

Dispersive estimates for Schrödinger operators with point interactions in \mathbb{R}^3

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Abstract The study of dispersive properties of Schrödinger operators with point interactions is a fundamental tool for understanding the behavior of many body quantum systems interacting with very short range potential, whose dynamics can be approximated by non linear Schrödinger equations with singular interactions. In this work we proved that, in the case of one point interaction in \mathbb{R}^3 , the perturbed Laplacian satisfies the same $L^p - L^q$ estimates of the free Laplacian in the smaller regime $q \in [2, 3)$. These estimates are implied by a recent result concerning the L^p boundedness of the wave operators for the perturbed Laplacian. Our approach, however, is more direct and relatively simple, and could potentially be useful to prove optimal weighted estimates also in the regime $q \geq 3$.

1 Introduction

In quantum mechanics, a huge variety of phenomena are described by system of quantum particles interacting with a very short range potentials, supported near away a discrete set of points in \mathbb{R}^d . This leads to the study of Hamiltonians which formally are defined as

$$H_{\mu, Y} = “-\Delta + \sum_{y \in Y} \mu_j \delta_y” \quad (1)$$

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where $-\Delta$ is the free Laplacian on \mathbb{R}^d , $Y := \{y_1, y_2, \dots\}$ is a countable discrete subset of \mathbb{R}^d and μ_{y_j} are real coupling constants. Thus H describes the motion of a quantum particle interacting with a “contact potentials”, created by point sources of strength μ_{y_j} centered at y_j . The first appearance of such Hamiltonians dates back to the celebrated paper of Kronig and Penney [14], where they consider the case $d = 1$, $Y = \mathbb{Z}$ and μ_y independent on y as a model of a nonrelativistic electron moving in a fixed crystal lattice. The mathematical rigorous study of $H_{\mu, Y}$ was initiated by Albeverio, Fenstad and Høegh-Khron [2], and subsequently continued by other authors (see for instance [21], [11], [12], [7]). In this work we focus on the case of finitely many point interactions on \mathbb{R}^3 . The rigorous definition of $H_{\mu, Y}$ is based on the theory of self adjoint extension of symmetric operators (see [3] for a complete and detailed discussion): one starts with

$$\tilde{H}_Y := -\Delta|_{\mathcal{C}_0^\infty(\mathbb{R}^3 \setminus \{Y\})}, \quad (2)$$

which is a densely defined, non-negative, symmetric operator on $L^2(\mathbb{R}^3)$, with deficiency indices (N, N) , and hence it admits a N^2 -parameter family of self adjoint extension. Among these, we find the important subfamily of the so called local extension, characterized by the following proposition (see [3], [8]):

Proposition 1. Fix $Y := (y_1, \dots, y_N) \subset \mathbb{R}^3$ and $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_N) \in (-\infty, +\infty]^N$. Given $z \in \mathbb{C}$, define

$$G_z(x) := \frac{e^{iz|x|}}{4\pi|x|}, \quad \tilde{G}_z(x) := \begin{cases} \frac{e^{iz|x|}}{4\pi|x|} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad (3)$$

and the $N \times N$ matrix

$$[\Gamma_{\alpha, Y}(z)]_{(j, l)} := \left[\left(\alpha_j - \frac{iz}{4\pi} \right) \delta_{j, l} - \tilde{G}_z(y_j - y_l) \right]_{(j, l)} \quad (4)$$

The meromorphic function $z \mapsto [\Gamma_{\alpha, Y}(z)]^{-1}$ has at most N poles in the upper half space \mathbb{C}^+ , which are all located along the positive imaginary semi-axis. We denote by \mathcal{E} the set of such poles. There exists a self adjoint extension $H_{\alpha, Y}$ of \tilde{H}_Y with the following properties:

- Given $z \in \mathbb{C}^+ \setminus \{\mathcal{E}\}$, the domain of $H_{\alpha, Y}$ can be written as:

$$\mathcal{D}(H_{\alpha, Y}) = \left\{ \psi := \phi_z + \sum_{j, l=1}^N (\Gamma_{\alpha, Y}(z)^{-1})_{j, l} \phi_z(j_l) G_z(\cdot - y_l), \phi_z \in H^2 \right\}. \quad (5)$$

The decomposition is unique for a given z .

- With respect to the decomposition (5), the action of $H_{\alpha, Y}$ is given by

$$(H_{\alpha, Y} - z^2)\psi := (-\Delta - z^2)\phi. \quad (6)$$

Remark 1. The family of self adjoint operator $H_{\alpha,Y}$ realizes in a rigorous way the heuristic definition given by expression (1). It is worth noticing the different roles played by parameters: while μ_j measures the strength of the point interactions at y_j , α_j is related to the scattering length. Indeed, a generic function $\psi \in \mathcal{D}(H_{\alpha,Y})$, $\alpha \neq 0$, satisfies the so called *Bethe-Peierls contact condition*

$$\psi(x) \underset{x \rightarrow y_i}{\sim} \frac{1}{|x - y_j|} + 4\pi\alpha_j, \quad j = 1, \dots, N \quad (7)$$

which is typical for the low-energy behavior of an eigenstate of the Schrödinger equation for a quantum particle subject to a very short range potentials, centered at y_j and with s -wave scattering length $-(4\pi\alpha)^{-1}$ (see the works of Bethe and Peierls [5], [6]). When $\alpha_j = +\infty$, no actual interactions take place at y_j (the s -wave has infinity scattering length); in particular when $\alpha = +\infty$ we recover the Friedrichs extension of \tilde{H}_Y , namely the free Laplacian on $L^2(\mathbb{R}^3)$.

The spectral properties of $H_{\alpha,Y}$ are well known and completely characterized; we encode them in the following proposition (see [3], [8]):

Proposition 2.

1. The spectrum $\sigma(H_{\alpha,Y})$ of $H_{\alpha,Y}$ consists of at most N negative eigenvalues and the absolutely continuous part $\sigma_{ac}(H_{\alpha,Y}) = [0, +\infty)$. Moreover, there exists a one to one correspondence between the poles $i\lambda \in \mathcal{E}$ and the negative eigenvalues $-\lambda^2$ of $H_{\alpha,Y}$, counted with multiplicity.
2. The resolvent of $H_{\alpha,Y}$ is a rank N perturbation of the free resolvent, and it is given by:

$$(H_{\alpha,Y} - z^2)^{-1} - (H_0 - z^2)^{-1} = \sum_{j,k=1}^N (\Gamma_{\alpha,Y}(z)^{-1})_{jk} G_z^{y_j} \otimes \overline{G_z^{y_k}}. \quad (8)$$

We conclude this introduction by observing that $H_{\alpha,Y}$ can be also realized as limit of scaled short range Schrödinger operator. Indeed we have the following Proposition (see [3]):

Proposition 3. Fix $\alpha \in (-\infty, +\infty]^N$ and $Y = \{y_1, \dots, y_N\} \subset \mathbb{R}^3$. There exists real valued potential V_1, \dots, V_N of finite Rollnik norm, and real analytic functions $\lambda_j : \mathbb{R} \rightarrow \mathbb{R}$, with $\lambda_j(0) = 1$, such that the family of Schrödinger operators

$$H_\varepsilon := -\Delta + \sum_{j=1}^N \frac{\lambda_j(\varepsilon)}{\varepsilon^2} V \left(\frac{x - y_j}{\varepsilon} \right) \quad (9)$$

converges in strong resolvent sense to $H_{\alpha,Y}$ as ε goes to zero. Moreover:

$$\alpha_j \neq +\infty \text{ for some } j \iff -\Delta + V_j \text{ has a zero energy resonance.} \quad (10)$$

Remark 2. Proposition 3 makes more convincing the idea of considering the Hamiltonian $H_{\alpha,Y}$ as an approximation of more realistic phenomena, governed by very short range interactions.

2 Dispersive properties of $H_{\alpha,Y}$

Since H_α is a self adjoint operator, it generates a unitary group of operators $e^{itH_{\alpha,Y}}$; in particular the L^2 norm is preserved by the evolution:

$$\|e^{itH_{\alpha,Y}} f\|_{L^2(\mathbb{R}^3)} = \|f\|_{L^2(\mathbb{R}^3)}. \quad (11)$$

It is natural to investigate the dispersive properties of $e^{itH_{\alpha,Y}}$. The first work in this direction is by D'ancona, Pierfelice and Teta [9], who proved weighted $L^1 - L^\infty$ estimates

$$\|w^{-1} e^{itH_{\alpha,Y}} P_{ac} f\|_{L^\infty(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}} \|wf\|_{L^1(\mathbb{R}^3)} \quad (12)$$

where P_{ac} is the projection onto the absolutely continuous spectrum of $H_{\alpha,Y}$ and

$$w(x) = \sum_{j=1}^N \left(1 + \frac{1}{|x - y_j|} \right), \quad (13)$$

under the following assumption:

Assumption 1 *The matrix $\Gamma_{\alpha,Y}(z)$ is invertible for $z \geq 0$, with locally bounded inverse.*

It is worth noticing that the presence of a weight in (12) is unavoidable, because of the singularities appearing in the domain of $H_{\alpha,Y}$. In the case of one single point interaction, assumption 1 is always satisfied except for $\alpha = 0$, in which case the perturbed Hamiltonian has a zero energy resonance. Nevertheless, exploiting the explicit formula for the propagator e^{itH} available in the case $N = 1$ (see [18] [1]), also the case $\alpha = 0$ was settle down in [9], by showing weighted dispersive inequality with a slower decay in t , a typical phenomenon for Schrödinger operators with zero energy resonances:

$$\|w^{-1} e^{itH_{0,Y}} f\|_{L^\infty(\mathbb{R}^3)} \lesssim t^{-\frac{1}{2}} \|wf\|_{L^1(\mathbb{R}^3)}. \quad (14)$$

Observe now that, interpolating (12) and (14) with the trivial bound (11), we get weighted dispersive inequalities in the full range $q \in [2, +\infty]$:

Proposition 4.

1. *Under assumption 1, the following estimates holds:*

$$\|w^{-\left(1-\frac{2}{q}\right)} e^{itH_{\alpha,Y}} P_{ac} f\|_{L^q(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|w^{\frac{2}{p}-1} f\|_{L^p(\mathbb{R}^3)} \quad (15)$$

where $q \in [2, +\infty]$ and p is the dual exponent of q .
 2. In the case $N = 1$, $\alpha = 0$ we have

$$\|w^{-(1-\frac{2}{q})} e^{itH_{0,Y}} f\|_{L^q(\mathbb{R}^3)} \lesssim t^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})} \|w^{\frac{2}{p}-1} f\|_{L^p(\mathbb{R}^3)} \quad (16)$$

where $q \in [2, +\infty]$ and p is the dual exponent of q .

However, since the singularities $G_i(x-y_j)$ belong to $L^q(\mathbb{R}^3)$ for $q \in [2, 3)$, one may hope, at least in principle, to prove an unweighted version of (15) and (16). This is true indeed, and it is a consequence of a recent result [8]:

Theorem 1. For any Y and α , the wave operators

$$W_{\alpha,Y}^{\pm} = s - \lim_{t \rightarrow +\infty} e^{itH_{\alpha,Y}} e^{it\Delta} \quad (17)$$

for the pair $(H_{\alpha,Y}, -\Delta)$ exists and are complete on $L^2(\mathbb{R}^3)$, and they are bounded on $L^q(\mathbb{R}^3)$ for $1 < q < 3$.

Remark 3. The restriction $1 < q < 3$ already emerges at level of approximating Schrödinger operators. Indeed, if $H = -\Delta + V$ has a zero energy resonance (which by Proposition 3 is a necessary condition for H_ε to converges to H), then the wave operators

$$W_V^{\pm} := s - \lim_{t \rightarrow +\infty} e^{it(-\Delta+V)} e^{it\Delta} \quad (18)$$

are bounded on L^q if and only if $1 < q < 3$ (see Yajima [20])

Owing to Theorem 1 and the intertwining property of wave operators, viz.

$$f(H_{\alpha,Y}) P_{ac} H_{\alpha,Y} = W_{\alpha,Y}^{\pm} f(-\Delta) (W_{\alpha,Y}^{\pm})^* \quad (19)$$

for any Borel function f on \mathbb{R}^3 , one can lift the classical dispersive estimates for the free Laplacian into analogous estimates for $H_{\alpha,Y}$, albeit for the restriction on the exponent q . Thus we find:

Proposition 5. For any α and Y , we have the estimate

$$\|e^{itH_{\alpha,Y}} P_{ac} f\|_{L^q(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^3)} \quad \text{for } q \in [2, 3). \quad (20)$$

Interpolating (20) respectively with (12) and (14), we deduce also the following:

Corollary 1. 1. Under assumption 1, we have

$$\|w^{-(1-\frac{3-\varepsilon}{q})} e^{itH_{\alpha,Y}} P_{ac} f\|_{L^q(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|w^{(1-\frac{3-\varepsilon}{q})} f\|_{L^p(\mathbb{R}^3)} \quad (21)$$

in the regime $q \in [3, +\infty]$.

2. When $N = 1$ and $\alpha = 0$, we have

$$\|w^{-(1-\frac{3-\varepsilon}{q})} e^{itH_{0,Y}} f\|_{L^q(\mathbb{R}^3)} \lesssim t^{-\frac{1}{2}+\frac{\varepsilon}{q}} \|w^{(1-\frac{3-\varepsilon}{q})} f\|_{L^p(\mathbb{R}^3)} \quad (22)$$

in the regime $q \in [3, +\infty]$.

We can see that the results in [8] improves the one in [9] in various ways:

1. In the regime $q \in [2, 3)$, with an arbitrary number of centers, both the weights and the hypothesis 1 on Γ are removed.
2. In the regime $q \in [3, +\infty]$, with an arbitrary number of centers and under the hypothesis 1, the weights are strengthened to be almost optimal (indeed we can not remove ε in estimate (21)).
3. In the case $N = 1$, $\alpha = 0$ and in the regime $q \in [2, 3)$, weights are removed and the time decay is strengthened.
4. In the case $N = 1$, $\alpha = 0$ and in the regime $q \in [3, +\infty]$, both the weights and the time decay are strengthened, and again the weights are almost optimal.

In this work we want to provide a new and simpler proof of Proposition 5 in the particular case $N = 1$, without using any information about the wave operators.

3 Proof of Proposition 5, case $N = 1$

The operators H_{α, y_1} and H_{α, y_2} are conjugated by translations, hence we can assume $y = 0$ and we will simply write H_α instead of $H_{\alpha, 0}$. We recall an useful factorization for the operator H_α (see [3]). Introducing spherical coordinates on \mathbb{R}^3 , we can decompose $L^2(\mathbb{R}^3)$ with respect to angular momenta:

$$L^2(\mathbb{R}^3) = L^2(\mathbb{R}^+, r^2 dr) \otimes L^2(S^2) \quad (23)$$

where S^2 is the unit sphere in \mathbb{R}^3 . Moreover, using the unitary transformation

$$U : L^2((0, +\infty), r^2 dr) \rightarrow L^2(\mathbb{R}^+, r dr), \quad (Uf)(r) = rf(r) \quad (24)$$

and decomposing $L^2(S^2)$ into spherical harmonics

$$\{Y_{l, m} \mid l \in \mathbb{N}, m = 0, \pm 1, \dots, \pm l\}, \quad (25)$$

we obtain

$$L^2(\mathbb{R}^3) = \bigoplus_{l=0}^{+\infty} U^{-1} L^2(\mathbb{R}^+, r dr) \otimes \langle Y_{l, -l}, \dots, Y_{l, l} \rangle. \quad (26)$$

With respect to this decomposition, the symmetric operator $\tilde{H} := \tilde{H}_{\{0\}}$ writes as

$$\tilde{H} = \bigoplus_{l=0}^{+\infty} U^{-1} h_l U \otimes 1 \quad (27)$$

where h_l , $l \geq 0$ are symmetric operators on $L^2(\mathbb{R}^+)$, with common domain $\mathcal{C}_0^{+\infty}(\mathbb{R}^+)$ and actions given by

$$h_l = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}, \quad r > 0. \quad (28)$$

It is well known [17] that h_l are essentially self adjoint for $l \geq 1$, while h_0 admits a one parameter family of selfadjoint extension $\dot{h}_{0,\alpha}$ such that

$$H_\alpha = \dot{h}_{0,\alpha} \oplus \bigoplus_{l=1}^{+\infty} U^{-1} \dot{h}_l U \otimes 1 \quad (29)$$

where \dot{h}_l is the unique self adjoint extension of h_l , for $l \geq 1$. Identity (29) tells us that H_α completely diagonalizes with respect to decomposition (26), and that it coincides with $-\Delta$ after projecting out the subspace of radial functions. Hence it immediately follows

Lemma 1. *Suppose $f \in L^2(\mathbb{R}^3)$ is orthogonal to the subspace of radial functions. Then*

$$e^{itH_\alpha} f = e^{-it\Delta} f \quad (30)$$

Lemma 1 has an important Corollary, which considerably simplifies our proof:

Corollary 2. *In the proof of Proposition 5 (in the special case $N = 1$) we can suppose f to be radial.*

Proof. Suppose (20) to be true for radial functions. Given a generic $f \in L^2(\mathbb{R}^3)$, we can decompose it as $f_1 + f_2$, where

$$f_1 := \frac{4\pi}{|y|^2} \int_{S_y} f(r, \omega) d\mathcal{H}^2(\omega) \quad (31)$$

is the orthogonal projection onto $L^2_{rad}(\mathbb{R}^3)$. By Lemma 1, we get

$$e^{itH_\alpha} f = e^{itH_\alpha} f_1 + e^{-it\Delta} f_2. \quad (32)$$

By hypothesis and using the dispersive estimates for the free Laplacian, we deduce

$$\|e^{itH_\alpha} f\|_{L^q} \leq t^{\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} (\|f_1\|_{L^p} + \|f_2\|_{L^p}). \quad (33)$$

Now, using Hölder inequality, we get

$$\begin{aligned} \|f_1\|_{L^p}^p &\leq \int_0^{+\infty} r^{2-2p} \left(\int_{S_r} |f(r, \omega)| d\mathcal{H}^2(\omega) \right)^p dr \\ &\leq \int_0^{+\infty} r^{2-2p+\frac{2p}{q}} \int_{S_r} |f(r, \omega)|^p d\mathcal{H}^2(\omega) dr = \\ &\int_0^{+\infty} \int_{S_r} |f(r, \omega)|^p d\mathcal{H}^2(\omega) = \|f\|_{L^p}^p \end{aligned}$$

and consequently

$$\|f_2\|_{L^p} \leq \|f\|_{L^p} + \|f_1\|_{L^p} \lesssim \|f\|_{L^p} \quad (34)$$

which concludes the proof.

Now we are in turn to prove our main result. As mentioned before, in the case $N = 1$, the propagator associated to H_α is explicitly known. In particular, Scarlatti and Teta [18] have proved the following characterization:

$$e^{itH_\alpha} f = \begin{cases} e^{-it\Delta} f + \lim_{R \rightarrow \infty} M_R f & \text{if } \alpha = 0 \\ e^{itH_0} f + \lim_{R \rightarrow \infty} M_{\alpha,R} f & \text{if } \alpha > 0 \\ e^{itH_0} f + \lim_{R \rightarrow \infty} \tilde{M}_{\alpha,R} f & \text{if } \alpha < 0 \end{cases} \quad (35)$$

where the limit is taken in the L^2 sense and

$$M_R f(x) := (4\pi it)^{-1/2} \frac{1}{4\pi|x|} \int_{B_R} \frac{\tilde{f}(|y|)}{|y|} e^{-i\frac{(|x|+|y|)^2}{4t}} dy, \quad (36)$$

$$M_{\alpha,R} f(x) := -(4\pi it)^{-1/2} \frac{\alpha}{|x|} \int_{\mathbb{R}^3} \frac{f(y)}{|y|} \int_0^{+\infty} e^{-4\pi\alpha s} e^{-i\frac{(|x|+|y|+s)^2}{4t}} ds dy, \quad (37)$$

$$\begin{aligned} \tilde{M}_{\alpha,R} f(x) := & \left(-\psi_\alpha(x) \int_{B_R} \psi_\alpha(y) f(y) e^{it(4\pi\alpha)^2} dy \right. \\ & \left. - \frac{\alpha}{|x|} (it\pi)^{-1/2} \int_{B_R} \frac{f(y)}{|y|} \int_0^{+\infty} e^{4\pi\alpha s} \exp\left(-\frac{(u-|x|-|y|)^2}{4it}\right) ds dy \right), \end{aligned} \quad (38)$$

and $\psi_\alpha(x) = \sqrt{-2\alpha} \frac{e^{4\pi\alpha|x|}}{|x|}$ is the normalized eigenfunction associated to the negative eigenvalue $-(4\pi\alpha)^2$ for $\alpha < 0$. We are going to show that the following estimates hold uniformly in $R > 0$:

$$\|M_R f\|_{L^q} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \quad (39)$$

$$\|M_{\alpha,R} f\|_{L^q} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}, \quad (40)$$

$$\|\tilde{M}_{\alpha,R} f\|_{L^q} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p}. \quad (41)$$

The latter inequalities are clearly sufficient to prove Proposition 5 in the special case $N = 1$. Let us start by proving inequality (39). Thanks to Corollary 2 we can suppose $f(y) = \tilde{f}(|y|)$ for some $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$. Using spherical coordinates in both variables x and y we get

$$\|M_R f\|_{L^q} \lesssim t^{-1/2} \left[\int_0^{+\infty} r^{2-q} \left| \int_0^R \exp\left(-i\frac{\rho r}{2t} - i\frac{\rho^2}{4t}\right) \rho \tilde{f}(\rho) d\rho \right|^q dr \right]^{1/q}. \quad (42)$$

Setting

$$h(\rho) := \begin{cases} e^{-i\rho^2/4t} \rho \tilde{f}(\rho) & 0 \leq \rho \leq R \\ 0 & \rho \in \mathbb{R} \setminus [0, R] \end{cases} \quad (43)$$

the latter expression becomes

$$t^{-1/2} \left[\int_0^{+\infty} r^{2-q} \left| \widehat{h} \left(\frac{r}{2t} \right) \right|^q dr \right]^{1/q}, \quad (44)$$

which is equal to

$$t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \left[\int_0^{+\infty} r^{2-q} |\widehat{h}(r)|^q dr \right]^{1/q}. \quad (45)$$

At this point we are ready to use a classical weighted Fourier transform norm inequality, also known in literature as Pitt's inequality. We state here the original theorem proved by Pitt in 1937 [16]:

Theorem 2. [Pitt's theorem] Let $1 < \gamma \leq \eta < \infty$, choose $0 < b < \frac{1}{\gamma}$ with $\frac{1}{\gamma} + \frac{1}{\gamma} = 1$, set $\beta = 1 - \frac{1}{\gamma} - \frac{1}{\eta} - b < 0$ and define $v(x) = |x|^{b\gamma}$ for all $x \in \mathbb{R}$. There is a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}} |\widehat{f}(\xi)|^\eta |\xi|^\beta d\xi \right)^{1/\eta} \leq C \left(\int_{\mathbb{R}} |f(x)|^\gamma |x|^{b\gamma} dx \right)^{1/\gamma}, \quad (46)$$

for all $f \in L_v^\gamma(\mathbb{R})$.

Since $q < 3$ we may use this Theorem in the case $\eta = q$, $\gamma = p$, $\beta = \frac{2-q}{q}$ and $b = \frac{2-p}{p}$ obtaining

$$t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \left[\int_0^{+\infty} r^{2-q} |\widehat{h}(r)|^q dr \right]^{1/q} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \left[\int_0^{+\infty} |h(r)|^p r^{2-p} dr \right]^{1/p}, \quad (47)$$

which essentially is the desired estimate, indeed

$$\left[\int_0^{+\infty} |h(r)|^p r^{2-p} dr \right]^{1/p} = \|f\|_{L^p}. \quad (48)$$

This concludes the proof of (39), which, together with the standard dispersive estimates for the free Laplacian, implies the dispersive estimates for the semigroup $\{e^{itH_0}\}_{t>0}$.

Let us turn in to proving (40). Since $q < 3$ the function $1/|y|$ belongs to $L^q(B_R)$, hence we can exchange the order of integration and use Minkowski inequality

$$\begin{aligned} \|M_R f\|_{L^q} &\lesssim \\ &\lesssim t^{-1/2} \int_0^{+\infty} \left\| \int_{B_R} \frac{1}{|x|} \exp \left(-4\pi\alpha s - i \frac{(|x| + |y| + s)^2}{4t} \right) \frac{\widetilde{f}(y)}{|y|} dy \right\|_{L^q} ds \\ &= t^{-1/2} \int_0^{+\infty} e^{-4\pi\alpha s} \left\| \int_{B_R} \frac{1}{|x|} \exp \left(-i \frac{|y|^2}{4t} - i \frac{|x||y|}{2t} - i \frac{s|y|}{2t} \right) \frac{f(y)}{|y|} dy \right\|_{L^q} ds, \end{aligned} \quad (49)$$

which, as before, in spherical coordinates is bounded, up to constants, by

$$t^{-1/2} \int_0^{+\infty} e^{-4\pi\alpha s} \left(\int_0^{+\infty} r^{2-q} \left| \int_0^R e^{-i\frac{r\rho}{2t}} h_s(\rho)(\rho) d\rho \right|^q dr \right) ds, \quad (50)$$

where

$$h_s(\rho) := \begin{cases} \exp\left(-i\frac{\rho^2}{4t} - i\frac{s\rho}{2t}\right) \rho \tilde{f}(\rho) & 0 \leq \rho \leq R \\ 0 & \rho \in \mathbb{R} \setminus [0, R] \end{cases}. \quad (51)$$

The quantity (50) is nothing but

$$t^{-1/2} \int_0^{+\infty} e^{-4\pi\alpha s} \left(\int_0^{+\infty} r^{2-q} \left| \widehat{h}_s\left(\frac{r}{2t}\right) \right|^q dr \right)^{1/q} ds, \quad (52)$$

which, arguing as before, is bounded by $t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}$. This concludes the proof of (40).

The proof of (41) is very similar, indeed after projecting f onto the the absolutely continuous spectrum of H_α , the first summand in the right hand side of (38) disappears and hence the remaining part can be treated exactly in the same way as done in the proof of (40).

4 Conclusions

The proof given in section 3 is quite direct and does not use any deep results from scattering theory for the perturbed Hamiltonian $H_{\alpha,Y}$. Nevertheless, it is worth noticing that the proof of Pitt's inequality, the main tool of our argument, requires some technical results from harmonic analysis such as Muckenaupt estimates ([10] [15]), which play an essential role also in the proof of the L^p boundedness of the wave operators W^\pm given in [8]. The main advantage of our approach is that, owing to more general weighted Fourier inequalities (see for instance [4], [13]) rather than Pitt's inequality (in which the weights are forced to be pure powers), it can potentially be adapted to prove optimal $L^p - L^q$ estimates also in the regime $q \geq 3$. In particular, we conjecture the following result:

Conjecture 1. Fix $q \in [3, +\infty]$, and let $w_q(x)$ a weight such that $w(x) \equiv 1$ outside a ball centered at the origin and $w_q^{-1} G_i \in L^q(\mathbb{R}^3)$. Then for every $\alpha \neq 0$ and $y \in \mathbb{R}^3$, the following estimates holds:

$$\|w_q(\cdot - y)^{-1} e^{itH_{\alpha,y}} P_{ac} f\|_{L^q(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|w_q(\cdot - y) f\|_{L^p(\mathbb{R}^3)}. \quad (53)$$

When $\alpha = 0$, a similar estimates holds but with a slower time decay:

$$\|w_q(\cdot - y)^{-1} e^{itH_{\alpha,y}} f\|_{L^q(\mathbb{R}^3)} \lesssim t^{-\frac{1}{2}} \|w_q(\cdot - y) f\|_{L^p(\mathbb{R}^3)}. \quad (54)$$

Remark 4. Conjecture 1 is motivated by the natural principle for which removing the local singularity is enough to get dispersive estimates, and it would improve the result in Corollary 1. For example when $q = 3$ we expect that a logarithmic weight would suffice, while in estimates (21) and (22) there appear polynomial weights.

An alternative conjecture can be expressed in term of weighted Lorentz space, in which context there are other generalizations of Pitt's inequality (see for instance [19]):

Conjecture 2. Given $q \in [3, +\infty]$, define the weight

$$w_q := 1 + |x|^{\frac{3}{q}-1}$$

Then for every $\alpha \neq 0$ and $y \in \mathbb{R}^3$, the following estimates holds:

$$\|w_q(\cdot - y)^{-1} e^{itH_{\alpha,y}} P_{ac} f\|_{L^{q,\infty}(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|w_q(\cdot - y)f\|_{L^{p,1}(\mathbb{R}^3)}. \quad (55)$$

When $\alpha = 0$, a similar estimates holds but with a slower time decay:

$$\|w_q(\cdot - y)^{-1} e^{itH_{\alpha,y}} f\|_{L^{q,\infty}(\mathbb{R}^3)} \lesssim t^{-\frac{1}{2}} \|w_q(\cdot - y)f\|_{L^{p,1}(\mathbb{R}^3)}. \quad (56)$$

Remark 5. The function $w_q^{-1}G_i$ belongs to $L^{q,\infty}(\mathbb{R}^3)$, hence the plausibility of the conjecture. Observe moreover that it would be enough to prove (55) and (56) when $q = 3$, the general case following by interpolation with $q = \infty$, in which case we recover the weighted $L^1 - L^\infty$ estimates (12) and (14) proved in [9].

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