \rho_{\delta_j}\|_{L^1} - \|\varphi_j \bar{p}^s\|_{M_b} + \|(\varphi_j \hat{p}^s) * \rho_{\delta_j}\|_{L^1} - \|\varphi_j \hat{p}^s\|_{M_b} \leq \varepsilon 2^{-j}, \quad (3.24)$$

$$\|(\varphi_j u_3) * \rho_{\delta_j} - \varphi_j u_3\|_{L^\infty} \leq \varepsilon 2^{-j}, \quad (3.25)$$

where we used the continuous embedding of  $BH(\omega)$  into  $C(\bar{\omega})$ . By (3.25) we immediately deduce (3.22). By (3.23) we have that

$$\sum_{j=1}^{\infty} (\varphi_j \bar{p}^a) * \rho_{\delta_j} \rightarrow \bar{p}^a \quad \text{strongly in } L^1(\omega; \mathbb{M}_{sym}^{2 \times 2}),$$

while by (3.24) we obtain that

$$\left\| \sum_{j=1}^{\infty} (\varphi_j \bar{p}^s) * \rho_{\delta_j} \right\|_{L^1} \leq \sum_{j=1}^{\infty} \|\varphi_j \bar{p}^s\|_{M_b} + \varepsilon = \sum_{j=1}^{\infty} \int_{\omega} \varphi_j d|\bar{p}^s| + \varepsilon = |\bar{p}^s|(\omega) + \varepsilon.$$

These two facts, together with (3.2), yield

$$\limsup_{\varepsilon \rightarrow 0} \|E\bar{u}^\varepsilon\|_{L^1} \leq \|\bar{e} + \bar{p}^a\|_{L^1} + |\bar{p}^s|(\omega) = \|E\bar{u}\|_{M_b}.$$

The opposite inequality follows from (3.1) by lower semicontinuity. A similar argument applies to (3.21). Finally, it is easy to see that (3.20)–(3.22) are preserved in the construction of Step 2, since the approximation result for  $\bar{u}$  entails strong convergence of  $(E\bar{u}^\varepsilon)$  in  $L^1(\omega; \mathbb{M}_{sym}^{2 \times 2})$ .

We now prove an approximation result for Kirchhoff-Love admissible triples in terms of smooth triples. We denote by  $C_c^\infty(\omega \cup \gamma_n; \mathbb{M}_{sym}^{2 \times 2})$  the set of smooth maps whose support is a compact subset of  $\omega \cup \gamma_n$ . Moreover, we introduce the set  $L_{\infty, c}^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  of all  $p \in L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  satisfying the following two conditions:

- (i)  $\partial_\alpha^k \partial_\beta^j p \in L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  for every  $k, j \in \mathbb{N} \cup \{0\}$ ,  $\alpha, \beta = 1, 2$ ;
- (ii) there exists  $U \subset\subset \omega \cup \gamma_n$  such that  $p = 0$  a.e. on  $\omega \setminus \bar{U} \times (-\frac{1}{2}, \frac{1}{2})$ .

Note that if  $p \in L_{\infty, c}^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$ , then  $p(\cdot, x_3) \in C_c^\infty(\omega \cup \gamma_n; \mathbb{M}_{sym}^{2 \times 2})$  for a.e.  $x_3 \in (-\frac{1}{2}, \frac{1}{2})$ .

**Theorem 3.9.** *Let  $H_0$  be as in Lemma 3.6. Let  $w \in W^{1,2}(\Omega, \mathbb{R}^3) \cap KL(\Omega)$  and let  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . Then, there exists a sequence of triples*

$$(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in (W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times L_{\infty, c}^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})) \cap \mathcal{A}_{KL}(w)$$

such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly}^* \text{ in } BD(\Omega), \quad (3.26)$$

$$e^\varepsilon \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (3.27)$$

$$p^\varepsilon \rightharpoonup p \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}), \quad (3.28)$$

$$\mathcal{H}_0(p^\varepsilon) \rightarrow \mathcal{H}_0(p). \quad (3.29)$$

*Proof.* Up to translating  $u$  by  $w$ , it is enough to prove the theorem for  $w \equiv 0$ . Moreover, by Lemma 3.7 and by the metrizable topology on bounded subsets of  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  we can reduce to the case where  $u \in W^{1,1}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and there exists  $q \in L^1(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  such that

$$p = q \quad \text{in } \Omega, \quad p = -u \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_d. \quad (3.30)$$

According to Remark 3.2, we identify  $e$  and  $p$  with a function in  $L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  and a measure in  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$ , respectively, and we perform the decomposition of Proposition 3.5. By Remark 3.1 we have that  $\bar{u} \in W^{1,1}(\omega; \mathbb{R}^2)$  and  $u_3 \in W^{2,1}(\omega)$ , while by (3.30) there exist  $\bar{q}, \hat{q} \in L^1(\omega; \mathbb{M}_{sym}^{2 \times 2})$  such that

$$\bar{p} = \bar{q} \quad \text{in } \omega, \quad \bar{p} = -\bar{u} \odot \nu_{\partial\omega} \mathcal{H}^1 \quad \text{on } \gamma_d, \quad (3.31)$$

and

$$\hat{p} = \hat{q} \quad \text{in } \omega, \quad \hat{p} = -\nabla u_3 \odot \nu_{\partial\omega} \mathcal{H}^1 \quad \text{on } \gamma_d. \quad (3.32)$$

Note also that  $u_3 = 0$  on  $\gamma_d$ .

For the sake of simplicity we split the proof into two steps.

**Step 1.** We claim that we can always reduce to the case where there exists an open set  $J \subset \partial\omega$  such that  $\gamma_d$  is compactly contained in  $J$  and  $u_3 = 0$  on  $J$  (topological notions refer here to the relative topology of  $\partial\omega$ ).

To prove the claim, it is enough to show that the triple  $(u, e, p)$  can be approximated in the sense of (3.26)–(3.29) by a sequence of triples  $(u^\delta, e^\delta, p^\delta)$  in  $\mathcal{A}_{KL}(w)$  satisfying the following property: for every  $\delta > 0$  there exists an open set  $J^\delta \subset \partial\omega$  such that  $\gamma_d$  is compactly contained in  $J^\delta$  and  $u_3^\delta = 0$  on  $J^\delta$ .

To this purpose, let  $\{U_i\}_{i=1,\dots,n}$  be a finite covering of  $\partial\omega$  such that for every  $i$ , up to a  $C^2$  change of coordinates,  $\partial\omega \cap U_i$  is the graph of a  $C^2$  map and  $\omega \cap U_i$  is the related subgraph. We also require the covering to be such that for  $\alpha = 1, 2$  there exists an open neighbourhood  $U_{P_\alpha}$  of the point  $P_\alpha$  satisfying

$$P_\alpha \in \overline{U_{P_\alpha}} \subset U_\alpha \text{ for } \alpha = 1, 2 \quad \text{and} \quad U_{P_\alpha} \cap U_\beta = \emptyset \text{ for } \alpha \neq \beta.$$

We recall that by assumption  $\partial|_{\partial\omega}\gamma_d = \{P_1, P_2\}$ . Finally, let  $U_0 \subset \mathbb{R}^2$  be an open set, compactly contained in  $\omega$ , such that  $\{U_i\}_{i=0,\dots,n}$  is a finite covering of  $\overline{\omega}$ , and let  $\{\varphi_i\}_{i=0,\dots,n}$  be a subordinate partition of unity,  $\varphi_i \in C_c^\infty(U_i)$ ,  $0 \leq \varphi_i \leq 1$  for  $i = 0, \dots, n$ , and

$$\sum_{i=0}^n \varphi_i = 1 \quad \text{in } \overline{\omega}. \quad (3.33)$$

The approximating sequence will be constructed by modifying  $u$  in the sets  $U_1$  and  $U_2$  and keeping it unchanged in the other sets.

We first consider the set  $U_1$ . By our choice of the covering there exist a map  $\phi \in C^2(U_1; \mathbb{R}^2)$  and a rectangle  $R_1 := (a, b) \times (c, d)$  such that  $\phi(U_1) = R_1$  and  $\phi^{-1} \in C^2(R_1; U_1)$ ; moreover, there exists  $h \in C^2(a, b)$  such that

$$\phi(U_1 \cap \partial\omega) = \{(s, h(s)) : s \in (a, b)\}, \quad \phi(U_1 \cap \omega) := \{(s, t) \in R_1 : t < h(s)\}.$$

We can assume that for a suitable  $s_1 \in (a, b)$

$$\phi(U_1 \cap \gamma_d) = \{(s, h(s)) : s \in (s_1, b)\}.$$

Let  $V_1$  be an open set in  $\mathbb{R}^2$  such that  $\text{supp } \varphi_1 \subset V_1 \subset\subset U_1$ . For  $\delta$  small enough we define  $\psi^\delta : \phi(V_1) \rightarrow R_1$  as

$$\psi^\delta(s, t) = (s + \delta, t - h(s) + h(s + \delta))$$

and  $\phi^\delta : V_1 \rightarrow U_1$  as

$$\phi^\delta := \phi^{-1} \circ \psi^\delta \circ \phi.$$

It is easy to see that for  $\delta$  small enough

$$\phi^\delta(V_1 \cap \omega) \subset U_1 \cap \omega, \quad \phi^\delta(V_1 \setminus \overline{\omega}) \subset U_1 \setminus \overline{\omega},$$

and

$$\phi^\delta(V_1 \cap \partial\omega) \subset U_1 \cap \partial\omega.$$

Moreover, setting  $K_1 := \text{supp } \varphi_1$ , we have that

$$\|\phi^\delta - \text{id}\|_{C^2(K_1)} \rightarrow 0, \quad \|(\phi^\delta)^{-1} - \text{id}\|_{C^2(K_1)} \rightarrow 0, \quad (3.34)$$

as  $\delta \rightarrow 0$ .

We consider the functions  $\bar{u}^{\delta,1} := \varphi_1(\bar{u} \circ \phi^\delta)$  and  $u_3^{\delta,1} := \varphi_1(u_3 \circ \phi^\delta)$ , which are well defined on  $V_1 \cap \omega$  and are extended to zero outside the support of  $\varphi_1$ . By construction  $\bar{u}^{\delta,1} \in W^{1,1}(\omega; \mathbb{R}^2)$ ,  $u_3^{\delta,1} \in W^{2,1}(\omega)$ , and

$$u_3^{\delta,1} = 0 \quad \text{on } J^{\delta,1}, \quad (3.35)$$

where  $J^{\delta,1} := (U_1 \cap \gamma_d) \cup (\phi^\delta)^{-1}(U_1 \cap \gamma_d)$ . Moreover, by (3.34) we obtain

$$\bar{u}^{\delta,1} \rightarrow \varphi_1 \bar{u} \quad \text{strongly in } W^{1,1}(\omega; \mathbb{R}^2), \quad (3.36)$$

$$u_3^{\delta,1} \rightarrow \varphi_1 u_3 \quad \text{strongly in } W^{2,1}(\omega). \quad (3.37)$$

Straightforward computations yield the equalities

$$E\bar{u}^{\delta,1} = (\bar{u} \circ \phi^\delta) \odot \nabla \varphi_1 + \varphi_1 \operatorname{sym}((D\bar{u} \circ \phi^\delta)D\phi^\delta), \quad (3.38)$$

$$\begin{aligned} D^2 u_3^{\delta,1} &= (u_3 \circ \phi^\delta)D^2 \varphi_1 + 2\nabla \varphi_1 \odot ((D\phi^\delta)^T(\nabla u_3 \circ \phi^\delta)) \\ &\quad + \varphi_1 \sum_{\alpha=1,2} (\partial_\alpha u_3 \circ \phi^\delta)D^2 \phi_\alpha^\delta + \varphi_1 (D\phi^\delta)^T(D^2 u_3 \circ \phi^\delta)D\phi^\delta. \end{aligned} \quad (3.39)$$

It is therefore natural to introduce the functions  $\bar{e}^{\delta,1}, \hat{e}^{\delta,1} \in L^2(\omega; \mathbb{M}_{sym}^{2 \times 2})$ , defined as

$$\begin{aligned} \bar{e}^{\delta,1} &:= (\bar{u} \circ \phi^\delta) \odot \nabla \varphi_1 + \varphi_1 \operatorname{sym}((\bar{e} \circ \phi^\delta)D\phi^\delta), \\ \hat{e}^{\delta,1} &:= -(u_3 \circ \phi^\delta)D^2 \varphi_1 - 2\nabla \varphi_1 \odot ((D\phi^\delta)^T(\nabla u_3 \circ \phi^\delta)) \\ &\quad - \varphi_1 \sum_{\alpha=1,2} (\partial_\alpha u_3 \circ \phi^\delta)D^2 \phi_\alpha^\delta + \varphi_1 (D\phi^\delta)^T(\hat{e} \circ \phi^\delta)D\phi^\delta \end{aligned}$$

and the functions  $\bar{q}^{\delta,1}, \hat{q}^{\delta,1} \in L^1(\omega; \mathbb{M}_{sym}^{2 \times 2})$ , defined as

$$\begin{aligned} \bar{q}^{\delta,1} &:= \varphi_1 \operatorname{sym}((\bar{q} \circ \phi^\delta)D\phi^\delta) + \varphi_1 \operatorname{sym}([(D\bar{u} - E\bar{u}) \circ \phi^\delta]D\phi^\delta), \\ \hat{q}^{\delta,1} &:= \varphi_1 (D\phi^\delta)^T(\hat{q} \circ \phi^\delta)D\phi^\delta. \end{aligned}$$

By (3.38) and (3.39) there holds

$$E\bar{u}^{\delta,1} = \bar{e}^{\delta,1} + \bar{q}^{\delta,1} \quad \text{in } \omega, \quad D^2 u_3^{\delta,1} = -(\hat{e}^{\delta,1} + \hat{q}^{\delta,1}) \quad \text{in } \omega. \quad (3.40)$$

By (3.34), (3.36), and (3.37) we deduce the following convergence properties:

$$\bar{e}^{\delta,1} \rightarrow \bar{u} \odot \nabla \varphi_1 + \varphi_1 \bar{e} \quad \text{strongly in } L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}), \quad (3.41)$$

$$\hat{e}^{\delta,1} \rightarrow -u_3 D^2 \varphi_1 - 2\nabla \varphi_1 \odot \nabla u_3 + \varphi_1 \hat{e} \quad \text{strongly in } L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}), \quad (3.42)$$

$$\bar{q}^{\delta,1} \rightarrow \varphi_1 \bar{q} \quad \text{strongly in } L^1(\omega; \mathbb{M}_{sym}^{2 \times 2}), \quad (3.43)$$

$$\hat{q}^{\delta,1} \rightarrow \varphi_1 \hat{q} \quad \text{strongly in } L^1(\omega; \mathbb{M}_{sym}^{2 \times 2}). \quad (3.44)$$

An analogous construction in the set  $U_2$  provides us with two triples

$$\begin{aligned} (\bar{u}^{\delta,2}, \bar{e}^{\delta,2}, \bar{q}^{\delta,2}) &\in W^{1,1}(\omega; \mathbb{R}^2) \times L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}) \times L^1(\omega; \mathbb{M}_{sym}^{2 \times 2}), \\ (u_3^{\delta,2}, \hat{e}^{\delta,2}, \hat{q}^{\delta,2}) &\in W^{2,1}(\omega) \times L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}) \times L^1(\omega; \mathbb{M}_{sym}^{2 \times 2}), \end{aligned}$$

such that

$$E\bar{u}^{\delta,2} = \bar{e}^{\delta,2} + \bar{q}^{\delta,2} \quad \text{in } \omega, \quad D^2 u_3^{\delta,2} = -(\hat{e}^{\delta,2} + \hat{q}^{\delta,2}) \quad \text{in } \omega, \quad (3.45)$$

and the following convergence properties hold:

$$\bar{u}^{\delta,2} \rightarrow \varphi_2 \bar{u} \quad \text{strongly in } W^{1,1}(\omega; \mathbb{R}^2), \quad (3.46)$$

$$u_3^{\delta,2} \rightarrow \varphi_2 u_3 \quad \text{strongly in } W^{2,1}(\omega), \quad (3.47)$$

$$\bar{e}^{\delta,2} \rightarrow \bar{u} \odot \nabla \varphi_2 + \varphi_2 \bar{e} \quad \text{strongly in } L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}), \quad (3.48)$$

$$\hat{e}^{\delta,2} \rightarrow -u_3 D^2 \varphi_2 - 2\nabla \varphi_2 \odot \nabla u_3 + \varphi_2 \hat{e} \quad \text{strongly in } L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}), \quad (3.49)$$

$$\bar{q}^{\delta,2} \rightarrow \varphi_2 \bar{q} \quad \text{strongly in } L^1(\omega; \mathbb{M}_{sym}^{2 \times 2}), \quad (3.50)$$

$$\hat{q}^{\delta,2} \rightarrow \varphi_2 \hat{q} \quad \text{strongly in } L^1(\omega; \mathbb{M}_{sym}^{2 \times 2}). \quad (3.51)$$

Moreover, the following boundary condition is satisfied:

$$u_3^{\delta,2} = 0 \quad \text{on } J^{\delta,2}, \quad (3.52)$$

where  $J^{\delta,2}$  is an open subset of  $\partial\omega$  strictly containing  $U_2 \cap \gamma_d$ .

To complete the construction of the approximating sequence we set

$$\bar{u}^\delta := \bar{u}^{\delta,1} + \bar{u}^{\delta,2} + \sum_{i \neq 1,2} \varphi_i \bar{u}, \quad u_3^\delta := u_3^{\delta,1} + u_3^{\delta,2} + \sum_{i \neq 1,2} \varphi_i u_3,$$

and

$$u_\alpha^\delta := \bar{u}_\alpha^\delta - x_3 \partial_\alpha u_3^\delta \quad (\alpha = 1, 2).$$



It is immediate to see that  $u^\delta \in W^{1,1}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ ; moreover, by (3.35) and (3.52) we have

$$u_3^\delta = 0 \quad \text{on } J^\delta,$$

where  $J^\delta := J^{\delta,1} \cup J^{\delta,2} \cup \gamma_d$  is an open subset of  $\partial\omega$  and satisfies  $\gamma_d \subset\subset J^\delta$ . By (3.33), (3.36), (3.37), (3.46), and (3.47) we also have

$$u^\delta \rightarrow u \quad \text{strongly in } W^{1,1}(\Omega; \mathbb{R}^3). \quad (3.53)$$

By the continuity of the trace operator the previous convergence entails

$$u^\delta \rightarrow u \quad \text{strongly in } L^1(\partial\Omega; \mathbb{R}^3). \quad (3.54)$$

Finally, we introduce the functions  $e^\delta \in L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  and  $q^\delta \in L^1(\Omega; \mathbb{M}_{sym}^{2 \times 2})$ , defined as

$$\begin{aligned} e^\delta &:= \bar{e}^{\delta,1} + \bar{e}^{\delta,2} + x_3(\hat{e}^{\delta,1} + \hat{e}^{\delta,2}) + (\varphi_1 + \varphi_2)e_\perp \\ &\quad + \sum_{i \neq 1,2} (\varphi_i e + \bar{u} \odot \nabla \varphi_i - x_3 u_3 D^2 \varphi_i - 2x_3 \nabla \varphi_i \odot \nabla u_3), \\ q^\delta &:= \bar{q}^{\delta,1} + \bar{q}^{\delta,2} + x_3(\hat{q}^{\delta,1} + \hat{q}^{\delta,2}) - (\varphi_1 + \varphi_2)e_\perp + \sum_{i \neq 1,2} \varphi_i q_i, \end{aligned}$$

and the measure  $p^\delta \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$ , defined as

$$p^\delta := q^\delta \quad \text{in } \Omega, \quad p^\delta := -u^\delta \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_d.$$

Clearly,  $(u^\delta, e^\delta, p^\delta) \in \mathcal{A}_{KL}(w)$ . Moreover, by (3.41)–(3.44) and (3.48)–(3.51) we obtain

$$e^\delta \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}), \quad (3.55)$$

$$q^\delta \rightarrow q \quad \text{strongly in } L^1(\Omega; \mathbb{M}_{sym}^{2 \times 2}). \quad (3.56)$$

From (3.54) and (3.56) it follows immediately that

$$p^\delta \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$$

and

$$\mathcal{H}_0(p^\delta) \rightarrow \mathcal{H}_0(p).$$

**Step 2.** By Step 1 we can assume that there exists an open set  $J \subset \partial\omega$  such that  $\gamma_d$  is compactly contained in  $J$  and  $u_3 = 0$  on  $J$ .

Let us consider a finite covering  $\{Q_i\}_{i=1,\dots,m}$  of  $\partial\omega$  made of open squares centered at points on  $\partial\omega$ , with a face orthogonal to some vector  $n_i \in \mathbb{S}^1$  and such that, for every  $i = 1, \dots, m$ , the set  $Q_i \cap \omega$  is a  $C^2$  subgraph in the direction  $n_i$ . We also require that for some  $m_0 \in \{1, \dots, m\}$

$$\gamma_d \subset\subset \bigcup_{i=1}^{m_0} Q_i \cap \partial\omega \subset\subset J$$

and

$$\text{dist}(Q_i, \gamma_d) > 0 \quad \text{for every } i = m_0 + 1, \dots, m.$$

Let also  $Q_0$  be an open set compactly contained in  $\omega$  such that the collection of open sets  $\{Q_i\}_{i=0,\dots,m}$  is a finite covering of  $\bar{\omega}$ . We consider a subordinate partition of unity  $\{\varphi_i\}_{i=0,\dots,m}$ , with  $0 \leq \varphi_i \leq 1$ ,  $\varphi_i \in C_c^\infty(Q_i)$  for every  $i = 1, \dots, m$ , and  $\sum_{i=0}^m \varphi_i = 1$  on  $\bar{\omega}$ .

Denoting by  $\tilde{\Omega}$  the set

$$\tilde{\Omega} := \Omega \cup \bigcup_{i=1}^{m_0} (Q_i \times (-\frac{1}{2}, \frac{1}{2})),$$

we extend the triple  $(u, e, p)$  to  $\tilde{\Omega}$  by setting

$$u := 0 \quad \text{in } \tilde{\Omega} \setminus \Omega, \quad e := 0 \quad \text{in } \tilde{\Omega} \setminus \Omega, \quad p := \begin{cases} -u \odot \nu_{\partial\Omega} \mathcal{H}^2 & \text{on } \tilde{\Omega} \cap \partial\Omega, \\ 0 & \text{in } \tilde{\Omega} \setminus \bar{\Omega}. \end{cases}$$

The extended maps satisfy

$$u \in BD(\tilde{\Omega}) \cap KL(\tilde{\Omega}), \quad e \in L^2(\tilde{\Omega}; \mathbb{M}_{sym}^{3 \times 3}), \quad p \in M_b(\tilde{\Omega}; \mathbb{M}_{sym}^{3 \times 3})$$

and

$$Eu = e + p \quad \text{in } \tilde{\Omega}.$$

Note, in particular, that since  $u_3 = 0$  and  $\nu_{\partial\Omega} = (\nu_{\partial\omega}, 0)$  on  $\tilde{\Omega} \cap \partial\Omega$ , we have that  $p_{i3} = 0$  in  $\tilde{\Omega}$  for  $i = 1, 2, 3$ . Thus, we can as usual identify  $e$  with a function in  $L^2(\tilde{\Omega}; \mathbb{M}_{sym}^{2 \times 2})$  and  $p$  with a measure in  $M_b(\tilde{\Omega}; \mathbb{M}_{sym}^{2 \times 2})$ .

For every  $i = 1, \dots, m_0$  we introduce the outward translations

$$\tau_{i,\varepsilon}(x') := x' + a_\varepsilon n_i \quad \text{for } x' \in \mathbb{R}^2,$$

while for  $i = m_0 + 1, \dots, m$  we consider the inward translations

$$\tau_{i,\varepsilon}(x') := x' - a_\varepsilon n_i \quad \text{for } x' \in \mathbb{R}^2,$$

where  $(a_\varepsilon)$  is a sequence converging to 0, as  $\varepsilon \rightarrow 0$ . We define

$$\bar{u}^\varepsilon := \sum_{i=1}^m (\varphi_i \bar{u}) \circ \tau_{i,\varepsilon} + \varphi_0 \bar{u}, \quad (3.57)$$

$$\bar{e}^\varepsilon := \sum_{i=1}^m (\varphi_i \bar{e}) \circ \tau_{i,\varepsilon} + \varphi_0 \bar{e} + \sum_{i=1}^m (\nabla \varphi_i \odot \bar{u}) \circ \tau_{i,\varepsilon} + \nabla \varphi_0 \odot \bar{u}, \quad (3.58)$$

$$\bar{p}^\varepsilon := \sum_{i=1}^m \tau_{i,\varepsilon}^\#(\varphi_i \bar{p}) + \varphi_0 \bar{p}, \quad (3.59)$$

where  $\tau_{i,\varepsilon}^\#(\varphi_i \bar{p})$  denotes the pull-back measure of  $\varphi_i \bar{p}$ . Notice that  $(\bar{u}^\varepsilon, \bar{e}^\varepsilon, \bar{p}^\varepsilon)$  is well defined in an open neighbourhood  $\omega_\varepsilon$  of  $\omega$ , that is,  $\bar{u}^\varepsilon \in BD(\omega_\varepsilon)$ ,  $\bar{e}^\varepsilon \in L^2(\omega_\varepsilon; \mathbb{M}_{sym}^{2 \times 2})$ ,  $\bar{p}^\varepsilon \in M_b(\omega_\varepsilon; \mathbb{M}_{sym}^{2 \times 2})$ , and

$$E\bar{u}^\varepsilon = \bar{e}^\varepsilon + \bar{p}^\varepsilon \quad \text{in } \omega_\varepsilon.$$

Moreover, by construction there exists an open set  $U_\varepsilon \subset \mathbb{R}^2$  such that  $\gamma_d \subset\subset U_\varepsilon$  and  $u^\varepsilon = 0$ ,  $e^\varepsilon = 0$ , and  $p^\varepsilon = 0$  in  $U_\varepsilon$ . Finally, we can choose  $a_\varepsilon \rightarrow 0$  in such a way that

$$\tau_{i,\varepsilon}^\#(\varphi_i \bar{p})(\partial\omega \cap Q_i) = 0 \quad \text{for } i = m_0 + 1, \dots, m,$$

so that

$$|\bar{p}^\varepsilon|(\partial\omega) = 0 \quad \text{for every } \varepsilon. \quad (3.60)$$

Let now  $(\rho_\delta) \subset C_c^\infty(\mathbb{R}^2)$  be a sequence of convolution kernels. For  $\delta < a_\varepsilon$  we consider the functions

$$\bar{u}^{\varepsilon,\delta} := \bar{u}^\varepsilon * \rho_\delta, \quad \bar{e}^{\varepsilon,\delta} := \bar{e}^\varepsilon * \rho_\delta, \quad \bar{p}^{\varepsilon,\delta} := \bar{p}^\varepsilon * \rho_\delta.$$

Clearly, we have  $\bar{u}^{\varepsilon,\delta} \in C^\infty(\bar{\omega}; \mathbb{R}^2)$  and  $\bar{e}^{\varepsilon,\delta}, \bar{p}^{\varepsilon,\delta} \in C^\infty(\bar{\omega}; \mathbb{M}_{sym}^{2 \times 2})$ , and

$$E\bar{u}^{\varepsilon,\delta} = \bar{e}^{\varepsilon,\delta} + \bar{p}^{\varepsilon,\delta} \quad \text{in } \omega.$$

Moreover, for  $\delta$  small enough there holds

$$\bar{u}^{\varepsilon,\delta} = 0 \quad \text{on } \gamma_d \quad \text{and} \quad \bar{e}^{\varepsilon,\delta}, \bar{p}^{\varepsilon,\delta} \in C_c^\infty(\omega \cup \gamma_n; \mathbb{M}_{sym}^{2 \times 2}). \quad (3.61)$$

We apply a similar construction to the normal component of  $u$  and to the first moments of  $e$  and  $p$ . We first introduce

$$\begin{aligned} u_3^\varepsilon &:= \sum_{i=1}^m (\varphi_i u_3) \circ \tau_{i,\varepsilon} + \varphi_0 u_3, \\ \hat{e}^\varepsilon &:= \sum_{i=1}^m (\varphi_i \hat{e}) \circ \tau_{i,\varepsilon} + \varphi_0 \hat{e} - 2 \sum_{i=1}^m (\nabla \varphi_i \odot \nabla u_3) \circ \tau_{i,\varepsilon} - 2 \nabla \varphi_0 \odot \nabla u_3 \\ &\quad - \sum_{i=1}^m (D^2 \varphi_i u_3) \circ \tau_{i,\varepsilon} - D^2 \varphi_0 u_3, \\ \hat{p}^\varepsilon &:= \sum_{i=1}^m \tau_{i,\varepsilon}^\# (\varphi_i \hat{p}) + \varphi_0 \hat{p}, \end{aligned}$$

and we then define for  $\delta < a_\varepsilon$

$$u_3^{\varepsilon,\delta} := u_3^\varepsilon * \rho_\delta, \quad \hat{e}^{\varepsilon,\delta} := \hat{e}^\varepsilon * \rho_\delta, \quad \hat{p}^{\varepsilon,\delta} := \hat{p}^\varepsilon * \rho_\delta.$$

As before, we can modify the choice of  $a_\varepsilon \rightarrow 0$  in such a way that

$$|\hat{p}^\varepsilon|(\partial\omega) = 0. \quad (3.62)$$

Moreover, for  $\delta$  small enough we have that  $u_3^{\varepsilon,\delta} \in C^\infty(\bar{\omega})$ ,  $\hat{e}^{\varepsilon,\delta}, \hat{p}^{\varepsilon,\delta} \in C_c^\infty(\omega \cup \gamma_n; \mathbb{M}_{sym}^{2 \times 2})$ , and  $u_3^{\varepsilon,\delta} = 0$  on  $\gamma_d$ ,  $\nabla u_3^{\varepsilon,\delta} = 0$  on  $\gamma_d$ . Finally, there holds

$$D^2 u_3^{\varepsilon,\delta} = -(\hat{e}^{\varepsilon,\delta} + \hat{p}^{\varepsilon,\delta}) \quad \text{in } \omega.$$

Analogously, we define

$$e_\perp^\varepsilon := \sum_{i=1}^m (\varphi_i e_\perp) \circ \tau_{i,\varepsilon} + \varphi_0 e_\perp, \quad e_\perp^{\varepsilon,\delta} := e_\perp^\varepsilon * \rho_\delta,$$

where, with an abuse of notation, the composition  $(\varphi_i e_\perp) \circ \tau_{i,\varepsilon}$  stands for the function

$$(\varphi_i e_\perp) \circ \tau_{i,\varepsilon}(x) = \varphi_i(\tau_{i,\varepsilon}(x')) e_\perp(\tau_{i,\varepsilon}(x'), x_3) \quad \text{for a.e. } x \in \Omega,$$

and the convolution is intended with respect to the variable  $x' \in \mathbb{R}^2$ . It is immediate to see that  $e_\perp^{\varepsilon,\delta} \in L_{\infty,c}^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$ . We now set

$$\begin{aligned} u_\alpha^{\varepsilon,\delta} &:= \bar{u}_\alpha^{\varepsilon,\delta} - x_3 \partial_\alpha u_3^{\varepsilon,\delta} \quad (\alpha = 1, 2), \\ e^{\varepsilon,\delta} &:= \bar{e}^{\varepsilon,\delta} + x_3 \hat{e}^{\varepsilon,\delta} + e_\perp^{\varepsilon,\delta}, \\ p^{\varepsilon,\delta} &:= \bar{p}^{\varepsilon,\delta} + x_3 \hat{p}^{\varepsilon,\delta} - e_\perp^{\varepsilon,\delta}. \end{aligned}$$

By construction we have

$$(u^{\varepsilon,\delta}, e^{\varepsilon,\delta}, p^{\varepsilon,\delta}) \in (W^{1,2}(\Omega; \mathbb{R}^3) \times L_{\infty,c}^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}) \times L_{\infty,c}^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})) \cap \mathcal{A}_{KL}(w).$$

It is convenient to introduce also the measure  $p^\varepsilon \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$ , defined as

$$p^\varepsilon := \bar{p}^\varepsilon \otimes \mathcal{L}^1 + x_3 \hat{p}^\varepsilon \otimes \mathcal{L}^1 - e_\perp^\varepsilon.$$

Lemma 3.6, together with equalities (3.60) and (3.62), guarantees that we can choose  $\delta = \delta(\varepsilon)$  small enough, so that

$$\begin{aligned} \|\bar{u}^{\varepsilon,\delta(\varepsilon)} - \bar{u}^\varepsilon\|_{L^2} &< \varepsilon, \quad \|u_3^{\varepsilon,\delta(\varepsilon)} - u_3^\varepsilon\|_{W^{1,2}} < \varepsilon, \\ \|\bar{e}^{\varepsilon,\delta(\varepsilon)} - \bar{e}^\varepsilon\|_{L^2} &< \varepsilon, \quad \|\hat{e}^{\varepsilon,\delta(\varepsilon)} - \hat{e}^\varepsilon\|_{L^2} < \varepsilon, \quad \|e_\perp^{\varepsilon,\delta(\varepsilon)} - e_\perp^\varepsilon\|_{L^2(\Omega)} < \varepsilon, \\ \|\bar{p}^{\varepsilon,\delta(\varepsilon)}\|_{L^1(\Omega)} &- |p^\varepsilon|(\Omega) < \varepsilon, \end{aligned}$$

and

$$\int_\Omega H_0(p^{\varepsilon,\delta(\varepsilon)}) dx \leq H_0(p^\varepsilon)(\Omega) + \varepsilon. \quad (3.63)$$

From the convergence properties above we deduce (3.26)–(3.28). To conclude the proof of the theorem it remains to prove (3.29). By (3.63) we have

$$\limsup_{\varepsilon \rightarrow 0} \mathcal{H}_0(p^{\varepsilon, \delta(\varepsilon)}) \leq \limsup_{\varepsilon \rightarrow 0} \mathcal{H}_0(p^\varepsilon).$$

On the other hand, since  $p$  has been extended to zero on the set  $\cup_{i=1}^{m_0} (Q_i \setminus \bar{\omega}) \times (-\frac{1}{2}, \frac{1}{2})$ , while for  $i = m_0 + 1, \dots, m$  the map  $\tau_{i, \varepsilon}$  is an inward translations, we have

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0} \mathcal{H}_0(p^\varepsilon) \\ & \leq \mathcal{H}_0(\varphi_0 p) + \limsup_{\varepsilon \rightarrow 0} \sum_{i=1}^m \int_{-\frac{1}{2}}^{\frac{1}{2}} H_0(\tau_{i, \varepsilon}^\#(\varphi_i \bar{p} + x_3 \varphi_i \hat{p} + \varphi_i e_\perp(\cdot, x_3)))(\omega \cup \gamma_d) dx_3 \\ & \leq \mathcal{H}_0(\varphi_0 p) + \sum_{i=1}^m \int_{-\frac{1}{2}}^{\frac{1}{2}} H_0(\varphi_i(\bar{p} + x_3 \hat{p} + e_\perp(\cdot, x_3)))(\omega \cup \gamma_d) dx_3 \\ & = \sum_{i=0}^m \mathcal{H}_0(\varphi_i p) = \sum_{i=0}^m \int_{\Omega \cup \Gamma_d} \varphi_i H_0\left(\frac{dp}{d|p|}\right) d|p| = \mathcal{H}_0(p). \end{aligned}$$

This, together with (3.28), completes the proof of (3.29) and of the theorem.  $\square$

**Remark 3.10.** Arguing as in Remark 3.8, one can modify the construction of the sequence  $(u^\varepsilon, e^\varepsilon, p^\varepsilon)$  in Theorem 3.9 in such a way that the convergence properties (3.19)–(3.22) are also satisfied. In particular, (3.22) is preserved, since the approximation argument for  $u_3$  involves only local translations and convolutions.

**Remark 3.11.** We point out that the approximation result provided by Lemma 3.7 is crucial in Step 1 of the proof of Theorem 3.9. Indeed, it is not in general true that, if  $v \in BD(\omega)$  and  $\Psi : U \rightarrow \omega$  is a smooth bijection with smooth inverse, the composition  $v \circ \Psi$  belongs to  $BD(U)$ . Lemma 3.7 allows us to assume  $\bar{u} \in W^{1,1}(\omega; \mathbb{R}^2)$  and this regularity guarantees that  $\bar{u} \circ \phi^\delta \in W^{1,1}(V_1; \mathbb{R}^2)$ , hence, in particular,  $\bar{u} \circ \phi^\delta \in BD(V_1)$ .

#### 4. $\Gamma$ -CONVERGENCE OF THE STATIC FUNCTIONALS

In this section we study the  $\Gamma$ -convergence of the rescaled energies  $(\mathcal{J}_\varepsilon)$ , as  $\varepsilon \rightarrow 0$ . We first introduce the limit functional.

Let  $\mathbb{M} : \mathbb{M}_{sym}^{2 \times 2} \rightarrow \mathbb{M}_{sym}^{3 \times 3}$  be the operator given by

$$\mathbb{M}\xi := \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1(\xi) \\ \xi_{12} & \xi_{22} & \lambda_2(\xi) \\ \lambda_1(\xi) & \lambda_2(\xi) & \lambda_3(\xi) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2}, \quad (4.1)$$

where for every  $\xi \in \mathbb{M}_{sym}^{2 \times 2}$  the triple  $(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi))$  is the unique solution to the minimum problem

$$\min_{\lambda_i \in \mathbb{R}} Q \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.$$

We observe that the triple  $(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi))$  can be characterized as the unique solution of the linear system

$$\mathbb{C}\mathbb{M}\xi : \begin{pmatrix} 0 & 0 & \zeta_1 \\ 0 & 0 & \zeta_2 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix} = 0 \quad (4.2)$$

for every  $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}$ . This implies that  $\mathbb{M}$  is a linear map.

Let  $Q_r : \mathbb{M}_{sym}^{2 \times 2} \rightarrow [0, +\infty)$  be the quadratic form given by

$$Q_r(\xi) := Q(\mathbb{M}\xi) \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2}.$$

By (2.3) it satisfies the estimates

$$r_{\mathbb{C}}|\xi|^2 \leq Q_r(\xi) \leq R_{\mathbb{C}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2}. \quad (4.3)$$

We also consider the linear operator  $\mathbb{C}_r : \mathbb{M}_{sym}^{2 \times 2} \rightarrow \mathbb{M}_{sym}^{3 \times 3}$  defined as

$$\mathbb{C}_r \xi := \mathbb{C} \mathbb{M} \xi \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2}. \quad (4.4)$$

By (4.2) we have

$$\mathbb{C}_r \xi : \zeta = \mathbb{C} \mathbb{M} \xi : \zeta = \mathbb{C} \mathbb{M} \xi : \mathbb{M} \zeta'' \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2}, \zeta \in \mathbb{M}_{sym}^{3 \times 3}, \quad (4.5)$$

where  $\zeta'' \in \mathbb{M}_{sym}^{2 \times 2}$  satisfies  $\zeta''_{\alpha\beta} = \zeta_{\alpha\beta}$  for  $\alpha, \beta = 1, 2$ . This implies that

$$Q_r(\xi) = \frac{1}{2} \mathbb{C}_r \xi : \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2}.$$

We introduce the functional  $\mathcal{Q}_r : L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}) \rightarrow [0, +\infty)$ , defined as

$$\mathcal{Q}_r(f) := \int_{\Omega} Q_r(f(z)) \, dz \quad \text{for every } f \in L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}).$$

It describes the limiting elastic energy of a configuration of the plate whose elastic strain is given by  $f$ .

We define  $H_r : \mathbb{M}_{sym}^{2 \times 2} \rightarrow [0, +\infty)$  as

$$H_r(\xi) := \min_{\lambda_1, \lambda_2 \in \mathbb{R}} H \begin{pmatrix} \xi_{11} & \xi_{12} & \lambda_1 \\ \xi_{12} & \xi_{22} & \lambda_2 \\ \lambda_1 & \lambda_2 & -(\xi_{11} + \xi_{22}) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2}. \quad (4.6)$$

It turns out that  $H_r$  is convex, positively one-homogeneous, and satisfies

$$r_K |\xi| \leq H_r(\xi) \leq \sqrt{3} R_K |\xi| \quad \text{for every } \xi \in \mathbb{M}_{sym}^{2 \times 2}. \quad (4.7)$$

For every  $\mu \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  we define

$$\mathcal{H}_r(\mu) := \int_{\Omega \cup \Gamma_d} H_r \left( \frac{d\mu}{d|\mu|} \right) d|\mu|. \quad (4.8)$$

With the previous notation, we introduce the functional  $\mathcal{J} : \mathcal{A}_{KL}(w) \rightarrow [0, +\infty]$ , defined as

$$\mathcal{J}(u, e, p) := \mathcal{Q}_r(e) + \mathcal{H}_r(p) \quad (4.9)$$

for every  $(u, e, p) \in \mathcal{A}_{KL}(w)$ , where we identify  $e$  with a function in  $L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  and  $p$  with a measure in  $M_b(\Omega; \mathbb{M}_{sym}^{2 \times 2})$ , according to Remark 3.2. We are now in a position to state the main result of the section.

**Theorem 4.1.** *Let  $\mathcal{J}_\varepsilon$  and  $\mathcal{J}$  be the functionals defined in (2.11) and (4.9). Let  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and for every  $\varepsilon > 0$  let  $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$  be a minimizer of  $\mathcal{J}_\varepsilon$ . Then there exist a subsequence (not relabelled) and a triple  $(u, e, p) \in \mathcal{A}_{KL}(w)$  such that*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \quad (4.10)$$

$$e^\varepsilon \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.11)$$

$$\Lambda_\varepsilon e^\varepsilon \rightarrow \mathbb{M}e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.12)$$

$$p^\varepsilon \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.13)$$

$$\mathcal{H}(\Lambda_\varepsilon p^\varepsilon) \rightarrow \mathcal{H}_r(p). \quad (4.14)$$

Moreover,  $(u, e, p)$  is a minimizer of  $\mathcal{J}$  and

$$\lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(u_\varepsilon, e_\varepsilon, p_\varepsilon) = \mathcal{J}(u, e, p). \quad (4.15)$$

**Remark 4.2.** The existence of a minimizer for  $\mathcal{J}_\varepsilon$  is guaranteed by [9, Theorem 3.3].

**Remark 4.3.** More general boundary conditions can be considered in Theorem 4.1. For instance, the thesis continues to hold if for every  $\varepsilon > 0$   $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$  is a minimizer of  $\mathcal{J}_\varepsilon$  in the class  $\mathcal{A}_\varepsilon(w^\varepsilon)$  and  $w^\varepsilon \in W^{1,2}(\Omega; \mathbb{R}^3)$  is such that  $w^\varepsilon \rightarrow w$  strongly in  $L^2(\Omega; \mathbb{R}^3)$  and  $\Lambda_\varepsilon E w^\varepsilon \rightarrow E w$  strongly in  $L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  with  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ .

The proof of Theorem 4.1 is based on  $\Gamma$ -convergence arguments. We first prove a compactness result and a liminf inequality for sequences of triples with equibounded energies.

**Theorem 4.4.** *Let  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and let  $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$  be such that*

$$\mathcal{J}_\varepsilon(u_\varepsilon, e_\varepsilon, p_\varepsilon) \leq C \quad \text{for every } \varepsilon > 0, \quad (4.16)$$

where  $C$  is a constant independent of  $\varepsilon$ . Then, there exist  $\tilde{e} \in L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  and  $\tilde{p} \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3})$  such that, up to subsequences,

$$\Lambda_\varepsilon e_\varepsilon \rightharpoonup \tilde{e} \quad \text{weakly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.17)$$

$$\Lambda_\varepsilon p_\varepsilon \rightharpoonup \tilde{p} \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3}). \quad (4.18)$$

Moreover, there exists  $(u, e, p) \in \mathcal{A}_{KL}(w)$ , with  $e_{\alpha\beta} = \tilde{e}_{\alpha\beta}$  and  $p_{\alpha\beta} = \tilde{p}_{\alpha\beta}$  for  $\alpha, \beta = 1, 2$ , such that, up to subsequences,

$$u_\varepsilon \rightharpoonup u \quad \text{weakly}^* \text{ in } BD(\Omega), \quad (4.19)$$

$$e_\varepsilon \rightharpoonup e \quad \text{weakly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.20)$$

$$p_\varepsilon \rightharpoonup p \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.21)$$

and

$$\mathcal{J}(u, e, p) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(u_\varepsilon, e_\varepsilon, p_\varepsilon). \quad (4.22)$$

*Proof.* By the energy estimate (4.16) and by (2.3) we deduce the bounds

$$\|e_\varepsilon\|_{L^2} \leq \|\Lambda_\varepsilon e_\varepsilon\|_{L^2} \leq C \quad \text{for every } \varepsilon. \quad (4.23)$$

Hence, there exist  $\tilde{e}, e \in L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  such that (4.17) and (4.20) hold up to subsequences, with  $e_{\alpha\beta} = \tilde{e}_{\alpha\beta}$  for  $\alpha, \beta = 1, 2$  and  $e_{i3} = 0$  for  $i = 1, 2, 3$ . By the lower semicontinuity of  $\mathcal{Q}$  with respect to weak convergence in  $L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  and by the definition of  $Q_r$  we also deduce

$$\mathcal{Q}_r(e) \leq \mathcal{Q}(\tilde{e}) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{Q}(\Lambda_\varepsilon e_\varepsilon). \quad (4.24)$$

By (4.16) and (2.5) we obtain analogously

$$\|p_\varepsilon\|_{M_b} \leq \|\Lambda_\varepsilon p_\varepsilon\|_{M_b} \leq C. \quad (4.25)$$

Therefore, there exist  $\tilde{p} \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3})$  and  $p \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  such that (4.18) and (4.21) hold up to subsequences, with  $p_{\alpha\beta} = \tilde{p}_{\alpha\beta}$  for  $\alpha, \beta = 1, 2$  and  $p_{i3} = 0$  for  $i = 1, 2, 3$ . By the lower semicontinuity of  $\mathcal{H}$  with respect to weak\* convergence in  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3})$  and by the definition of  $H_r$ , we have

$$\mathcal{H}_r(p) \leq \mathcal{H}(\tilde{p}) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{H}(\Lambda_\varepsilon p_\varepsilon), \quad (4.26)$$

which, together with (4.24), gives (4.22).

Since  $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$ , for every  $\varepsilon$  there holds

$$Eu_\varepsilon = e_\varepsilon + p_\varepsilon \quad \text{in } \Omega, \quad (4.27)$$

and

$$p_\varepsilon = (w - u_\varepsilon) \odot \nu_{\partial\Omega} \mathcal{H}^2 \quad \text{on } \Gamma_d. \quad (4.28)$$

By (4.23), (4.25), and (4.27), the sequence  $(Eu_\varepsilon)$  is bounded in  $M_b(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ . By (4.25) and (4.28), the traces of  $(u_\varepsilon)$  are uniformly bounded in  $L^1(\Gamma_d; \mathbb{R}^3)$ . Hence, by (2.2) the sequence  $(u_\varepsilon)$  is bounded in  $BD(\Omega)$  and (4.19) holds up to subsequences. Moreover, it is immediate to see that  $Eu = e + p$  in  $\Omega$ , hence  $u \in KL(\Omega)$ .

To conclude the proof, it remains to check that  $p = (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2$  on  $\Gamma_d$ . To this purpose we argue as in [9, Lemma 2.1]. Since  $\gamma_d$  is an open subset of  $\partial\omega$ , there exists an open set  $A \subset \mathbb{R}^2$  such that  $\gamma_d = A \cap \partial\omega$ . We set  $U := (\omega \cup A) \times (-\frac{1}{2}, \frac{1}{2})$  and we extend the triples  $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$  to the set  $U$  in the following way:

$$v_\varepsilon := \begin{cases} u_\varepsilon & \text{in } \Omega, \\ w & \text{in } U \setminus \Omega, \end{cases} \quad f_\varepsilon := \begin{cases} e_\varepsilon & \text{in } \Omega, \\ Ew & \text{in } U \setminus \Omega, \end{cases} \quad q_\varepsilon := \begin{cases} p_\varepsilon & \text{in } \Omega \cup \Gamma_d, \\ 0 & \text{otherwise.} \end{cases}$$

The symmetric part of the gradient of  $v_\varepsilon$  satisfies

$$Ev_\varepsilon = \begin{cases} Eu_\varepsilon & \text{in } \Omega, \\ (w - u_\varepsilon) \odot \nu_{\partial\Omega} \mathcal{H}^2 & \text{on } \Gamma_d, \\ Ew & \text{in } U \setminus \bar{\Omega}. \end{cases}$$

Therefore, by (4.19), up to subsequences,  $v_\varepsilon \rightharpoonup v$  weakly\* in  $BD(U)$ , where

$$v := \begin{cases} u & \text{in } \Omega, \\ w & \text{in } U \setminus \Omega, \end{cases} \quad \text{and} \quad Ev = \begin{cases} Eu & \text{in } \Omega, \\ (w - u) \odot \nu_{\partial\Omega} \mathcal{H}^2 & \text{on } \Gamma_d, \\ Ew & \text{in } U \setminus \bar{\Omega}. \end{cases} \quad (4.29)$$

Analogously, up to subsequences,  $f_\varepsilon \rightharpoonup f$  weakly in  $L^2(U; \mathbb{M}_{sym}^{3 \times 3})$  and, since the restrictions to  $\Omega \cup \Gamma_d$  of functions in  $C_0(U; \mathbb{M}_{sym}^{3 \times 3})$  belong to  $C_0(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ , there holds  $q_\varepsilon \rightharpoonup q$  weakly\* in  $M_b(U; \mathbb{M}_{sym}^{3 \times 3})$ , where

$$f := \begin{cases} e & \text{in } \Omega, \\ Ew & \text{in } U \setminus \Omega, \end{cases} \quad \text{and} \quad q := \begin{cases} p & \text{in } \Omega \cup \Gamma_d, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $Ev_\varepsilon = f_\varepsilon + q_\varepsilon$  in  $U$  for every  $\varepsilon$ , we deduce that  $Ev = f + q$  in  $U$ . The thesis follows now from (4.29).  $\square$

In the next theorem we show that the lower bound established in Theorem 4.4 is optimal by exhibiting a recovery sequence.

**Theorem 4.5.** *Let  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and let  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . Then, there exists a sequence of triples  $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$  such that*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly* in } BD(\Omega), \quad (4.30)$$

$$e^\varepsilon \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.31)$$

$$p^\varepsilon \rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.32)$$

$$\Lambda_\varepsilon e^\varepsilon \rightarrow \mathbb{M}e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (4.33)$$

$$\mathcal{H}(\Lambda_\varepsilon p^\varepsilon) \rightarrow \mathcal{H}_r(p), \quad (4.34)$$

and

$$\lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) = \mathcal{J}(u, e, p). \quad (4.35)$$

*Proof.* By Theorem 3.9 and by the metrizable topology on bounded subsets of  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  we can reduce to the case where

$$(u, e, p) \in (W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times L_{\infty, c}^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})) \cap \mathcal{A}_{KL}(w).$$

In particular,  $u = w$  on  $\Gamma_d$  and  $p = 0$   $\mathcal{H}^2$ -a.e. on  $\Gamma_d$ .

Let now  $\phi_1, \phi_2, \phi_3 \in L^2(\Omega)$  be such that

$$\mathbb{M}e = \begin{pmatrix} e_{11} & e_{12} & \phi_1 \\ e_{12} & e_{22} & \phi_2 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix}.$$

Since  $p \in L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ , by the measurable selection lemma (see, e.g., [14]) and by (2.5) and (4.7) there exist  $\eta_1, \eta_2, \eta_3 \in L^2(\Omega)$  such that

$$\mathcal{H}_r(p) = \mathcal{H} \begin{pmatrix} p_{11} & p_{12} & \eta_1 \\ p_{12} & p_{22} & \eta_2 \\ \eta_1 & \eta_2 & -(p_{11} + p_{22}) \end{pmatrix}. \quad (4.36)$$

We argue as in [25, Proposition 4.1] and we approximate the maps  $\phi_i$  and  $\eta_i$  by means of elliptic regularizations. For every  $\varepsilon$  we define  $\phi_i^\varepsilon \in W_0^{1,2}(\Omega)$ ,  $i = 1, 2, 3$ , as the solution of the

elliptic boundary value problem

$$\begin{cases} -\varepsilon\Delta\phi_i^\varepsilon + \phi_i^\varepsilon = \phi_i & \text{in } \Omega, \\ \phi_i^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

and  $\eta_\alpha^\varepsilon \in W_0^{1,2}(\Omega)$ ,  $\alpha = 1, 2$ , as the solution of

$$\begin{cases} -\varepsilon\Delta\eta_\alpha^\varepsilon + \eta_\alpha^\varepsilon = \eta_\alpha & \text{in } \Omega, \\ \eta_\alpha^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The standard theory of elliptic equations gives

$$\phi_i^\varepsilon \rightarrow \phi_i \quad \text{strongly in } L^2(\Omega), \quad (4.37)$$

$$\eta_\alpha^\varepsilon \rightarrow \eta_\alpha \quad \text{strongly in } L^2(\Omega), \quad (4.38)$$

as  $\varepsilon \rightarrow 0$ , and

$$\|\nabla\phi_i^\varepsilon\|_{L^2} \leq C\varepsilon^{-\frac{1}{2}}, \quad \|\nabla\eta_\alpha^\varepsilon\|_{L^2} \leq C\varepsilon^{-\frac{1}{2}}. \quad (4.39)$$

We also introduce the function  $f^\varepsilon \in L^2(\omega; \mathbb{M}_{sym}^{3 \times 3})$ , defined componentwise as

$$\begin{aligned} f_{\alpha\alpha}^\varepsilon(x') &:= 2\varepsilon \int_0^{x_3} (\partial_\alpha\phi_\alpha^\varepsilon(x', s) + \partial_\alpha\eta_\alpha^\varepsilon(x', s)) ds \quad (\alpha = 1, 2), & f_{33}^\varepsilon(x') &:= 0, \\ f_{12}^\varepsilon(x') &:= \varepsilon \int_0^{x_3} (\partial_2\phi_1^\varepsilon(x', s) + \partial_2\eta_1^\varepsilon(x', s) + \partial_1\phi_2^\varepsilon(x', s) + \partial_1\eta_2^\varepsilon(x', s)) ds, \\ f_{\alpha 3}^\varepsilon(x') &:= \frac{\varepsilon^2}{2} \int_0^{x_3} (\partial_\alpha\phi_3^\varepsilon(x', s) - \partial_\alpha p_{11}(x', s) - \partial_\alpha p_{22}(x', s)) ds \quad (\alpha = 1, 2) \end{aligned}$$

for a.e.  $x' \in \omega$ .

We are now in a position to define the recovery sequence. Let

$$\begin{aligned} u_\alpha^\varepsilon &:= u_\alpha + 2\varepsilon \int_0^{x_3} (\phi_\alpha^\varepsilon(x', s) + \eta_\alpha^\varepsilon(x', s)) ds \quad (\alpha = 1, 2), \\ u_3^\varepsilon &:= u_3 + \varepsilon^2 \int_0^{x_3} (\phi_3^\varepsilon(x', s) - p_{11}(x', s) - p_{22}(x', s)) ds, \end{aligned}$$

and

$$e^\varepsilon := e + \begin{pmatrix} 0 & 0 & \varepsilon\phi_1^\varepsilon \\ 0 & 0 & \varepsilon\phi_2^\varepsilon \\ \varepsilon\phi_1^\varepsilon & \varepsilon\phi_2^\varepsilon & \varepsilon^2\phi_3^\varepsilon \end{pmatrix} + f^\varepsilon, \quad p^\varepsilon := p + \begin{pmatrix} 0 & 0 & \varepsilon\eta_1^\varepsilon \\ 0 & 0 & \varepsilon\eta_2^\varepsilon \\ \varepsilon\eta_1^\varepsilon & \varepsilon\eta_2^\varepsilon & -\varepsilon^2(p_{11} + p_{22}) \end{pmatrix}.$$

Since  $u = w$  on  $\Gamma_d$ ,  $p \in L_{\infty,c}^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ , and  $\phi_i^\varepsilon, \eta_\alpha^\varepsilon \in W_0^{1,2}(\Omega)$ , we have that  $u_\varepsilon = w$  on  $\Gamma_d$ . It is also easy to check that  $(u_\varepsilon, e_\varepsilon, p_\varepsilon) \in \mathcal{A}_\varepsilon(w)$ . From (4.37) and (4.38) it follows that  $u^\varepsilon \rightarrow u$  strongly in  $L^2(\Omega; \mathbb{R}^3)$ . By (4.37) and (4.39) we deduce (4.31) and (4.33), while by (4.38) we obtain

$$p^\varepsilon \rightarrow p \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}),$$

hence (4.32) and (4.30) follow. Finally, by (4.36) we have (4.34), which, together with (4.33), implies the convergence of the energies.  $\square$

We are now in a position to prove Theorem 4.1.

*Proof of Theorem 4.1.* Since  $(w, Ew, 0) \in \mathcal{A}_\varepsilon(w)$  for every  $\varepsilon > 0$ , by minimality we have that

$$\mathcal{J}_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon) \leq \mathcal{J}_\varepsilon(w, Ew, 0) \leq R_{\mathbb{C}}\|Ew\|_{L^2}^2,$$

where the last inequality follows from (2.3) and the fact that  $w \in KL(\Omega)$ . By Theorem 4.4 we deduce that there exists  $(u, e, p) \in \mathcal{A}_{KL}(\Omega)$  such that, up to subsequences,

$$\begin{aligned} u^\varepsilon &\rightharpoonup u \quad \text{weakly* in } BD(\Omega), \\ e^\varepsilon &\rightharpoonup e \quad \text{weakly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \\ p^\varepsilon &\rightharpoonup p \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}), \end{aligned}$$



and

$$\mathcal{J}(u, e, p) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon). \quad (4.40)$$

Let now  $(v, f, q) \in \mathcal{A}_{KL}(\Omega)$ . By Theorem 4.5 there exists a sequence of triples  $(v^\varepsilon, f^\varepsilon, q^\varepsilon) \in \mathcal{A}_\varepsilon(w)$  such that

$$\mathcal{J}(v, f, q) = \lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(v^\varepsilon, f^\varepsilon, q^\varepsilon) \geq \limsup_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(u^\varepsilon, e^\varepsilon, p^\varepsilon), \quad (4.41)$$

where the last inequality follows from the minimality of  $(u_\varepsilon, e_\varepsilon, p_\varepsilon)$ . Combining (4.41) with (4.40), we deduce that  $(u, e, p)$  is a minimizer of  $\mathcal{J}$  and by choosing  $(v, f, q) = (u, e, p)$  in (4.41) we obtain (4.15).

It remains to prove (4.11), (4.12), and (4.14). By the lower semicontinuity of  $\mathcal{Q}$  and  $\mathcal{H}$  with respect to weak convergence in  $L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  and weak\* convergence in  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ , respectively, and by the definition of  $\mathcal{Q}_r$  and  $\mathcal{H}_r$  we have

$$\mathcal{Q}_r(e) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon), \quad \mathcal{H}_r(p) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{H}(\Lambda_\varepsilon p^\varepsilon). \quad (4.42)$$

Combining (4.15) and (4.42) yields

$$\lim_{\varepsilon \rightarrow 0} \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon) = \mathcal{Q}_r(e), \quad \lim_{\varepsilon \rightarrow 0} \mathcal{H}(\Lambda_\varepsilon p^\varepsilon) = \mathcal{H}_r(p),$$

so that (4.14) is proved. On the other hand, we remark that by (4.5)

$$\begin{aligned} \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon - \mathbb{M}e) &= \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon) + \mathcal{Q}_r(e) - \int_{\Omega} \mathbb{C}\mathbb{M}e : \Lambda_\varepsilon e^\varepsilon \, dx \\ &= \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon) + \mathcal{Q}_r(e) - \int_{\Omega} \mathbb{C}\mathbb{M}e : e^\varepsilon \, dx \end{aligned} \quad (4.43)$$

Therefore, passing to the limit in (4.43) and applying again (4.5), we obtain

$$\lim_{\varepsilon \rightarrow 0} \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon - \mathbb{M}e) = 0,$$

so that (4.12) follows now from (2.3). Finally, convergence (4.11) is an immediate consequence of (4.12).  $\square$

## 5. CONVERGENCE OF QUASISTATIC EVOLUTIONS

In this section we focus on the quasistatic evolution problems associated with the functionals  $\mathcal{J}_\varepsilon$  and  $\mathcal{J}$ , introduced in the previous section. To this purpose, for every  $t \in [0, T]$  we prescribe a boundary datum  $w(t) \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and assume the map  $t \mapsto w(t)$  to be absolutely continuous from  $[0, T]$  into  $W^{1,2}(\Omega; \mathbb{R}^3)$ .

Let  $s_1, s_2 \in [0, T]$ ,  $s_1 \leq s_2$ . For every function  $t \mapsto \mu(t)$  of bounded variation from  $[0, T]$  into  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3})$ , we define the *dissipation* of  $t \mapsto \mu(t)$  in  $[s_1, s_2]$  as

$$\mathcal{D}(\mu; s_1, s_2) := \sup \left\{ \sum_{j=1}^n \mathcal{H}(\mu(t_j) - \mu(t_{j-1})) : s_1 = t_0 \leq t_1 \leq \dots \leq t_n = s_2, n \in \mathbb{N} \right\}.$$

Analogously, for every function  $t \mapsto \mu(t)$  of bounded variation from  $[0, T]$  into  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  we define the *reduced dissipation* of  $t \mapsto \mu(t)$  in  $[s_1, s_2]$  as

$$\mathcal{D}_r(\mu; s_1, s_2) := \sup \left\{ \sum_{j=1}^n \mathcal{H}_r(\mu(t_j) - \mu(t_{j-1})) : s_1 = t_0 \leq t_1 \leq \dots \leq t_n = s_2, n \in \mathbb{N} \right\}$$

for every  $s_1, s_2 \in [0, T]$ ,  $s_1 \leq s_2$ .

**Definition 5.1.** Let  $\varepsilon > 0$ . An  $\varepsilon$ -*quasistatic evolution* for the boundary datum  $w(t)$  is a function  $t \mapsto (u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$  from  $[0, T]$  into  $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  that satisfies the following conditions:

(qs1) for every  $t \in [0, T]$  we have  $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t)) \in \mathcal{A}_\varepsilon(w(t))$  and

$$\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) \leq \mathcal{Q}(\Lambda_\varepsilon f) + \mathcal{H}(\Lambda_\varepsilon q - \Lambda_\varepsilon p^\varepsilon(t)) \quad (5.1)$$

for every  $(v, f, q) \in \mathcal{A}_\varepsilon(w(t))$ ;

(qs2) the function  $t \mapsto p^\varepsilon(t)$  from  $[0, T]$  into  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  has bounded variation and for every  $t \in [0, T]$

$$\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) + \mathcal{D}(\Lambda_\varepsilon p^\varepsilon; 0, t) = \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(0)) + \int_0^t \int_\Omega \mathbb{C} \Lambda_\varepsilon e^\varepsilon(s) : E\dot{w}(s) \, dx ds. \quad (5.2)$$

**Definition 5.2.** A *reduced quasistatic evolution* for the boundary datum  $w(t)$  is a function  $t \mapsto (u(t), e(t), p(t))$  from  $[0, T]$  into  $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  that satisfies the following conditions:

(qs1)<sub>r</sub> for every  $t \in [0, T]$  we have  $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$  and

$$\mathcal{Q}_r(e(t)) \leq \mathcal{Q}_r(f) + \mathcal{H}_r(q - p(t)) \quad (5.3)$$

for every  $(v, f, q) \in \mathcal{A}_{KL}(w(t))$ ;

(qs2)<sub>r</sub> the function  $t \mapsto p(t)$  from  $[0, T]$  into  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  has bounded variation and for every  $t \in [0, T]$

$$\mathcal{Q}_r(e(t)) + \mathcal{D}_r(p; 0, t) = \mathcal{Q}_r(e(0)) + \int_0^t \int_\Omega \mathbb{C}_r e(s) : E\dot{w}(s) \, dx ds. \quad (5.4)$$

**Remark 5.3.** Since the functions  $t \mapsto p^\varepsilon(t)$  and  $t \mapsto p(t)$  from  $[0, T]$  into  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  have bounded variation, they are bounded and the set of their discontinuity points (in the strong topology) is at most countable. By Lemma 5.9 below the same properties hold for the functions  $t \mapsto e^\varepsilon(t)$  and  $t \mapsto e(t)$  from  $[0, T]$  into  $L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ , and for the functions  $t \mapsto u^\varepsilon(t)$  and  $t \mapsto u(t)$  from  $[0, T]$  into  $BD(\Omega)$ . Therefore,  $t \mapsto e^\varepsilon(t)$  and  $t \mapsto e(t)$  belong to  $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}))$ , while  $t \mapsto u^\varepsilon(t)$  and  $t \mapsto u(t)$  belong to  $L^\infty([0, T]; BD(\Omega))$ . As  $t \mapsto E\dot{w}(t)$  belongs to  $L^1([0, T]; L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}))$ , the integrals on the right-hand side of (5.2) and (5.4) are well defined.

We are now in a position to state the main result of the article.

**Theorem 5.4.** *Let  $t \mapsto w(t)$  be absolutely continuous from  $[0, T]$  into  $W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ . Let  $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$  be such that*

$$\mathcal{Q}_r(e_0) \leq \mathcal{Q}_r(f) + \mathcal{H}_r(q - p_0) \quad (5.5)$$

for every  $(v, f, q) \in \mathcal{A}_{KL}(w(0))$ . Assume there exists a sequence of triples  $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon) \in \mathcal{A}_\varepsilon(w(0))$  such that

$$\mathcal{Q}(\Lambda_\varepsilon e_0^\varepsilon) \leq \mathcal{Q}(\Lambda_\varepsilon f) + \mathcal{H}(\Lambda_\varepsilon q - \Lambda_\varepsilon p_0^\varepsilon) \quad (5.6)$$

for every  $(v, f, q) \in \mathcal{A}_\varepsilon(w(0))$  and every  $\varepsilon > 0$ , and

$$u_0^\varepsilon \rightharpoonup u_0 \quad \text{weakly}^* \text{ in } BD(\Omega), \quad (5.7)$$

$$e_0^\varepsilon \rightarrow e_0 \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (5.8)$$

$$\Lambda_\varepsilon e_0^\varepsilon \rightarrow \mathbb{M}e_0 \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (5.9)$$

$$p_0^\varepsilon \rightharpoonup p_0 \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}), \quad (5.10)$$

$$\|\Lambda_\varepsilon p_0^\varepsilon\|_{M_b} \leq C, \quad (5.11)$$

where  $\mathbb{M}$  is the operator introduced in (4.1) and  $C$  is a constant independent of  $\varepsilon$ . Finally, for every  $\varepsilon > 0$  let  $t \mapsto (u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$  be an  $\varepsilon$ -quasistatic evolution for the boundary datum  $w(t)$  such that  $u^\varepsilon(0) = u_0^\varepsilon$ ,  $e^\varepsilon(0) = e_0^\varepsilon$ , and  $p^\varepsilon(0) = p_0^\varepsilon$ . Then, there exists a reduced quasistatic evolution  $t \mapsto (u(t), e(t), p(t))$  for the boundary datum  $w(t)$  such that  $u(0) = u_0$ ,  $e(0) = e_0$  and  $p(0) = p_0$ , and, up to subsequences,

$$u^\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly}^* \text{ in } BD(\Omega), \quad (5.12)$$

$$e^\varepsilon(t) \rightarrow e(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (5.13)$$

$$\Lambda_\varepsilon e^\varepsilon(t) \rightarrow \mathbb{M}e(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (5.14)$$

$$p^\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly}^* \text{ in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}) \quad (5.15)$$

for every  $t \in [0, T]$ . Moreover, the functions  $t \mapsto u(t)$ ,  $t \mapsto e(t)$ , and  $t \mapsto p(t)$  are absolutely continuous from  $[0, T]$  into  $BD(\Omega)$ ,  $L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ , and  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ , respectively.

**Remark 5.5.** From [9, Theorem 4.5] it follows that for every triple  $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon) \in \mathcal{A}_\varepsilon(w(0))$  satisfying (5.6) there exists an  $\varepsilon$ -quasistatic evolution  $t \mapsto (u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t))$  such that  $u^\varepsilon(0) = u_0^\varepsilon$ ,  $e^\varepsilon(0) = e_0^\varepsilon$ , and  $p^\varepsilon(0) = p_0^\varepsilon$ . Moreover, by [9, Theorem 5.2] the functions  $t \mapsto u^\varepsilon(t)$ ,  $t \mapsto e^\varepsilon(t)$ , and  $t \mapsto p^\varepsilon(t)$  are absolutely continuous from  $[0, T]$  into  $BD(\Omega)$ ,  $L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ , and  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ , respectively, and for a.e.  $t \in [0, T]$  we have

$$\|\Lambda_\varepsilon \dot{e}^\varepsilon(t)\|_{L^2} \leq C_1 \|E\dot{w}(t)\|_{L^2}, \quad (5.16)$$

$$\|\Lambda_\varepsilon \dot{p}^\varepsilon(t)\|_{M_b} \leq C_2 \|E\dot{w}(t)\|_{L^2}, \quad (5.17)$$

where  $C_1$  and  $C_2$  are positive constants depending on  $R_K$ ,  $r_{\mathbb{C}}$ ,  $R_{\mathbb{C}}$ ,  $\sup_{t \in [0, T]} \|\Lambda_\varepsilon e^\varepsilon(t)\|_{L^2}$ , and  $\sup_{t \in [0, T]} \|\Lambda_\varepsilon p^\varepsilon(t)\|_{M_b}$ . We notice that these results are proven in [9] under the assumption of a reference configuration of class  $C^2$ , but, as observed in [15], Lipschitz regularity is enough in the absence of external loads.

**Remark 5.6.** The set of admissible initial data for Theorem 5.4 is nonempty. Indeed, for every  $\varepsilon > 0$  let  $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon) \in \mathcal{A}_\varepsilon(w(0))$  be a minimizer of the functional  $\mathcal{J}_\varepsilon$ , that is,

$$\mathcal{Q}(\Lambda_\varepsilon e_0^\varepsilon) + \mathcal{H}(\Lambda_\varepsilon p_0^\varepsilon) \leq \mathcal{Q}(\Lambda_\varepsilon f) + \mathcal{H}(\Lambda_\varepsilon q)$$

for every  $(v, f, q) \in \mathcal{A}_\varepsilon(w(0))$ . Since by (2.6)

$$\mathcal{H}(\Lambda_\varepsilon q) \leq \mathcal{H}(\Lambda_\varepsilon q - \Lambda_\varepsilon p_0^\varepsilon) + \mathcal{H}(\Lambda_\varepsilon p_0^\varepsilon),$$

we deduce that  $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon)$  satisfies (5.6) for every  $\varepsilon > 0$ .

By Theorem 4.1 we infer the existence of a triple  $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$  such that (5.7)–(5.10) are satisfied,  $(u_0, e_0, p_0)$  is a minimizer of the functional  $\mathcal{J}$ , and

$$\lim_{\varepsilon \rightarrow 0} \mathcal{H}(\Lambda_\varepsilon p_0^\varepsilon) = \mathcal{H}_r(p_0). \quad (5.18)$$

Since  $H_r$  satisfies the triangle inequality, the triple  $(u_0, e_0, p_0)$  satisfies (5.5). Finally, (5.11) is an immediate consequence of (5.18) and (2.5).

**Remark 5.7.** Theorem 5.4 ensures, in particular, the existence of an absolutely continuous reduced quasistatic evolution for every initial datum  $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$  that satisfies (5.5) and is approximable in the sense of (5.6)–(5.11). We mention here that existence of solutions can be actually proved for every initial datum  $(u_0, e_0, p_0) \in \mathcal{A}_{KL}(w(0))$  satisfying (5.5) by applying the abstract method à la Mielke [27], namely by discretizing time and by solving suitable incremental minimum problems. Moreover, arguing as in [9, Theorem 5.2], one can show that every reduced quasistatic evolution is absolutely continuous from  $[0, T]$  into  $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ .

To prove Theorem 5.4 we need two technical lemmas concerning some consequences of the minimality condition  $(qs1)_r$ .

**Lemma 5.8.** *Let  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ . A triple  $(u, e, p) \in \mathcal{A}_{KL}(w)$  is a solution of the minimum problem*

$$\min \{ \mathcal{Q}_r(f) + \mathcal{H}_r(q - p) : (v, f, q) \in \mathcal{A}_{KL}(w) \} \quad (5.19)$$

if and only if

$$-\mathcal{H}_r(q) \leq \int_{\Omega} \mathbb{C}_r e : f \, dx \quad (5.20)$$

for every  $(v, f, q) \in \mathcal{A}_{KL}(0)$ .

*Proof.* Let  $(u, e, p) \in \mathcal{A}_{KL}(w)$  be a solution to (5.19) and let  $(v, f, q) \in \mathcal{A}_{KL}(0)$ . For every  $\eta \in \mathbb{R}$  the triple  $(u + \eta v, e + \eta f, p + \eta q)$  belongs to  $\mathcal{A}_{KL}(w)$ , hence

$$0 \leq \mathcal{Q}_r(e + \eta f) + \mathcal{H}_r(\eta q) - \mathcal{Q}_r(e).$$

Using the positive homogeneity of  $H_r$ , we obtain

$$0 \leq \pm \eta \int_{\Omega} \mathbb{C}_r e : f \, dx + \eta^2 \mathcal{Q}_r(f) + \eta \mathcal{H}_r(\pm q),$$

for every  $\eta > 0$ . Dividing by  $\eta$  and sending  $\eta$  to 0 yield (5.20).

The converse implication is true by convexity.  $\square$

**Lemma 5.9.** *Let  $w_1, w_2 \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  and for  $\alpha = 1, 2$  let  $(u_\alpha, e_\alpha, p_\alpha) \in \mathcal{A}_{KL}(w_\alpha)$  be a solution of the minimum problem*

$$\min \{ \mathcal{Q}_r(f) + \mathcal{H}_r(q - p_\alpha) : (v, f, q) \in \mathcal{A}_{KL}(w_\alpha) \}. \quad (5.21)$$

*Then there exists a positive constant  $C$ , depending only on  $R_K, r_C, R_C, \Omega$ , and  $\Gamma_d$ , such that*

$$\|e_2 - e_1\|_{L^2} \leq C\theta_{12}, \quad (5.22)$$

$$\|Eu_1 - Eu_2\|_{M_b} \leq C\theta_{12}, \quad (5.23)$$

$$\|u_1 - u_2\|_{L^1} \leq C(\theta_{12} + \|w_1 - w_2\|_{L^2}), \quad (5.24)$$

where  $\theta_{12}$  is given by

$$\theta_{12} := \|p_1 - p_2\|_{M_b} + \|p_1 - p_2\|_{M_b}^{\frac{1}{2}} + \|Ew_1 - Ew_2\|_{L^2}.$$

*Proof.* Since  $(u_2 - u_1 - w_2 + w_1, e_2 - e_1 - Ew_2 + Ew_1, p_2 - p_1) \in \mathcal{A}_{KL}(0)$ , by Lemma 5.8 we have

$$\begin{aligned} -\mathcal{H}_r(p_2 - p_1) &\leq \int_{\Omega} \mathbb{C}_r e_1 : (e_2 - e_1 - Ew_2 + Ew_1) dx, \\ -\mathcal{H}_r(p_1 - p_2) &\leq \int_{\Omega} \mathbb{C}_r e_2 : (e_1 - e_2 - Ew_1 + Ew_2) dx. \end{aligned}$$

Adding term by term, changing sign, and applying (4.7) yield

$$\int_{\Omega} \mathbb{C}_r (e_2 - e_1) : (e_2 - e_1) dx \leq \int_{\Omega} \mathbb{C}_r (e_2 - e_1) : (Ew_2 - Ew_1) dx + 2\sqrt{3}R_K \|p_2 - p_1\|_{M_b}.$$

By (4.3) we deduce

$$r_C \|e_2 - e_1\|_{L^2}^2 \leq R_C \|e_2 - e_1\|_{L^2} \|Ew_2 - Ew_1\|_{L^2} + 2\sqrt{3}R_K \|p_2 - p_1\|_{M_b},$$

which implies (5.22) by the Cauchy inequality. Since  $Eu_i = e_i + p_1$  in  $\Omega$ , Hölder inequality gives

$$\|Eu_2 - Eu_1\|_{M_b} \leq \mathcal{L}^3(\Omega)^{1/2} \|e_2 - e_1\|_{L^2} + \|p_2 - p_1\|_{M_b},$$

so that (5.23) follows from (5.22). Finally, since  $p_2 - p_1 = (w_2 - w_1 - u_2 + u_1) \odot \nu_{\partial\Omega} \mathcal{H}^2$  on  $\Gamma_d$ , we have

$$\|u_2 - u_1\|_{L^1(\Gamma_d)} \leq \|w_2 - w_1\|_{L^1(\Gamma_d)} + \|p_2 - p_1\|_{M_b} \leq C \|w_2 - w_1\|_{W^{1,2}} + \|p_2 - p_1\|_{M_b}$$

where we used the continuity of the trace operator from  $W^{1,2}(\Omega; \mathbb{R}^3)$  into  $L^1(\partial\Omega; \mathbb{R}^3)$ . Inequality (5.24) now follows from (2.2) and (5.23).  $\square$

We are now in a position to prove Theorem 5.4.

*Proof of Theorem 5.4.* For convenience of the reader we split the proof into four steps.

**Step 1. Compactness estimates.** Let us prove that there exists a constant  $C$ , depending only on the data, such that

$$\sup_{t \in [0, T]} \|\Lambda_\varepsilon e^\varepsilon(t)\|_{L^2} \leq C, \quad \sup_{t \in [0, T]} \|\Lambda_\varepsilon p^\varepsilon(t)\|_{M_b} \leq C \quad (5.25)$$

for every  $\varepsilon$ . As  $t \mapsto w(t)$  is absolutely continuous with values in  $W^{1,2}(\Omega; \mathbb{R}^3)$ , the function  $t \mapsto \|E\dot{w}(t)\|_2$  is integrable on  $[0, T]$ . This fact, together with (2.3), (2.4), and (5.2), implies that

$$r_C \|\Lambda_\varepsilon e^\varepsilon(t)\|_{L^2}^2 \leq R_C \|\Lambda_\varepsilon e^\varepsilon(0)\|_{L^2}^2 + 2R_C \sup_{t \in [0, T]} \|\Lambda_\varepsilon e^\varepsilon(t)\|_{L^2} \int_0^T \|E\dot{w}(s)\|_2 ds \quad (5.26)$$

for every  $t \in [0, T]$ . The former inequality in (5.25) follows now from (5.9) and Cauchy inequality. As for the latter, by (5.2), (5.26), and (5.9) we deduce that

$$\mathcal{D}(\Lambda_\varepsilon p^\varepsilon; 0, T) \leq C.$$

By definition of  $\mathcal{D}$  and (2.5) we infer that

$$r_K \|\Lambda_\varepsilon p^\varepsilon(t) - \Lambda_\varepsilon p_0^\varepsilon\|_{M_b} \leq \mathcal{H}(\Lambda_\varepsilon p^\varepsilon(t) - \Lambda_\varepsilon p^\varepsilon(0)) \leq \mathcal{D}(\Lambda_\varepsilon p^\varepsilon; 0, t) \leq C$$

for every  $t \in [0, T]$ , which implies the second inequality in (5.25) by (5.11).

Combining (5.16), (5.17), and (5.25), we obtain

$$\begin{aligned} \|\Lambda_\varepsilon e^\varepsilon(t_1) - \Lambda_\varepsilon e^\varepsilon(t_2)\|_{L^2} &\leq C \int_{t_1}^{t_2} \|E\dot{w}(s)\|_{L^2} ds \\ \|\Lambda_\varepsilon p^\varepsilon(t_1) - \Lambda_\varepsilon p^\varepsilon(t_2)\|_{M_b} &\leq C \int_{t_1}^{t_2} \|E\dot{w}(s)\|_{L^2} ds \end{aligned}$$

for every  $0 \leq t_1 \leq t_2 \leq T$ , where  $C$  is a constant depending only on the data. Therefore, by Ascoli-Arzelà Theorem there exist two subsequences, still denoted  $\Lambda_\varepsilon e^\varepsilon$  and  $\Lambda_\varepsilon p^\varepsilon$ , and two absolutely continuous functions  $\tilde{e} : [0, T] \rightarrow L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  and  $\tilde{p} : [0, T] \rightarrow M_b(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3})$  such that

$$\Lambda_\varepsilon e^\varepsilon(t) \rightharpoonup \tilde{e}(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (5.27)$$

$$\Lambda_\varepsilon p^\varepsilon(t) \rightharpoonup \tilde{p}(t) \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_D^{3 \times 3}) \quad (5.28)$$

for every  $t \in [0, T]$ .

Let  $e : [0, T] \rightarrow L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$  be defined as

$$e_{\alpha\beta}(t) = \tilde{e}_{\alpha\beta}(t) \quad (\alpha, \beta = 1, 2) \quad \text{and} \quad e_{i3}(t) = 0 \quad (i = 1, 2, 3)$$

for every  $t \in [0, T]$  and let  $p : [0, T] \rightarrow M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$  be defined as

$$p_{\alpha\beta}(t) = \tilde{p}_{\alpha\beta}(t) \quad (\alpha, \beta = 1, 2) \quad \text{and} \quad p_{i3}(t) = 0 \quad (i = 1, 2, 3) \quad (5.29)$$

for every  $t \in [0, T]$ . Then  $t \mapsto e(t)$  is absolutely continuous from  $[0, T]$  into  $L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3})$ ,  $t \mapsto p(t)$  is absolutely continuous from  $[0, T]$  into  $M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3})$ , and by (5.27) and (5.28) we have

$$e^\varepsilon(t) \rightharpoonup e(t) \quad \text{weakly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}), \quad (5.30)$$

$$p^\varepsilon(t) \rightharpoonup p(t) \quad \text{weakly* in } M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{3 \times 3}) \quad (5.31)$$

for every  $t \in [0, T]$ . Using (2.2) and the fact that  $(u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t)) \in \mathcal{A}_\varepsilon(w(t))$  for every  $\varepsilon > 0$ , it is easy to see that there exists an absolutely continuous function  $u : [0, T] \rightarrow BD(\Omega)$  such that

$$u^\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly* in } BD(\Omega)$$

for every  $t \in [0, T]$ . Moreover, arguing as in the proof of Theorem 4.4, one can show that  $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ . The initial condition  $u(0) = u_0$ ,  $e(0) = e_0$ , and  $p(0) = p_0$  is also clearly fulfilled.

**Step 2. Reduced stability.** We now show that the triple  $(u(t), e(t), p(t))$  is a solution to the minimum problem

$$\min \{ \mathcal{Q}_r(f) + \mathcal{H}_r(q - p(t)) : (v, f, q) \in \mathcal{A}_{KL}(w(t)) \} \quad (5.32)$$

for every  $t \in [0, T]$ .

Let us fix  $t \in [0, T]$ . By Lemma 5.8 it is enough to prove condition (5.20). Let  $(v, f, q) \in \mathcal{A}_{KL}(0)$ . By Theorem 4.5 there exists a sequence of triples  $(v^\varepsilon, f^\varepsilon, q^\varepsilon) \in \mathcal{A}_\varepsilon(0)$  such that

$$\Lambda_\varepsilon f^\varepsilon \rightarrow \mathbb{M}f \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{3 \times 3}) \quad (5.33)$$

and

$$\mathcal{H}(\Lambda_\varepsilon q^\varepsilon) \rightarrow \mathcal{H}_r(q). \quad (5.34)$$

By [9, Theorem 3.6] the minimality condition (5.1) is equivalent to

$$-\mathcal{H}(\Lambda_\varepsilon \check{q}) \leq \int_\Omega \mathbb{C} \Lambda_\varepsilon e^\varepsilon(t) : \Lambda_\varepsilon \check{f} \, dx \quad (5.35)$$

for every  $(\check{v}, \check{f}, \check{q}) \in \mathcal{A}_\varepsilon(0)$ . Therefore, we have that

$$-\mathcal{H}(\Lambda_\varepsilon e^\varepsilon) \leq \int_{\Omega} \mathbb{C} \Lambda_\varepsilon e^\varepsilon(t) : \Lambda_\varepsilon f^\varepsilon dx$$

for every  $\varepsilon > 0$ ; hence, combining (5.27), (5.33), and (5.34), we obtain

$$-\mathcal{H}_r(q) \leq \int_{\Omega} \mathbb{C} \tilde{e}(t) : \mathbb{M} f dx.$$

Since  $\mathbb{C} \tilde{e}(t) : \mathbb{M} f = \mathbb{C} \mathbb{M} e(t) : \mathbb{M} f = \mathbb{C}_r e(t) : f$  a.e. in  $\Omega$  by (4.5), the inequality above reduces to (5.20).

**Step 3. Identification of the limiting scaled elastic strain.** We shall prove that the function  $\tilde{e}(t)$  in (5.27) satisfies

$$\tilde{e}(t) = \mathbb{M} e(t) \tag{5.36}$$

for every  $t \in [0, T]$ .

For every  $\psi \in W^{1,2}(\Omega; \mathbb{R}^3)$  with  $\psi = 0$  on  $\Gamma_d$  we can consider the triples  $(\pm\psi, \pm E\psi, 0)$  as test functions in (5.35). This leads to the condition

$$\int_{\Omega} \mathbb{C} \Lambda_\varepsilon e^\varepsilon(t) : \Lambda_\varepsilon E\psi dx = 0 \tag{5.37}$$

for every  $\psi \in W^{1,2}(\Omega; \mathbb{R}^3)$  with  $\psi = 0$  on  $\Gamma_d$  and for every  $\varepsilon$ .

Let now  $U \subset \omega$ ,  $(a, b) \subset (-\frac{1}{2}, \frac{1}{2})$ , and  $\lambda_i \in \mathbb{R}$ ,  $i = 1, 2, 3$ . Let us denote the characteristic functions of the sets  $U$  and  $(a, b)$  by  $\chi_U$  and  $\chi_{(a,b)}$ , respectively. Finally, let  $(\varphi_i^k) \subset C_c^1(\omega)$  and  $(\xi^k) \subset C^1([-\frac{1}{2}, \frac{1}{2}])$  be such that  $\varphi_i^k \rightarrow \lambda_i \chi_U$  strongly in  $L^4(\omega)$ ,  $i = 1, 2, 3$ , and  $(\xi^k)' \rightarrow \chi_{(a,b)}$  strongly in  $L^4(-\frac{1}{2}, \frac{1}{2})$ . For every  $\varepsilon$  and  $k \in \mathbb{N}$  we consider the function

$$\psi^{\varepsilon,k}(x) := \begin{pmatrix} 2\varepsilon \xi^k(x_3) \varphi_1^k(x') \\ 2\varepsilon \xi^k(x_3) \varphi_2^k(x') \\ \varepsilon^2 \xi^k(x_3) \varphi_3^k(x') \end{pmatrix}$$

for every  $x \in \Omega$ . Since  $\psi^{\varepsilon,k} \in W^{1,2}(\Omega; \mathbb{R}^3)$  and  $\psi^{\varepsilon,k} = 0$  on  $\Gamma_d$ , by (5.37) we have

$$\int_{\Omega} \mathbb{C} \Lambda_\varepsilon e^\varepsilon(t) : \Lambda_\varepsilon E\psi^{\varepsilon,k} dx = 0$$

for every  $\varepsilon$ . Passing to the limit with respect to  $\varepsilon \rightarrow 0$  and then to  $k \rightarrow \infty$ , we deduce

$$\int_{U \times (a,b)} \mathbb{C} \tilde{e}(t) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} dx = 0.$$

Since  $U$  and  $(a, b)$  are arbitrary, we conclude that for every  $\lambda_i \in \mathbb{R}$ .

$$\mathbb{C} \tilde{e}(t) : \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} = 0,$$

a.e. in  $\Omega$ . This implies (5.36) by (4.2).

**Step 4. Reduced energy balance.** By (5.2) and lower semicontinuity we have

$$\begin{aligned} \mathcal{Q}_r(e(t)) + \mathcal{D}(p; 0, t) &\leq \lim_{\varepsilon \rightarrow 0} \left\{ \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(0)) + \int_0^t \int_{\Omega} \mathbb{C} \Lambda_\varepsilon e^\varepsilon(s) : E\dot{w}(s) dx ds \right\} \\ &= \mathcal{Q}_r(e_0) + \int_0^t \int_{\Omega} \mathbb{C}_r e(s) : E\dot{w}(s) dx ds, \end{aligned}$$

where the last equality follows from (5.9), (5.25), (5.27), (5.36), and the dominated convergence theorem. Since by (5.29) and the definition of  $\mathcal{H}_r$  there holds

$$\mathcal{D}_r(p; 0, t) \leq \mathcal{D}(\tilde{p}; 0, t) \tag{5.38}$$

for every  $t \in [0, T]$ , we conclude that

$$\mathcal{Q}_r(e(t)) + \mathcal{D}_r(p; 0, t) \leq \mathcal{Q}_r(e_0) + \int_0^t \int_{\Omega} \mathbb{C}_r e(s) : E \dot{w}(s) \, dx ds. \quad (5.39)$$

As it is standard in the variational theory for rate-independent processes, the converse energy inequality follows from the minimality condition  $(qs1)_r$ . We omit the proof as it follows closely those of [9, Theorem 4.7] and of [27, Theorem 4.4].

Combining  $(qs2)$ ,  $(qs2)_r$ , and the fact that the right-hand side of  $(qs2)$  converges to the right-hand side of  $(qs2)_r$ , we deduce that

$$\mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) + \mathcal{D}(\Lambda_\varepsilon p^\varepsilon; 0, t) \rightarrow \mathcal{Q}_r(e(t)) + \mathcal{D}_r(p; 0, t) \quad (5.40)$$

for every  $t \in [0, T]$ . On the other hand, by lower semicontinuity of  $\mathcal{Q}_r$  and of  $\mathcal{D}_r$  we have

$$\mathcal{Q}_r(e(t)) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) \quad (5.41)$$

and

$$\mathcal{D}_r(p; 0, t) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{D}(\Lambda_\varepsilon p^\varepsilon; 0, t) \quad (5.42)$$

for every  $t \in [0, T]$ . From (5.40)–(5.42) it follows that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{Q}(\Lambda_\varepsilon e^\varepsilon(t)) = \mathcal{Q}_r(e(t)) = \mathcal{Q}(Me(t))$$

for every  $t \in [0, T]$ . This, together with (5.27) and (5.36), implies strong convergence of the scaled strains  $\Lambda_\varepsilon e_\varepsilon(t)$ , and consequently of the strains  $e_\varepsilon(t)$ , for every  $t \in [0, T]$ . This concludes the proof of the theorem.  $\square$

## 6. CHARACTERIZATION OF REDUCED QUASISTATIC EVOLUTIONS

In the following we shall consider the space  $\Pi_{\Gamma_d}(\Omega)$  of admissible plastic strains, defined as the class of all  $p \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  for which there exist  $u \in BD(\Omega)$ ,  $e \in L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$ , and  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  such that  $(u, e, p) \in \mathcal{A}_{KL}(w)$ .

We shall also use the set

$$\Sigma(\Omega) := \{\sigma \in L^\infty(\Omega; \mathbb{M}_{sym}^{2 \times 2}) : \operatorname{div}_{x'} \bar{\sigma} \in L^2(\omega; \mathbb{R}^2), \operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma} \in M_b(\omega)\},$$

where  $\bar{\sigma}, \hat{\sigma} \in L^\infty(\omega; \mathbb{M}_{sym}^{2 \times 2})$  are the zero-th and first order moments of  $\sigma$ , defined according to Definition 3.3. In the first subsection we shall introduce a duality pairing between stresses  $\sigma \in \Sigma(\Omega)$  and plastic strains  $p \in \Pi_{\Gamma_d}(\Omega)$ . In the second subsection we shall use this duality pairing to deduce a weak formulation of the classical flow rule for a reduced quasistatic evolution. In the last subsection we discuss some examples, where reduced quasistatic evolutions can be characterized in terms of two-dimensional quantities.

**6.1. Stress-strain duality.** We first introduce a notion of duality for the zero-th order moments of the stress and the plastic strain. We essentially follow the theory developed in [21] and [9, Subsection 2.3].

For every  $\sigma \in \Sigma(\Omega)$  we can define the trace  $[\bar{\sigma} \nu_{\partial\omega}] \in L^\infty(\partial\omega; \mathbb{R}^2)$  of its zero-th order moment  $\bar{\sigma}$  through the formula

$$\int_{\partial\omega} [\bar{\sigma} \nu_{\partial\omega}] \cdot \varphi \, d\mathcal{H}^1 := \int_{\omega} \operatorname{div}_{x'} \bar{\sigma} \cdot \varphi \, dx' + \int_{\omega} \bar{\sigma} : E \varphi \, dx' \quad (6.1)$$

for every  $\varphi \in W^{1,1}(\omega; \mathbb{R}^2)$ . This is well defined since  $W^{1,1}(\omega; \mathbb{R}^2)$  is embedded into  $L^2(\omega; \mathbb{R}^2)$ .

Let  $\sigma \in \Sigma(\Omega)$  and  $\xi \in BD(\omega)$ . We define the distribution  $[\bar{\sigma} : E\xi]$  on  $\omega$  by

$$\langle [\bar{\sigma} : E\xi], \varphi \rangle := - \int_{\omega} \varphi \operatorname{div}_{x'} \bar{\sigma} \cdot \xi \, dx' - \int_{\omega} \bar{\sigma} : (\nabla \varphi \odot \xi) \, dx' \quad (6.2)$$

for every  $\varphi \in C_c^\infty(\omega)$ . From [21, Theorem 3.2] it follows that  $[\bar{\sigma} : E\xi]$  is a bounded measure on  $\omega$ , whose variation satisfies

$$|[\bar{\sigma} : E\xi]| \leq \|\bar{\sigma}\|_{L^\infty} |E\xi| \quad \text{in } \omega. \quad (6.3)$$

We can now define a duality between the zero-th order moments of elements in  $\Sigma(\Omega)$  and  $\Pi_{\Gamma_d}(\Omega)$ . Given  $\sigma \in \Sigma(\Omega)$  and  $p \in \Pi_{\Gamma_d}(\Omega)$ , we fix  $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}) \times (W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega))$  such that  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . Let  $\bar{u} \in BD(\omega)$ ,  $u_3 \in BH(\omega)$  and  $\bar{w} \in W^{1,2}(\omega; \mathbb{R}^2)$ ,  $w_3 \in W^{2,2}(\omega)$  be the Kirchhoff-Love components of  $u$  and  $w$ , respectively. We then define the measure  $[\bar{\sigma} : \bar{p}] \in M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  by setting

$$[\bar{\sigma} : \bar{p}] := \begin{cases} [\bar{\sigma} : E\bar{u}] - \bar{\sigma} : \bar{e} & \text{in } \omega, \\ [\bar{\sigma} \nu_{\partial\omega}] \cdot (\bar{w} - \bar{u}) \mathcal{H}^1 & \text{on } \gamma_d, \end{cases}$$

so that

$$\int_{\omega \cup \gamma_d} \varphi d[\bar{\sigma} : \bar{p}] = \int_{\omega} \varphi d[\bar{\sigma} : E\bar{u}] - \int_{\omega} \varphi \bar{\sigma} : \bar{e} dx' + \int_{\gamma_d} [\bar{\sigma} \nu_{\partial\omega}] \cdot \varphi (\bar{w} - \bar{u}) d\mathcal{H}^1 \quad (6.4)$$

for every  $\varphi \in C(\bar{\omega})$ .

**Remark 6.1.** Arguing as in [9], one can prove that the definition of  $[\bar{\sigma} : \bar{p}]$  is independent of the choice of the triple  $(u, e, w)$ . Moreover, if  $\bar{\sigma} \in C^1(\bar{\omega}; \mathbb{M}_{sym}^{2 \times 2})$ , then

$$\int_{\omega \cup \gamma_d} \varphi d[\bar{\sigma} : \bar{p}] = \int_{\omega \cup \gamma_d} \varphi \bar{\sigma} : d\bar{p}$$

for every  $\varphi \in C^1(\bar{\omega})$ . One can prove by approximation that the same equality is true for every  $\bar{\sigma} \in C(\bar{\omega}; \mathbb{M}_{sym}^{2 \times 2})$  and  $\varphi \in C(\bar{\omega})$ .

The following integration by parts formula can be proved.

**Proposition 6.2.** *Let  $\sigma \in \Sigma(\Omega)$ ,  $w \in W^{1,2}(\omega; \mathbb{R}^3) \cap KL(\Omega)$ , and  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . Let also  $\bar{u} \in BD(\omega)$  and  $\bar{w} \in W^{1,2}(\omega; \mathbb{R}^2)$  be the tangential Kirchhoff-Love components of  $u$  and  $w$ . Then*

$$\begin{aligned} \int_{\omega \cup \gamma_d} \varphi d[\bar{\sigma} : \bar{p}] + \int_{\omega} \varphi \bar{\sigma} : (\bar{e} - E\bar{w}) dx' + \int_{\omega} \bar{\sigma} : (\nabla \varphi \odot (\bar{u} - \bar{w})) dx' \\ = - \int_{\omega} \operatorname{div}_{x'} \bar{\sigma} \cdot \varphi (\bar{u} - \bar{w}) dx' + \int_{\gamma_n} [\bar{\sigma} \nu_{\partial\omega}] \cdot \varphi (\bar{u} - \bar{w}) d\mathcal{H}^1 \end{aligned} \quad (6.5)$$

for every  $\varphi \in C^1(\bar{\omega})$ .

*Proof.* The result is a corollary of [9, Proposition 2.2].  $\square$

We now introduce a notion of duality for the first order moments of the stress and of the plastic strain. We follow the lines of [10, Subsection 3.2] and [12, Subsection 2.3].

We start with a proposition concerning the traces of the first order moment of a stress in  $\Sigma(\Omega)$ . To this purpose we introduce the space

$$\hat{\Sigma}(\omega) := \{ \vartheta \in L^\infty(\omega; \mathbb{M}_{sym}^{2 \times 2}) : \operatorname{div}_{x'} \operatorname{div}_{x'} \vartheta \in M_b(\omega) \},$$

endowed with the norm  $\|\vartheta\|_{L^\infty} + \|\operatorname{div}_{x'} \operatorname{div}_{x'} \vartheta\|_{M_b}$ . We also denote by  $T_{\partial\omega} : W^{2,1}(\omega) \rightarrow W^{1,1}(\partial\omega)$  the trace operator on  $W^{2,1}(\omega)$ . We recall that  $T_{\partial\omega}(W^{2,1}(\omega)) \neq W^{1,1}(\partial\omega)$ , see [11, Théorème 2].

**Proposition 6.3.** *There exists a surjective continuous linear operator*

$$\begin{aligned} \mathcal{L} : \hat{\Sigma}(\omega) &\rightarrow (T_{\partial\omega}(W^{2,1}(\omega)))' \times L^\infty(\partial\omega) \\ \vartheta &\mapsto (b_0(\vartheta), b_1(\vartheta)) \end{aligned}$$

such that for every  $\vartheta \in \hat{\Sigma}(\omega)$  and  $v \in W^{2,1}(\omega)$  there holds

$$\int_{\omega} \vartheta : D^2 v dx' - \int_{\omega} v d(\operatorname{div}_{x'} \operatorname{div}_{x'} \vartheta) = -\langle b_0(\vartheta), v \rangle + \int_{\partial\omega} b_1(\vartheta) \frac{\partial v}{\partial \nu_{\partial\omega}} d\mathcal{H}^1, \quad (6.6)$$



where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $(T_{\partial\omega}(W^{2,1}(\omega)))'$  and  $T_{\partial\omega}(W^{2,1}(\omega))$ . Moreover, if  $\vartheta \in C^2(\bar{\omega}; \mathbb{M}_{sym}^{2 \times 2})$ , then

$$b_0(\vartheta) = \operatorname{div}_{x'} \vartheta \cdot \nu_{\partial\omega} - \frac{\partial}{\partial \tau_{\partial\omega}} (\vartheta \nu_{\partial\omega} \cdot \tau_{\partial\omega}), \quad (6.7)$$

$$b_1(\vartheta) = \vartheta \nu_{\partial\omega} \cdot \nu_{\partial\omega}, \quad (6.8)$$

where  $\tau_{\partial\omega}$  is the tangent vector to  $\partial\omega$ .

*Proof.* See [10, Théorème 2.3].  $\square$

**Remark 6.4.** The second integral on the left-handside of (6.6) is well defined because of the embedding of  $W^{2,1}(\omega)$  into  $C(\bar{\omega})$  (see [3, Theorem 4.12]).

Let  $\sigma \in \Sigma(\Omega)$  and  $v \in BH(\omega)$ . We define the distribution  $[\hat{\sigma} : D^2v]$  on  $\omega$  by

$$\langle [\hat{\sigma} : D^2v], \varphi \rangle := \int_{\omega} \varphi v d(\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}) - 2 \int_{\omega} \hat{\sigma} : (\nabla \varphi \odot \nabla v) dx' - \int_{\omega} v \hat{\sigma} : \nabla^2 \varphi dx'$$

for every  $\varphi \in C_c^\infty(\omega)$ . From [12, Proposition 2.1] it follows that  $[\hat{\sigma} : D^2v]$  is a bounded measure on  $\omega$ , whose variation satisfies

$$|[\hat{\sigma} : D^2v]| \leq \|\hat{\sigma}\|_{L^\infty} |D^2v| \quad \text{in } \omega.$$

We can now define a duality between the first order moments of elements in  $\Sigma(\Omega)$  and  $\Pi_{\Gamma_d}(\Omega)$ . Given  $\sigma \in \Sigma(\Omega)$  and  $p \in \Pi_{\Gamma_d}(\Omega)$ , we fix  $(u, e, w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}) \times (W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega))$  such that  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . We then define the measure  $[\hat{\sigma} : \hat{p}] \in M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  by setting

$$[\hat{\sigma} : \hat{p}] := \begin{cases} -[\bar{\sigma} : D^2u_3] - \hat{\sigma} : \hat{e} & \text{in } \omega, \\ b_1(\hat{\sigma}) \frac{\partial(u_3 - w_3)}{\partial \nu_{\partial\omega}} \mathcal{H}^1 & \text{on } \gamma_d, \end{cases}$$

so that

$$\int_{\omega \cup \gamma_d} \varphi d[\hat{\sigma} : \hat{p}] = - \int_{\omega} \varphi d[\bar{\sigma} : D^2u_3] - \int_{\omega} \varphi \hat{\sigma} : \hat{e} dx' + \int_{\gamma_d} \varphi b_1(\hat{\sigma}) \frac{\partial(u_3 - w_3)}{\partial \nu_{\partial\omega}} d\mathcal{H}^1$$

for every  $\varphi \in C(\bar{\omega})$ .

**Remark 6.5.** The definition of  $[\hat{\sigma} : \hat{p}]$  does not depend on the choice of the triple  $(u, e, w)$ . Moreover, if  $\hat{\sigma} \in C^2(\bar{\omega}; \mathbb{M}_{sym}^{2 \times 2})$  and  $p \in \Pi_{\Gamma_d}(\Omega)$ , then

$$\int_{\omega \cup \gamma_d} \varphi d[\hat{\sigma} : \hat{p}] = \int_{\omega \cup \gamma_d} \varphi \hat{\sigma} : d\hat{p} \quad (6.9)$$

for every  $\varphi \in C^2(\bar{\omega})$ . This follows from the equality

$$\int_{\gamma_d} \varphi b_1(\hat{\sigma}) \frac{\partial(u_3 - w_3)}{\partial \nu_{\partial\omega}} d\mathcal{H}^1 = \int_{\gamma_d} \varphi \hat{\sigma} : (\nabla(u_3 - w_3) \odot \nu_{\partial\omega}) d\mathcal{H}^1,$$

which, in turn, is a consequence of (6.8). By an approximation argument one can show that (6.9) holds true for every  $\hat{\sigma} \in C(\bar{\omega}; \mathbb{M}_{sym}^{2 \times 2})$  and  $\varphi \in C(\bar{\omega})$ .

As a corollary of [12, Proposition 2.1], we have the following integration by parts formula.

**Proposition 6.6.** *Let  $\sigma \in \Sigma(\Omega)$ ,  $w \in W^{1,2}(\omega; \mathbb{R}^3) \cap KL(\Omega)$ , and  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . Then*

$$\begin{aligned} & \int_{\omega \cup \gamma_d} \varphi d[\hat{\sigma} : \hat{p}] + \int_{\omega} \varphi \hat{\sigma} : (\hat{e} + D^2w_3) dx' \\ & \quad - 2 \int_{\omega} \hat{\sigma} : (\nabla \varphi \odot \nabla(u_3 - w_3)) dx' - \int_{\omega} (u_3 - w_3) \hat{\sigma} : \nabla^2 \varphi dx' \\ & = - \int_{\omega} \varphi (u_3 - w_3) d(\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}) + \langle b_0(\hat{\sigma}), \varphi(u_3 - w_3) \rangle - \int_{\gamma_n} b_1(\hat{\sigma}) \frac{\partial(\varphi(u_3 - w_3))}{\partial \nu_{\partial\omega}} d\mathcal{H}^1 \end{aligned} \quad (6.10)$$

for every  $\varphi \in C^2(\bar{\omega})$ , where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing between  $(T_{\partial\omega}(W^{2,1}(\omega)))'$  and  $T_{\partial\omega}(W^{2,1}(\omega))$ .

**Remark 6.7.** The duality product  $\langle b_0(\hat{\sigma}), \varphi(u_3 - w_3) \rangle$  in (6.10) is well defined, since one can show that  $T_{\partial\omega}(BH(\omega)) = T_{\partial\omega}(W^{2,1}(\omega))$  (see, e.g., [11, Section 2]).

We are now in a position to introduce a duality pairing between  $\Sigma(\Omega)$  and  $\Pi_{\Gamma_d}(\Omega)$ . For every  $\sigma \in \Sigma(\Omega)$  and  $p \in \Pi_{\Gamma_d}(\Omega)$  we define the measure  $[\sigma : p] \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  as

$$[\sigma : p] := [\bar{\sigma} : \bar{p}] \otimes \mathcal{L}^1 + \frac{1}{12} [\hat{\sigma} : \hat{p}] \otimes \mathcal{L}^1 - \sigma_{\perp} : e_{\perp}. \quad (6.11)$$

By Remarks 6.1 and 6.5 we have that

$$\int_{\Omega \cup \Gamma_d} \varphi d[\sigma : p] = \int_{\omega} \varphi \bar{\sigma} : d\bar{p} + \frac{1}{12} \int_{\omega} \varphi \hat{\sigma} : d\hat{p} - \int_{\Omega} \varphi \sigma_{\perp} : e_{\perp} dx \quad (6.12)$$

for every  $\sigma \in \Sigma(\Omega)$  with  $\bar{\sigma}, \hat{\sigma} \in C(\bar{\omega}; \mathbb{M}_{sym}^{2 \times 2})$  and every  $\varphi \in C(\bar{\omega})$ . In particular, this implies that

$$\int_{\Omega \cup \Gamma_d} \varphi d[\sigma : p] = \int_{\Omega} \varphi \sigma : dp \quad (6.13)$$

for every  $\sigma \in \Sigma(\Omega) \cap C(\bar{\Omega}; \mathbb{M}_{sym}^{2 \times 2})$  and every  $\varphi \in C(\bar{\omega})$ .

Following [9], for every  $\sigma \in \Sigma(\Omega)$  and  $p \in \Pi_{\Gamma_d}(\Omega)$  we consider the duality pairings

$$\langle \bar{\sigma}, \bar{p} \rangle := [\bar{\sigma} : \bar{p}](\omega \cup \gamma_d), \quad \langle \hat{\sigma}, \hat{p} \rangle := [\hat{\sigma} : \hat{p}](\omega \cup \gamma_d),$$

and

$$\langle \sigma, p \rangle := [\sigma : p](\Omega \cup \Gamma_d) = \langle \bar{\sigma}, \bar{p} \rangle + \frac{1}{12} \langle \hat{\sigma}, \hat{p} \rangle - \int_{\Omega} \sigma_{\perp} : e_{\perp} dx. \quad (6.14)$$

We shall now discuss the connection between the duality (6.14) and the functional  $\mathcal{H}_r$  introduced in (4.8). To this purpose, we consider the set

$$K_r := \{ \sigma \in \mathbb{M}_{sym}^{2 \times 2} : \sigma : \xi \leq H_r(\xi) \text{ for every } \xi \in \mathbb{M}_{sym}^{2 \times 2} \},$$

which coincides with the subdifferential of  $H_r$  at the origin. We also set

$$\mathcal{K}_r(\Omega) := \{ \sigma \in L^{\infty}(\Omega; \mathbb{M}_{sym}^{2 \times 2}) : \sigma(x) \in K_r \text{ for a.e. } x \in \Omega \}.$$

By (2.1) we have that for every  $\mu \in M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$

$$\mathcal{H}_r(\mu) = \sup \left\{ \int_{\Omega \cup \Gamma_d} \tau : d\mu : \tau \in C_0(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2}) \cap \mathcal{K}_r(\Omega) \right\}.$$

A variant of this equality can be proved using the duality defined in (6.14).

**Proposition 6.8.** *Let  $p \in \Pi_{\Gamma_d}(\Omega)$ . Then the following equalities hold:*

$$\mathcal{H}_r(p) = \sup \{ \langle \sigma, p \rangle : \sigma \in \Sigma(\Omega) \cap \mathcal{K}_r(\Omega) \} \quad (6.15)$$

$$= \sup \{ \langle \sigma, p \rangle : \sigma \in \Theta(\Omega) \}, \quad (6.16)$$

where  $\Theta(\Omega)$  is the set of all  $\sigma \in \Sigma(\Omega) \cap \mathcal{K}_r(\Omega)$  such that  $[\bar{\sigma} \nu_{\partial\omega}] = 0$  on  $\gamma_n$ ,  $b_1(\hat{\sigma}) = 0$  on  $\gamma_n$ , and  $\langle b_0(\hat{\sigma}), v \rangle = 0$  for every  $v \in W^{2,1}(\omega)$  with  $v = 0$  on  $\gamma_d$ .

*Proof.* Let us set  $\Gamma_0 := \Gamma_n \cup (\omega \times \{ \pm \frac{1}{2} \})$ . By [36, Chapter II, Section 4] and (6.13) we have that

$$\begin{aligned} \mathcal{H}_r(p) &= \sup \left\{ \int_{\Omega \cup \Gamma_d} \sigma : dp : \sigma \in C^{\infty}(\mathbb{R}^3; \mathbb{M}_{sym}^{2 \times 2}) \cap \mathcal{K}_r(\Omega), \text{ supp } \sigma \cap \Gamma_0 = \emptyset \right\} \\ &\leq \sup \{ \langle \sigma, p \rangle : \sigma \in \Theta(\Omega) \} \\ &\leq \sup \{ \langle \sigma, p \rangle : \sigma \in \Sigma(\Omega) \cap \mathcal{K}_r(\Omega) \}. \end{aligned} \quad (6.17)$$

To prove the converse inequality, let  $w \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ ,  $u \in KL(\Omega)$ , and  $e \in L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$  be such that  $(u, e, p) \in \mathcal{A}_{KL}(w)$ . By Theorem 3.9 we can construct a sequence of triples  $(u^\varepsilon, e^\varepsilon, p^\varepsilon) \in (W^{1,2}(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}) \times L_{\infty,c}^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})) \cap \mathcal{A}_{KL}(w)$  such that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly}^* \text{ in } BD(\Omega), \quad (6.18)$$

$$e^\varepsilon \rightarrow e \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}), \quad (6.19)$$

$$\mathcal{H}_r(p^\varepsilon) \rightarrow \mathcal{H}_r(p). \quad (6.20)$$

By Remark 3.10 we can also assume that

$$\bar{u}^\varepsilon \rightarrow \bar{u} \quad \text{strongly in } L^2(\omega; \mathbb{R}^2), \quad \|E\bar{u}^\varepsilon\|_{L^1} \rightarrow \|E\bar{u}\|_{M_b}, \quad (6.21)$$

$$u_3^\varepsilon \rightarrow u_3 \quad \text{in } C(\bar{\omega}), \quad \|D^2 u_3^\varepsilon\|_{L^1} \rightarrow \|D^2 u_3\|_{M_b}. \quad (6.22)$$

Let now  $\sigma \in \mathcal{K}_r(\Omega) \cap \Sigma(\Omega)$ . It is clear that

$$\int_{\Omega} \sigma : p^\varepsilon \, dx \leq \mathcal{H}_r(p^\varepsilon). \quad (6.23)$$

We now claim that

$$\int_{\Omega} \sigma : p^\varepsilon \, dx \rightarrow \langle \sigma, p \rangle. \quad (6.24)$$

If the claim is proved, then passing to the limit in (6.23) and applying (6.20) yield

$$\langle \sigma, p \rangle \leq \mathcal{H}_r(p),$$

which, together with (6.17), implies the thesis.

We now prove (6.24). Since  $\bar{u}^\varepsilon \in W^{1,2}(\omega; \mathbb{R}^2)$  and  $E\bar{u}^\varepsilon = \bar{e}^\varepsilon + \bar{p}^\varepsilon$  in  $\omega$ , the following equalities hold:

$$\begin{aligned} \int_{\omega} \bar{\sigma} : \bar{p}^\varepsilon \, dx' &= - \int_{\omega} \bar{\sigma} : (\bar{e}^\varepsilon - E\bar{w}) \, dx' + \int_{\omega} \bar{\sigma} : (E\bar{u}^\varepsilon - E\bar{w}) \, dx' \\ &= - \int_{\omega} \bar{\sigma} : (\bar{e}^\varepsilon - E\bar{w}) \, dx' - \int_{\omega} \operatorname{div}_{x'} \bar{\sigma} \cdot (\bar{u}^\varepsilon - \bar{w}) \, dx' + \int_{\gamma_n} [\bar{\sigma} \nu_{\partial\omega}] \cdot (\bar{u}^\varepsilon - \bar{w}) \, d\mathcal{H}^1, \end{aligned}$$

where we have used (6.1) and the fact that  $\bar{u}^\varepsilon = \bar{w}$  on  $\gamma_d$ . From (6.21) it follows that  $\bar{u}^\varepsilon \rightarrow \bar{u}$  strongly in  $L^1(\partial\omega; \mathbb{R}^2)$  (see, e.g., [36, Chapter II, Theorem 3.1]). By (6.19) and (6.21) we can therefore pass to the limit in the identity above and by (6.5) we deduce that

$$\int_{\omega} \bar{\sigma} : \bar{p}^\varepsilon \, dx' \rightarrow \langle \bar{\sigma}, \bar{p} \rangle. \quad (6.25)$$

Similarly, since  $u_3^\varepsilon \in W^{2,2}(\omega)$  and  $D^2 u_3^\varepsilon = -(\hat{e}^\varepsilon + \hat{p}^\varepsilon)$  in  $\omega$ , we have

$$\begin{aligned} \int_{\omega} \hat{\sigma} : \hat{p}^\varepsilon \, dx' &= - \int_{\omega} \hat{\sigma} : (\hat{e}^\varepsilon + D^2 w_3) \, dx' - \int_{\omega} \hat{\sigma} : (D^2 u_3^\varepsilon - D^2 w_3) \, dx' \\ &= - \int_{\omega} \hat{\sigma} : (\hat{e}^\varepsilon + D^2 w_3) \, dx' - \int_{\omega} (u_3^\varepsilon - w_3) \, d(\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}) \\ &\quad + \langle b_0(\hat{\sigma}), u_3^\varepsilon - w_3 \rangle - \int_{\gamma_n} b_1(\hat{\sigma}) \frac{\partial(u_3^\varepsilon - w_3)}{\partial\nu_{\partial\omega}} \, d\mathcal{H}^1, \end{aligned}$$

where we have used (6.6) and the fact that  $\nabla u_3^\varepsilon = \nabla w_3$  on  $\gamma_d$ . By (6.22) and [10, Theorem 3.4] we can pass to the limit in the boundary terms. Therefore, by (6.19), (6.22), and (6.10), we conclude that

$$\int_{\omega} \hat{\sigma} : \hat{p}^\varepsilon \, dx' \rightarrow \langle \hat{\sigma}, \hat{p} \rangle. \quad (6.26)$$

Claim (6.24) follows now by combining the identity

$$\int_{\Omega} \sigma : p^\varepsilon \, dx = \int_{\omega} \bar{\sigma} : \bar{p}^\varepsilon \, dx' + \frac{1}{12} \int_{\omega} \hat{\sigma} : \hat{p}^\varepsilon \, dx' - \int_{\Omega} \sigma_{\perp} : e_{\perp}^\varepsilon \, dx$$

with (6.14) and the convergence properties (6.19), (6.25), and (6.26).  $\square$

We are now in a position to show a further equivalent characterization of the minimality condition (qs1)<sub>r</sub>.

**Proposition 6.9.** *Let  $\sigma \in L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2})$ . The following conditions are equivalent:*

- (a)  $-\mathcal{H}_r(q) \leq \int_{\Omega} \sigma : f \, dx$  for every  $(v, f, q) \in \mathcal{A}_{KL}(0)$ ,
- (b)  $\sigma \in \Theta(\Omega)$ ,  $\operatorname{div}_{x'} \bar{\sigma} = 0$  in  $\omega$ , and  $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma} = 0$  in  $\omega$ .

*Proof.* Assume (a). Let  $B \subset \Omega$  be a Borel set and let  $\chi_B$  denote its characteristic function. Let  $\xi \in \mathbb{M}_{sym}^{2 \times 2}$  and let  $f := \chi_B \xi$ . Since  $(0, -f, f) \in \mathcal{A}_{KL}(0)$ , by (a) we obtain

$$\sigma(x) : \xi \leq H_r(\xi) \quad \text{for a.e. } x \in B.$$

Since  $B$  is arbitrary, we deduce that  $\sigma \in \mathcal{K}_r(\Omega)$ .

We observe that  $(\pm v, \pm E v, 0) \in \mathcal{A}_{KL}(0)$  for every  $v \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  such that  $v = 0$  on  $\Gamma_d$ . Hence, by (a) we have that

$$\int_{\Omega} \sigma : E v \, dx = 0 \tag{6.27}$$

for every  $v \in W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$  with  $v = 0$  on  $\Gamma_d$ . Let now  $\bar{v} \in W^{1,2}(\omega; \mathbb{R}^2)$  with  $\bar{v} = 0$  on  $\gamma_d$ . Choosing  $v_{\alpha} = \bar{v}_{\alpha}$  for  $\alpha = 1, 2$  and  $v_3 = 0$ , we deduce by (6.27) that

$$\int_{\omega} \bar{\sigma} : E \bar{v} \, dx' = 0 \tag{6.28}$$

for every  $\bar{v} \in W^{1,2}(\omega; \mathbb{R}^2)$  with  $\bar{v} = 0$  on  $\gamma_d$ . Since this is true, in particular, for  $\bar{v} \in C_c^{\infty}(\omega; \mathbb{R}^2)$ , we conclude that  $\operatorname{div}_{x'} \bar{\sigma} = 0$  in  $\omega$ . Moreover, by (6.1), (6.28), and the subsequent Lemma 6.10, we obtain that  $[\bar{\sigma} \nu_{\partial \omega}] = 0$  on  $\gamma_n$ .

Let us now consider the function

$$v(x) = \begin{pmatrix} -x_3 \nabla v_3(x') \\ v_3(x') \end{pmatrix} \quad \text{for a.e. } x \in \Omega,$$

where  $v_3 \in W^{2,2}(\omega)$  is such that  $v_3 = 0$  and  $\nabla v_3 = 0$  on  $\gamma_d$ . Equation (6.27) yields

$$\int_{\omega} \hat{\sigma} : D^2 v_3 \, dx' = 0 \tag{6.29}$$

for every  $v_3 \in W^{2,2}(\omega)$  with  $v_3 = 0$  and  $\nabla v_3 = 0$  on  $\gamma_d$ . Since (6.29) is satisfied, in particular, for every  $v_3 \in C_c^{\infty}(\omega)$ , we deduce that  $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma} = 0$  in  $\omega$ . Moreover, by (6.6), (6.29), and Lemma 6.10, we obtain that

$$-\langle b_0(\hat{\sigma}), v_3 \rangle + \int_{\gamma_n} b_1(\hat{\sigma}) \frac{\partial v_3}{\partial \nu_{\partial \omega}} \, d\mathcal{H}^1 = 0$$

for every  $v_3 \in W^{2,1}(\omega)$  such that  $v_3 = 0$  and  $\nabla v_3 = 0$  on  $\gamma_d$ . By [11, Théorème 1] the trace operator from  $W^{2,1}(\omega)$  into  $T_{\partial \omega}(W^{2,1}(\omega)) \times L^1(\partial \omega)$  that associates to  $u$  the traces of  $u$  and of  $\frac{\partial u}{\partial \nu_{\partial \omega}}$  on  $\partial \omega$  is surjective. We deduce that  $b_1(\hat{\sigma}) = 0$  on  $\gamma_n$  and  $\langle b_0(\hat{\sigma}), v_3 \rangle = 0$  for every  $v_3 \in W^{2,1}(\omega)$  with  $v_3 = 0$  on  $\gamma_d$ , hence  $\sigma \in \Theta(\Omega)$ . This concludes the proof of (b).

Assume now (b). Choosing  $\varphi \equiv 1$  in (6.5) and (6.10) yields

$$\langle \bar{\sigma}, \bar{q} \rangle = - \int_{\omega} \bar{\sigma} : \bar{f} \, dx', \quad \langle \hat{\sigma}, \hat{q} \rangle = - \int_{\omega} \hat{\sigma} : \hat{f} \, dx'$$

for every  $(v, f, q) \in \mathcal{A}_{KL}(0)$ . Therefore, by (6.14)

$$\langle \sigma, q \rangle = - \int_{\Omega} \sigma : f \, dx.$$

Condition (a) follows now from Proposition 6.8.  $\square$

We conclude this subsection with an approximation lemma, that was needed in the proof of Proposition 6.9.

**Lemma 6.10.** (i) Let  $\bar{v} \in W^{1,1}(\omega; \mathbb{R}^2)$  with  $\bar{v} = 0$  on  $\gamma_d$ . Then there exists a sequence  $(\bar{v}^\varepsilon) \subset W^{1,2}(\omega; \mathbb{R}^2)$  such that  $\bar{v}^\varepsilon = 0$  on  $\gamma_d$  for every  $\varepsilon > 0$  and  $\bar{v}^\varepsilon \rightarrow \bar{v}$  strongly in  $W^{1,1}(\omega; \mathbb{R}^2)$ .  
(ii) Let  $v \in W^{2,1}(\omega)$  with  $v = 0$  and  $\nabla v = 0$  on  $\gamma_d$ . Then there exists a sequence  $(v^\varepsilon) \subset W^{2,2}(\omega)$  such that  $v^\varepsilon = 0$  and  $\nabla v^\varepsilon = 0$  on  $\gamma_d$ , and  $v^\varepsilon \rightarrow v$  strongly in  $W^{2,1}(\omega)$ .

*Proof.* We only sketch the proof of (i). Statement (ii) can be proved by similar arguments.

Arguing as in Step 1 of the proof of Theorem 3.9, we can reduce, without loss of generality, to the case where there exists an open set  $J \subset \partial\omega$  such that  $\gamma_d$  is compactly contained in  $J$  and  $\bar{v} = 0$  on  $J$ . As in Step 2 of the proof of Theorem 3.9 we consider the open covering  $\{Q_i\}_{i=0,\dots,m}$  of  $\bar{\omega}$ , a subordinate partition of unity  $\{\varphi_i\}_{i=0,\dots,m}$ , and the outward and inward translations  $\tau_{i,\varepsilon}$  with  $a_\varepsilon = \varepsilon$ . We set

$$\tilde{\omega} := \omega \cup \bigcup_{i=1}^{m_0} Q_i$$

and we extend  $\bar{v}$  to  $\tilde{\omega}$  by setting  $\bar{v} = 0$  outside  $\bar{\omega}$ , so that  $\bar{v} \in W^{1,1}(\tilde{\omega}; \mathbb{R}^2)$ . We define

$$\bar{v}^\varepsilon := \left( \sum_{i=1}^m (\varphi_i \bar{v}) \circ \tau_{i,\varepsilon} + \varphi_0 \bar{v} \right) * \rho_{\delta(\varepsilon)},$$

where  $\rho_{\delta(\varepsilon)}$  is a mollifier and  $\delta(\varepsilon) < \varepsilon$  is chosen small enough in such a way that  $\bar{v}^\varepsilon = 0$  on  $\gamma_d$ . It is now easy to check that the sequence  $(\bar{v}^\varepsilon)$  has all the required properties.  $\square$

**6.2. Equivalent formulations in rate form.** From here to the end of the section we will assume  $t \mapsto w(t)$  to be absolutely continuous from  $[0, T]$  into  $W^{1,2}(\Omega; \mathbb{R}^3) \cap KL(\Omega)$ . This implies that the maps  $t \mapsto \bar{w}(t)$  and  $t \mapsto w_3(t)$  are absolutely continuous from  $[0, T]$  into  $W^{1,2}(\omega; \mathbb{R}^2)$  and  $W^{2,2}(\omega)$ , respectively.

We first prove some preliminary results. An easy adaptation of [9, Lemma 5.5] provides us with the following lemma.

**Lemma 6.11.** Let  $t \mapsto (u(t), e(t), p(t))$  be an absolutely continuous function from  $[0, T]$  into  $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  with  $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$  for every  $t \in [0, T]$ . Then  $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}_{KL}(\dot{w}(t))$  for a.e.  $t \in [0, T]$ .

For absolutely continuous triples the energy balance can be equivalently written as a balance of powers, as shown in the next proposition.

**Proposition 6.12.** Let  $t \mapsto (u(t), e(t), p(t))$  be an absolutely continuous function from  $[0, T]$  into  $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  and let  $\sigma(t) := \mathbb{C}_r e(t)$ . Then, the following conditions are equivalent:

(a) for every  $t \in [0, T]$

$$\mathcal{Q}_r(e(t)) + \mathcal{D}_r(p; 0, t) = \mathcal{Q}_r(e(0)) + \int_0^t \int_\Omega \sigma(s) : E\dot{w}(s) \, dx \, ds;$$

(b) for a.e.  $t \in [0, T]$

$$\int_\Omega \sigma(t) : \dot{e}(t) \, dx + \mathcal{H}_r(\dot{p}(t)) = \int_\Omega \sigma(t) : E\dot{w}(t) \, dx.$$

*Proof.* Since  $t \mapsto p(t)$  is absolutely continuous, by [9, Theorem 7.1] we have

$$\mathcal{D}_r(p; 0, t) = \int_0^t \mathcal{H}_r(\dot{p}(s)) \, ds.$$

The equivalence of (a) and (b) follows now by differentiation of (a) and integration of (b).  $\square$

We are finally in a position to state the main result of this section.

**Theorem 6.13.** Let  $t \mapsto (u(t), e(t), p(t))$  be a function from  $[0, T]$  into  $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{2 \times 2}) \times M_b(\Omega \cup \Gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  and let  $\sigma(t) := \mathbb{C}_r e(t)$ . Then the following conditions are equivalent:

(a)  $t \mapsto (u(t), e(t), p(t))$  is a reduced quasistatic evolution for the boundary datum  $w(t)$ ;

- (b)  $t \mapsto (u(t), e(t), p(t))$  is absolutely continuous and  
 (b1) for every  $t \in [0, T]$  we have  $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ ,  $\sigma(t) \in \Theta(\Omega)$ ,  
 $\operatorname{div}_{x'} \bar{\sigma}(t) = 0$  in  $\omega$ , and  $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}(t) = 0$  in  $\omega$ ,  
 (b2) for a.e.  $t \in [0, T]$  there holds

$$\mathcal{H}_r(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t) \rangle = \langle \bar{\sigma}(t), \dot{\bar{p}}(t) \rangle + \frac{1}{12} \langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t);$$

- (c)  $t \mapsto (u(t), e(t), p(t))$  is absolutely continuous and  
 (c1) for every  $t \in [0, T]$  we have  $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$ ,  $\sigma(t) \in \Theta(\Omega)$ ,  
 $\operatorname{div}_{x'} \bar{\sigma}(t) = 0$  in  $\omega$ , and  $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}(t) = 0$  in  $\omega$ ,  
 (c2) for a.e.  $t \in [0, T]$  and for every  $\tau \in \Theta(\Omega)$  there holds

$$\langle \sigma(t) - \tau, \dot{p}(t) \rangle \geq 0;$$

- (d)  $t \mapsto (u(t), e(t))$  is absolutely continuous and  
 (d1) for every  $t \in [0, T]$  we have  $\sigma(t) \in \Theta(\Omega)$ ,  $\operatorname{div}_{x'} \bar{\sigma}(t) = 0$  in  $\omega$ , and  $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}(t) = 0$  in  $\omega$ ,  
 (d2) for a.e.  $t \in [0, T]$  and for every  $\tau \in \Theta(\Omega)$  there holds

$$\begin{aligned} & \int_{\Omega} (\tau - \sigma(t)) : \dot{e}(t) dx + \int_{\omega} \operatorname{div}_{x'} \bar{\tau} \cdot \dot{u}(t) dx' + \frac{1}{12} \int_{\omega} \dot{u}_3(t) d(\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\tau}) \\ & \geq \int_{\gamma_d} [(\bar{\tau} - \bar{\sigma}) \nu_{\partial\omega}] \cdot \dot{w}(t) d\mathcal{H}^1 + \frac{1}{12} \langle b_0(\hat{\tau} - \hat{\sigma}(t)), \dot{w}_3(t) \rangle - \frac{1}{12} \int_{\gamma_d} b_1(\hat{\tau} - \hat{\sigma}(t)) \frac{\partial \dot{w}_3(t)}{\partial \nu_{\partial\omega}} d\mathcal{H}^1, \end{aligned}$$

- (d3) for every  $t \in [0, T]$ ,  $p(t) = Eu(t) - e(t)$  on  $\Omega$  and  $p(t) = (w(t) - u(t)) \odot \nu_{\partial\Omega} \mathcal{H}^2$  on  $\Gamma_d$ .

**Remark 6.14.** The duality products  $\langle \sigma(t), \dot{p}(t) \rangle$  and  $\langle \sigma(t) - \tau, \dot{p}(t) \rangle$  in conditions (b) and (c) are well defined since  $\dot{p}(t) \in \Pi_{\Gamma_d}(\Omega)$  by Lemma 6.11.

*Proof of Theorem 6.13.* We first show that (a) is equivalent to (b). By Remark 5.7 every reduced quasistatic evolution is absolutely continuous, while Proposition 6.9 and Remark 5.8 yield the equivalence of  $(\text{qs1})_r$  and (b1). Hence, by Proposition 6.12 it is enough to show that for every absolutely continuous function satisfying either (b1) or  $(\text{qs1})_r$ , (b2) is equivalent to the following condition: for a.e.  $t \in [0, T]$

$$\int_{\Omega} \sigma(t) : \dot{e}(t) dx + \mathcal{H}_r(\dot{p}(t)) = \int_{\Omega} \sigma(t) : E\dot{w}(t) dx.$$

This follows from Propositions 6.2 and 6.6, once we notice that  $(\dot{u}(t), \dot{e}(t), \dot{p}(t)) \in \mathcal{A}_{KL}(\dot{w}(t))$  by Lemma 6.11.

To show that (b) and (c) are equivalent, it is enough to prove that, if (b1) holds, then (b2) is equivalent to (c2). Indeed, condition (c2) is equivalent to

$$\langle \sigma(t), \dot{p}(t) \rangle \geq \sup_{\tau \in \Theta(\Omega)} \langle \tau, \dot{p}(t) \rangle.$$

On the other hand, by (b1) there holds

$$\langle \sigma(t), \dot{p}(t) \rangle \leq \sup_{\tau \in \Theta(\Omega)} \langle \tau, \dot{p}(t) \rangle.$$

By Proposition 6.8 we deduce the thesis.

To conclude the proof of the theorem, we show that (c) is equivalent to (d). We first remark that if  $t \mapsto (u(t), e(t))$  is absolutely continuous and (d3) holds, then  $t \mapsto p(t)$  is absolutely continuous and  $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$  for every  $t \in [0, T]$ . Hence, it remains only to prove that, if (c1) holds, then (c2) is equivalent to (d2). By Propositions 6.2 and 6.6 there

holds

$$\begin{aligned} \langle \sigma(t) - \tau, \dot{p}(t) \rangle &= \int_{\Omega} (\tau - \sigma(t)) : (\dot{e}(t) - E\dot{w}(t)) \, dx \\ &\quad + \int_{\omega} \operatorname{div}_{x'} \bar{\tau} \cdot (\dot{u} - \dot{w}) \, dx' + \frac{1}{12} \int_{\omega} (\dot{u}_3 - \dot{w}_3) \, d(\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\tau}), \end{aligned}$$

therefore (c2) is equivalent to

$$\begin{aligned} \int_{\Omega} (\tau - \sigma(t)) : (\dot{e}(t) - E\dot{w}(t)) \, dx \\ + \int_{\omega} \operatorname{div}_{x'} \bar{\tau} \cdot (\dot{u} - \dot{w}) \, dx' + \frac{1}{12} \int_{\omega} (\dot{u}_3 - \dot{w}_3) \, d(\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\tau}) \geq 0 \end{aligned} \quad (6.30)$$

for a.e.  $t \in [0, T]$  and every  $\tau \in \Theta(\Omega)$ . By (c1), (6.1), and (6.6) we deduce that

$$\int_{\omega} (\bar{\tau} - \bar{\sigma}(t)) : E\dot{w}(t) \, dx' = \int_{\gamma_d} [(\bar{\tau} - \bar{\sigma}(t)) \nu_{\partial\omega}] \cdot \dot{w}(t) \, d\mathcal{H}^1 - \int_{\omega} \operatorname{div}_{x'} \bar{\tau} \cdot \dot{w}(t) \, dx',$$

and

$$\begin{aligned} \int_{\omega} (\hat{\tau} - \hat{\sigma}(t)) : D^2 \dot{w}_3(t) \, dx' \\ = -\langle b_0(\hat{\tau} - \hat{\sigma}(t)), \dot{w}_3(t) \rangle + \int_{\gamma_d} b_1(\hat{\tau} - \hat{\sigma}(t)) \frac{\partial \dot{w}_3(t)}{\partial \nu_{\partial\omega}} \, d\mathcal{H}^1 + \int_{\omega} \dot{w}_3 \, d(\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\tau}). \end{aligned}$$

Therefore, (6.30) is in turn equivalent to (d2) and the proof of the theorem is complete.  $\square$

**6.3. Two-dimensional characterizations.** In this subsection we show that, under some additional hypotheses on the boundary datum and the initial data, a reduced quasistatic evolution can be written in terms of two-dimensional quantities only. The first proposition concerns a quasistatic evolution  $(u(t), e(t), p(t))$  with “in-plane” boundary datum and initial data. In this case, the triple given by the tangential component of  $u(t)$  and the zero-th order moments of  $e(t)$  and  $p(t)$  is a two-dimensional quasistatic evolution in  $\omega$  in the sense of [9]. It is convenient to introduce the following notation: for every  $\bar{w} \in W^{1,2}(\omega; \mathbb{R}^2)$  we denote by  $\bar{\mathcal{A}}_{KL}(\bar{w})$  the class of all triples  $(v, f, q)$  in  $BD(\omega) \times L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}) \times M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  such that  $E v = f + q$  in  $\omega$  and  $q = (\bar{w} - v) \odot \nu_{\partial\omega} \mathcal{H}^1$  on  $\gamma_d$ . Moreover, we introduce the space

$$\bar{\Sigma}(\omega) := \{ \sigma \in L^\infty(\omega; \mathbb{M}_{sym}^{2 \times 2}) : \operatorname{div}_{x'} \sigma \in L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}) \}$$

and the set

$$\mathcal{K}_r(\omega) := \{ \sigma \in L^\infty(\omega; \mathbb{M}_{sym}^{2 \times 2}) : \sigma(x') \in K_r \text{ for a.e. } x' \in \omega \}.$$

**Proposition 6.15.** *Let  $t \mapsto \bar{w}(t)$  be absolutely continuous from  $[0, T]$  into  $W^{1,2}(\omega; \mathbb{R}^2)$  and let*

$$w(t, x) := \begin{pmatrix} \bar{w}(t, x') \\ 0 \end{pmatrix} \quad \text{for every } t \in [0, T] \text{ and a.e. } x \in \Omega.$$

Let  $(\bar{u}_0, \bar{e}_0, \bar{p}_0) \in \bar{\mathcal{A}}_{KL}(\bar{w}(0))$  and let

$$u_0(x) := \begin{pmatrix} \bar{u}_0(x') \\ 0 \end{pmatrix}, \quad e_0(x) := \bar{e}_0(x') \quad \text{for a.e. } x \in \Omega, \quad p_0 := \bar{p}_0 \otimes \mathcal{L}^1.$$

Finally, let  $t \mapsto (u(t), e(t), p(t))$  be a reduced quasistatic evolution for the boundary value  $w(t)$  such that  $u(0) = u_0$ ,  $e(0) = e_0$ , and  $p(0) = p_0$ , and let  $\sigma(t) := \mathbb{C}_r e(t)$ . Then the map  $t \mapsto (\bar{u}(t), \bar{e}(t), \bar{p}(t))$  satisfies the following conditions:

- (i)  $t \mapsto (\bar{u}(t), \bar{e}(t), \bar{p}(t))$  is absolutely continuous from  $[0, T]$  into  $BD(\omega) \times L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}) \times M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  and  $\bar{u}(0) = \bar{u}_0$ ,  $\bar{e}(0) = \bar{e}_0$ , and  $\bar{p}(0) = \bar{p}_0$ ;
- (ii) for every  $t \in [0, T]$  we have  $(\bar{u}(t), \bar{e}(t), \bar{p}(t)) \in \bar{\mathcal{A}}_{KL}(\bar{w}(t))$ ,  $\bar{\sigma}(t) \in \bar{\Sigma}(\omega) \cap \mathcal{K}_r(\omega)$ ,  $\operatorname{div}_{x'} \bar{\sigma}(t) = 0$  in  $\omega$ , and  $[\bar{\sigma} \nu_{\partial\omega}] = 0$  on  $\gamma_n$ ;

(iii) for a.e.  $t \in [0, T]$  there holds

$$\mathcal{H}_r(\dot{\hat{p}}(t)) = \langle \bar{\sigma}(t), \dot{\hat{p}}(t) \rangle. \quad (6.31)$$

*Proof.* Condition (i) follows from Remark 5.7. By condition (b1) of Theorem 6.13 and the convexity of  $K_r$  we deduce condition (ii).

By property (b2) of Theorem 6.13 and Proposition 6.8 we have

$$\begin{aligned} \mathcal{H}_r(\dot{p}(t)) &= \langle \bar{\sigma}(t), \dot{\hat{p}}(t) \rangle + \frac{1}{12} \langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t) dx \\ &\leq \mathcal{H}_r(\dot{\hat{p}}(t)) + \frac{1}{12} \langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t) dx \\ &= \mathcal{H}_r(\dot{\hat{p}}(t)) - \frac{1}{12} \int_{\omega} \hat{\sigma}(t) : \dot{e}(t) dx - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t) dx, \end{aligned} \quad (6.32)$$

where the last inequality follows from (6.10) with  $\varphi \equiv 1$  and from the fact that  $\sigma(t) \in \Theta(\Omega)$  and  $w_3(t) = 0$  for every  $t \in [0, T]$ . On the other hand, setting

$$\lambda(t) := |\dot{\hat{p}}(t)| + |\dot{\hat{p}}(t)| + \mathcal{L}^2$$

for a.e.  $t \in [0, T]$ , we have that the measure  $\dot{\hat{p}}(t) + x_3 \dot{\hat{p}}(t) - \dot{e}_{\perp}(\cdot, x_3)$  on  $\omega \cup \gamma_d$  is absolutely continuous with respect to  $\lambda(t)$  for a.e.  $x_3 \in (-\frac{1}{2}, \frac{1}{2})$ . Therefore, by Jensen inequality we obtain

$$\begin{aligned} \mathcal{H}_r(\dot{p}(t)) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\omega \cup \gamma_d} H_r \left( \frac{d(\dot{\hat{p}}(t) + x_3 \dot{\hat{p}}(t) - \dot{e}_{\perp}(\cdot, x_3))}{d\lambda(t)} \right) d\lambda(t) dx_3 \\ &\geq \int_{\omega \cup \gamma_d} H_r \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d(\dot{\hat{p}}(t) + x_3 \dot{\hat{p}}(t) - \dot{e}_{\perp}(\cdot, x_3))}{d\lambda(t)} dx_3 \right) d\lambda(t) \\ &= \int_{\omega \cup \gamma_d} H_r \left( \frac{d\dot{\hat{p}}(t)}{d\lambda(t)} \right) d\lambda(t) = \mathcal{H}_r(\dot{\hat{p}}(t)) \end{aligned} \quad (6.33)$$

for a.e.  $t \in [0, T]$ . Combining (6.32) and (6.33), we deduce that

$$-\frac{d}{dt} \left( \frac{1}{12} \mathcal{Q}_r(\hat{e}(t)) + \mathcal{Q}_r(e_{\perp}(t)) \right) = -\frac{1}{12} \int_{\omega} \hat{\sigma}(t) : \dot{e}(t) dx - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t) dx \geq 0.$$

In particular, this implies that

$$\frac{1}{12} \mathcal{Q}_r(\hat{e}(t)) + \mathcal{Q}_r(e_{\perp}(t)) \leq \frac{1}{12} \mathcal{Q}_r(\hat{e}(0)) + \mathcal{Q}_r(e_{\perp}(0)) = 0,$$

hence  $\hat{e}(t) = 0$  and  $e_{\perp}(t) = 0$  for every  $t \in [0, T]$ . This, together with (6.32) and (6.33), yields (6.31).  $\square$

In this last proposition we consider a quasistatic evolution  $(u(t), e(t), p(t))$  with “out-of-plane” boundary datum and initial data and we prove that the triple given by the normal component of  $u(t)$  and the first order moment of  $e(t)$  and  $p(t)$  is a two-dimensional quasistatic evolution in  $\omega$  in the sense of [12, Definition 4.1]. To this purpose, for every  $w_3 \in W^{2,2}(\omega)$  we define the class  $\hat{\mathcal{A}}_{KL}(w_3)$  as the set of all triples  $(v, f, q) \in BH(\omega) \times L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}) \times M_b(\omega; \mathbb{M}_{sym}^{2 \times 2})$  such that  $D^2 v = -(f + q)$  in  $\omega$ ,  $v = w_3$  on  $\gamma_d$ , and  $q = (\nabla v - \nabla w_3) \odot \nu_{\partial\omega} \mathcal{H}^1$  on  $\gamma_d$ .

**Proposition 6.16.** *Assume the function  $H$  to be homogeneous of degree one, i.e.,*

$$H(\lambda \xi) = |\lambda| H(\xi) \quad \text{for every } \lambda \in \mathbb{R}, \xi \in \mathbb{M}_{sym}^{3 \times 3}. \quad (6.34)$$

Let  $t \mapsto w_3(t)$  be absolutely continuous from  $[0, T]$  into  $W^{2,2}(\omega)$  and let

$$w(t, x) := \begin{pmatrix} -x_3 \nabla w_3(t, x') \\ w_3(t, x') \end{pmatrix} \quad \text{for every } t \in [0, T] \text{ and a.e. } x \in \Omega.$$

Let  $(v_0, \hat{e}_0, \hat{p}_0) \in \hat{\mathcal{A}}_{KL}(w_3(0))$  and let

$$u_0(x) := \begin{pmatrix} -x_3 \nabla v_0(x') \\ v_0(x') \end{pmatrix}, \quad e_0(x) := x_3 \hat{e}_0(x') \quad \text{for a.e. } x \in \Omega, \quad p_0 := x_3 \hat{p}_0 \otimes \mathcal{L}^1.$$



Finally, let  $t \mapsto (u(t), e(t), p(t))$  be a reduced quasistatic evolution for the boundary value  $w(t)$  such that  $u(0) = u_0$ ,  $e(0) = e_0$ , and  $p(0) = p_0$ , and let  $\sigma(t) := \mathbb{C}_r e(t)$ . Then the map  $t \mapsto (u_3(t), \hat{e}(t), \hat{p}(t))$  satisfies the following conditions:

- (i)  $t \mapsto (u_3(t), \hat{e}(t), \hat{p}(t))$  is absolutely continuous from  $[0, T]$  into  $BH(\omega) \times L^2(\omega; \mathbb{M}_{sym}^{2 \times 2}) \times M_b(\omega \cup \gamma_d; \mathbb{M}_{sym}^{2 \times 2})$  and  $u_3(0) = v_0$ ,  $\hat{e}(0) = \hat{e}_0$ , and  $\hat{p}(0) = \hat{p}_0$ ;
- (ii) for every  $t \in [0, T]$  we have  $(u_3(t), \hat{e}(t), \hat{p}(t)) \in \hat{\mathcal{A}}_{KL}(w_3(t))$ ,  $\hat{\sigma}(t) \in \hat{\Sigma}(\omega) \cap \mathcal{K}_r(\omega)$ ,  $\operatorname{div}_{x'} \operatorname{div}_{x'} \hat{\sigma}(t) = 0$  in  $\omega$ ,  $b_1(\hat{\sigma}(t)) = 0$  on  $\gamma_n$ , and  $\langle b_0(\hat{\sigma}(t)), v \rangle = 0$  for every  $v \in W^{2,1}(\omega)$  with  $v = 0$  on  $\gamma_d$ ;
- (iii) for a.e.  $t \in [0, T]$  there holds

$$\mathcal{H}_r(\dot{\hat{p}}(t)) = \langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle. \quad (6.35)$$

*Proof.* We first remark that (6.34) implies that the same property is fulfilled by  $H_r$ . This latter condition is in turn equivalent to saying that the set  $K_r$  is symmetric with respect to the origin.

Condition (i) follows from Remark 5.7. By property (b1) of Theorem 6.13 we have that  $\sigma(t) \in \mathcal{K}_r(\Omega)$  for every  $t \in [0, T]$ . Since  $K_r$  is convex and symmetric with respect with the origin, this implies that  $\hat{\sigma}(t) \in \mathcal{K}_r(\omega)$  for every  $t \in [0, T]$ . All the other conditions in (ii) follow from Theorem 6.13.

By property (b2) of Theorem 6.13 and Proposition 6.8 we have

$$\begin{aligned} \mathcal{H}_r(\dot{\hat{p}}(t)) &= \langle \bar{\sigma}(t), \dot{\hat{p}}(t) \rangle + \frac{1}{12} \langle \hat{\sigma}(t), \dot{\hat{p}}(t) \rangle - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t) dx \\ &\leq \frac{1}{12} \mathcal{H}_r(\dot{\hat{p}}(t)) + \langle \bar{\sigma}(t), \dot{\hat{p}}(t) \rangle - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t) dx \\ &= \frac{1}{12} \mathcal{H}_r(\dot{\hat{p}}(t)) - \int_{\omega} \bar{\sigma}(t) : \dot{e}(t) dx - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t) dx, \end{aligned} \quad (6.36)$$

where the last inequality follows from (6.5) with  $\varphi \equiv 1$  and from the fact that  $\sigma(t) \in \Theta(\Omega)$  and  $\bar{w}(t) = 0$  for every  $t \in [0, T]$ . On the other hand, setting

$$\lambda(t) := |\dot{\hat{p}}(t)| + |\dot{\hat{p}}(t)| + \mathcal{L}^2$$

for a.e.  $t \in [0, T]$  and applying (6.34) and Jensen inequality, we obtain

$$\begin{aligned} \mathcal{H}_r(\dot{\hat{p}}(t)) &\geq \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\omega \cup \gamma_d} |x_3| H_r \left( \frac{d(\dot{\hat{p}}(t) + x_3 \dot{\hat{p}}(t) - \dot{e}_{\perp}(\cdot, x_3))}{d\lambda(t)} \right) d\lambda(t) dx_3 \\ &\geq \int_{\omega \cup \gamma_d} H_r \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 \frac{d(\dot{\hat{p}}(t) + x_3 \dot{\hat{p}}(t) - \dot{e}_{\perp}(\cdot, x_3))}{d\lambda(t)} dx_3 \right) d\lambda(t) \\ &= \frac{1}{12} \mathcal{H}_r(\dot{\hat{p}}(t)) \end{aligned} \quad (6.37)$$

for a.e.  $t \in [0, T]$ . Combining (6.36) and (6.37), we deduce that

$$-\frac{d}{dt} (\mathcal{Q}_r(\bar{e}(t)) + \mathcal{Q}_r(e_{\perp}(t))) = - \int_{\omega} \bar{\sigma}(t) : \dot{e}(t) dx - \int_{\Omega} \sigma_{\perp}(t) : \dot{e}_{\perp}(t) dx \geq 0.$$

In particular, this implies that

$$\mathcal{Q}_r(\bar{e}(t)) + \mathcal{Q}_r(e_{\perp}(t)) \leq \mathcal{Q}_r(\bar{e}(0)) + \mathcal{Q}_r(e_{\perp}(0)) = 0,$$

hence  $\bar{e}(t) = 0$  and  $e_{\perp}(t) = 0$  for every  $t \in [0, T]$ . This, together with (6.36) and (6.37), yields (6.35).  $\square$

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