UNIVERSALITY IN THE 2D QUASI-PERIODIC ISING MODEL AND HARRIS-LUCK IRRELEVANCE

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Abstract. We prove that in the 2d Ising Model with a weak bidimensional quasi-periodic disorder in the interaction, the critical behavior is the same as in the non-disordered case, that is the critical exponents are identical and no logarithmic corrections are present. The result establishes the validity of the prediction based on the Harris-Luck criterion and it provides the first rigorous proof of universality in the Ising model in presence of quasi-periodic disorder. The proof combines Renormalization Group approaches with direct methods used to deal with small divisors in KAM theory.

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1. Introduction

1.1. **Universality and Harris-Luck Criterion.**

A certain number of macroscopic properties close to phase transitions shows a remarkable independence from microscopic details. In particular, it is both predicted theoretically and observed experimentally that the critical exponents, describing the singularities of thermodynamical functions, are the same in systems sharing only few general features, but having different inter-molecular forces, atomic weight or lattice structure. This phenomenon is known as *universality* and the Renormalization Group, introduced by Kadanoff [25] and Wilson [38], provides an explanation introducing the concepts of: scaling dimension, dimensionally relevant, marginal or irrelevant interactions, and universality classes. The fact that interactions are dimensionally relevant or marginal does not imply by itself that they can change the critical behavior: the neat effect on critical exponents is ruled by an effective dimension which can be different from the scaling dimension due to cancellations or other mechanisms.

A paradigmatic model where universality can be investigated is the Ising model, describing a system of spins with nearest-neighbor interactions and showing a phase transition in dimensions $d \geqslant 2$ described by certain values of the critical exponents. One can perturb such model with finite-ranged or higher spin interactions, or consider it on different lattices, and ask what happens to the critical behavior. In $d \geqslant 4$ universality is proven in the context of the strictly related ϕ^4 models, see *e.g.* [6] and references therein, where it has been rigorously shown that the value of the exponents is equal to the mean field ones, *e.g.* the correlation length exponent is $\nu = 1/2$ and the specific heat exponent $\alpha = (4-d)/2$. We remark that, however, while in $d \geqslant 5$ the behavior is exactly the same as in mean field, in $d = 4$ logarithmic corrections are present; the difference is that in the first case the interaction is irrelevant in the Renormalization Group sense, while in the second is marginal (or, more precisely, marginally irrelevant).

In $d = 2$ the Ising model with nearest neighbor interaction on a square lattice was solved by Onsager [35]. His solution proves that the value of the critical exponents ($\nu = 1, \alpha = 0$) is different from the ones obtained with mean field approximation. Having universality in mind, it is natural to ask whether these values are robust under perturbations. One can ask, for example, if the addition of a next-to-nearest neighbor interaction or a non quadratic one leaves the system in the Onsager universality class or not. In this case, it is not convenient to use ϕ^4 models, but one can use the representation in terms of Grassmann integrals, at the basis of the exact solution. Using this strategy it has been proved in [36, 30, 21] that for an Ising model with a short ranged or quartic perturbation the exponents are the same as in the pure Ising model (*e.g.* $\nu = 1, \alpha = 0$) and no logarithmic corrections are present; the asymptotic behavior is the same as in the unperturbed one (see also [20, 5] for more recent results) and the perturbation effectively irrelevant. The situation is indeed different if one considers two Ising models with a nearest neighbor or quartic interaction; in that case the interaction remains marginal, exponents are different with respect to the Ising ones and another notion of universality can be established [30, 8, 9, 22]. The issue of universality in Ising models can also be investigated considering nearest neighbor interactions but different lattices, where it has been established in [12, 13], and more general interactions have been considered in [4].

Another situation where the issue of universality can be posed in the Ising model is when *disorder* is considered. Disorder can be introduced either in the magnetic field [1, 2, 3, 26] or in the interaction, and we focus here on this second case, for which much less is known at a rigorous level. Typically, one can consider two kinds of disorder in the interaction: random or quasi-periodic. The first describes the effect of impurities, while the second is realized in quasi-crystals or cold atoms experiments. First investigations were done in the 2d random Ising model, where the disorder is marginal and the critical behavior is modified with respect to the pure case. In particular, the specific heat is found continuous (instead of logarithmically divergent) with a correlated disorder on a single line by an exact solution [34], while double log behavior is found in the general case with the replica approach [16]. In more general cases, a criterion was formulated for the irrelevance of disorder. Harris [24] proposed a criterion to predict when random disorder is irrelevant or not. Briefly, his argument can be summarized as follows. One can see the system as composed by independent subsystems of correlation length size ξ^d, and the square of the number of defects, described by the disorder, can be assumed proportional to $\Delta^2\xi^d$, if Δ^2 is the variance of the random disorder; therefore the density of defects is $\sqrt{\Delta^2/\xi^d}$ and the condition for irrelevance is that such quantity is smaller than the deviation of temperature from criticality $|\beta - \beta_c|$. Since, close to criticality, the relation $\xi \sim |\beta - \beta_c|^{-\nu}$ holds, with ν critical exponent, irrelevance is predicted for $\nu d/2 > 1$, see [24], while relevance is expected for $\nu d/2 < 1$. According to this criterion, irrelevance is predicted for $d \geqslant 5$ ($\nu = 1/2 > 2/d$) and relevance $d = 3$ (conformal bootstrap predicts $\nu = 0.627 \cdots < 2/3$, see [37]). In the marginal cases $d = 4$ ($\nu = 1/2$) and $d = 2$ ($\nu = 1$), Harris criterion gives no predictions. A generalization of Harris result was rigorously proved in [11] where it was shown that in all systems with continuous transitions $\tilde{\nu} \geqslant 2/d$ with $\tilde{\nu}$ the index of the disordered system.

While the Harris criterion regards the case of random hopping, later on Luck [28] showed that the criterion can be extended and applied in the $2d$ Ising model with *quasi-periodic* disorder. In this case the Harris-Luck criterion predicts irrelevance if the disorder is weak and analytic [28] so that the system belongs to the same universality class as in the pure case, while it is predicted that the disorder is relevant in the strong or unbounded case. Such conjectures have been confirmed by numerical investigations, see *e.g.* [23, 19, 14]. In particular, in [14] it is numerically found that Ising with weak quasi-periodic disorder remains in the Onsager class, while evidence of a new universality class is found at stronger disorder. On the analytical side, quasi-periodic potential in 2d has been rigorously analyzed in the Schrödinger equation [7, 27] but not in the context of the Dirac equation, which would be quite related to the Ising model.

In this paper we consider the 2d Ising model with quasi-periodic interaction in the infinite volume limit $J^{(j)}_{\mathbf{x}} = J^{(j)}(1 + \lambda\phi^{(j)}(\omega_0 x_0 + \theta_{j,0}, \omega_1 x_1 + \theta_{j,1}))$ with λ a small parameter and $\phi^{(j)}(t_0, t_1)$ 2π periodic and analytic with ω_0, ω_1 irrational. In order to take into account the periodic bounday conditions we consider a sequence of Ising models in boxes with interactions periodic with a period equal to the side of the boxes, and converging in the limit to quasi-periodic interactions. We prove that the critical behavior is the same as in the pure case, that is the critical exponents are the same and no logarithmic corrections are present. Despite the fact that disorder is dimensionally relevant in the Renormalization Group sense, it is effectively irrelevant due to improvement following from number theoretical conditions on the frequencies. The result proves the validity of the prediction based on the Harris-Luck criterion and it provides the first rigorous proof of universality in Ising model in presence of quasi-periodic disorder in the interaction. The proof combines Renormalization Group approaches used in interacting Ising models [30, 21, 33] with methods used to deal with small divisors in KAM Lindstedt series [17, 18] or Fermionic systems [10, 31, 32].

1.2. **Main Result.** The Hamiltonian of the 2d quasi-periodic Ising model is

$$H = -\sum_{\mathbf{x}} \left[J^{(1)}_{\mathbf{x}} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_1} + J^{(0)}_{\mathbf{x}} \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_0} \right] \tag{1.1}$$

where $\mathbf{e}_0 = (1, 0)$, $\mathbf{e}_1 = (0, 1)$, $\mathbf{x} = (x_0, x_1)$, $\mathbf{x} \in \Lambda_i$, $\Lambda_i = [0, L_{0,i}] \times [0, L_{1,i}] \cap \mathbb{Z}^2$, $\sigma_{\mathbf{x}} = \pm$ and periodic boundary conditions. Moreover $L_{0,i}, L_{1,i}$ is a sequence of integers such that $\lim_{i\to\infty} L_{0,i} = \infty$ and $\lim_{i\to\infty} L_{1,i} = \infty$ and finally

$$J^j_{\mathbf{x}} = J^{(j)}\Big(1 + \lambda\phi^{(j)}(\omega_{0,i}x_0 + \theta_{j,0}, \omega_{1,j}x_1 + \theta_{j,1})\Big) \qquad j = 0, 1\,. \tag{1.2}$$

where $\phi^{(j)}(t_0, t_1)$ a real analytic 2π-periodic function $\mathbb{T}^2 \to \mathbb{R}$, $\omega_{j,i} = p_{j,i}/L_{j,i}$ with $p_{j,i}$, $L_{j,i}$ relatively prime integers, $\omega_{j,i} < 1$ and

$$\lim_{i\to\infty} \omega_{0,i} = \omega_0\,, \qquad \lim_{i\to\infty} \omega_{1,i} = \omega_1\,, \tag{1.3}$$

with ω_0, ω_1 irrational. We assume that ω_j are Diophantine numbers ω_j verifying

$$|\omega_j n - p| \geqslant c_j |n_j|^{-\rho_j} \tag{1.4}$$

for any $p, n \in \mathbb{Z}$, $n \neq 0$, $\rho_j \geqslant 1, c_j > 0$, and we choose the sequence of $\omega_{j,i} = p_{j,i}/L_{j,i}$ its *best approximants* [15]. Finally we assume that $\lim_{i\to\infty} L_{0,i}/L_{1,i} = c$ with $0 < c < \infty$.

If $E_{\mathbf{x},j} = \sigma_{\mathbf{x}}\sigma_{\mathbf{x}+\mathbf{e}_j}$ the energy correlations are defined as

$$\langle E_{\mathbf{x},j_1}; E_{\mathbf{x},j_2}\rangle_i^T = \langle E_{\mathbf{x},j_1} E_{\mathbf{x},j_2}\rangle_i - \langle E_{\mathbf{x},j_1}\rangle_i \langle E_{\mathbf{x},j_2}\rangle_i \,, \tag{1.5}$$

with

$$\langle O\rangle_i = \frac{1}{Z} \sum_{\{\sigma_{\mathbf{x}}\}\in\{\pm\}^{\Lambda_i}} e^{-\beta H} O \,, \qquad Z = \sum_{\{\sigma_{\mathbf{x}}\}\in\{\pm\}^{\Lambda_i}} e^{-\beta H} \,, \tag{1.6}$$

with $\mathcal{Z}$ the partition function at inverse temperature $\beta > 0$.

Our main result is the following

Theorem 1.1. *There exists $\lambda_0 > 0$ such that, for $|\lambda| \leqslant \lambda_0$ there exists an analytic function $\beta_c(\lambda)$ such that $\lim_{i\to\infty} \langle E_{\mathbf{x},j_1}; E_{\mathbf{y},j_2}\rangle_i^T = \langle E_{\mathbf{x},j_1}; E_{\mathbf{y},j_2}\rangle^T$ exists and for $\beta \neq \beta_c(\lambda)$ decays as $O\big(\frac{1}{(|\phi(\mathbf{x},\mathbf{y})|\xi^{-1})^N}\big)$ for any $N \in \mathbb{N}$ with $\xi = O(|\beta - \beta_c|^{-1})$ while for $\beta \to \beta_c(\lambda)$*

$$\langle E_{\mathbf{x},j_1}; E_{\mathbf{y},j_2}\rangle^T = \frac{A_{\mathbf{x},\mathbf{y}}}{|\phi(\mathbf{x},\mathbf{y})|^2} + O\Big(\frac{1}{|\phi(\mathbf{x},\mathbf{y})|^{2+\theta}}\Big) \,, \tag{1.7}$$

with $\phi(\mathbf{x},\mathbf{y}) = a_0 x_0 + \mathrm{i} a_1 x_1$, $a_j = (\tanh \beta J^{(j)})^{-1} + \lambda b_j$, $b_j \in \mathbb{C}$, $A_{\mathbf{x},\mathbf{y}} = A_0 + \lambda A_1(\mathbf{x},\mathbf{y})$, $\beta_c(\lambda) = \beta_c(0) + \lambda B$, $\sinh(2\beta_c(0)J^{(0)})\sinh(2\beta_c(0)J^{(1)}) = 1$, b_j, A_0, A_1, B bounded by $O(1)$ constants and $1/2 < \theta < 1$.

The disorder produces a finite renormalization of the critical temperature and of the velocities; in addition the amplitude is also renormalized and has the from $A = (1 - t^2_{\mathbf{x},j_1})(1 - t^2_{\mathbf{y},j_2}) + O(\lambda)$ with $t_{\mathbf{x},j} = \tanh \beta J^{(j)}_{\mathbf{x}}$. Such effects do not alter the long distance behavior; in particular, the critical exponent of the specific heat is $\alpha = 0$ and the one of the correlation length is $\nu = 1$, and no logarithmic corrections are present.

1.3. **Contents.** The proof is based on the representation of the quasi-periodic Ising model as a Grassmann integral, which is recalled in Section 2. In this representation, there are two kinds of Grassmann variables, the critical and the non-critical, and the latter are integrated out in Section 3. In Section 4 we perform the integration of the critical modes, which is performed by Renormalization Group. The presence of quasi-periodic potential produce infinitely many relevant terms quadratic in the Grassmann variables, with momenta $\mathbf{k}_a, \mathbf{k}_b$ such that $k_{a,j} - k_{b,j} + 2\pi\omega_j + 2\pi m_j = 0$ with m_j positive or negative integers and $j = 0, 1$ denotes the components. The terms such that $k_{a,j} = k_{b,j}$ produces a renormalization of the velocities and a shift of the critical temperature. There are, however, infinitely many terms such that $2\pi\omega_j + 2\pi m_j$ is almost (but not exactly) vanishing and being dimensionally relevant they could alter the critical behavior; this is the way in which the problem of accumulation of small divisors appears in the Renormalization Group approach. The proof of universality consists in showing that such dimensionally relevant terms are indeed irrelevant by using the Diophantine condition (1.4). Finally the decay of the energy correlations is analyzed in Section 5.

2. Grassmann Representation

The energy correlations can be written as

$$\langle E_{\mathbf{x},j_1}; E_{\mathbf{y},j_2}\rangle_i^T = \frac{\partial^2}{\partial A_{\mathbf{x},j_1}\partial A_{\mathbf{y},j_2}} \log Z(A)\big|_{A=0} \tag{2.1}$$

with

$$Z(A) = \frac{1}{2} \sum_{\boldsymbol{\alpha}\in\{\pm\}^2} \tau_{\boldsymbol{\alpha}} Z_{\boldsymbol{\alpha}}(A) \,, \tag{2.2}$$

with $\tau_{+,-} = \tau_{-,+} = \tau_{-,-} = -\tau_{+,+} = 1$ and

$$Z_{\boldsymbol{\alpha}}(A) = \Big[\prod_{\mathbf{x}\in\Lambda}\prod_{j=0}^{1}\cosh(\beta J^{(j)}_{\mathbf{x}} + A_{\mathbf{x},j})\Big]\int D\Phi\, e^{S(\Phi,A)} \tag{2.3}$$

with

$$\begin{aligned} S(\Phi, A) = \sum_{\mathbf{x}\in\Lambda}\big[\tanh(\beta J^{(1)}_{\mathbf{x}} + A_{\mathbf{x},1})\overline{H}_{\mathbf{x}+\mathbf{e}_1}H_{\mathbf{x}} + \tanh(\beta J^{(0)}_{\mathbf{x}} + A_{\mathbf{x},0})\overline{V}_{\mathbf{x}+\mathbf{e}_0}V_{\mathbf{x}}\big]+ \\ + \sum_{\mathbf{x}\in\Lambda}\big[\overline{H}_{\mathbf{x}}H_{\mathbf{x}} + \overline{V}_{\mathbf{x}}V_{\mathbf{x}} + \overline{V}_{\mathbf{x}}\overline{H}_{\mathbf{x}} + V_{\mathbf{x}}\overline{H}_{\mathbf{x}} + H_{\mathbf{x}}\overline{V}_{\mathbf{x}} + V_{\mathbf{x}}H_{\mathbf{x}}\big] \end{aligned} \tag{2.4}$$

Here $\overline{H}_{\mathbf{x}}, H_{\mathbf{x}}, \overline{V}_{\mathbf{x}}, V_{\mathbf{x}}$ are independent *Grassmann variables*, four for each lattice site, and $E_{\mathbf{x},1} = \overline{H}_{\mathbf{x}}H_{\mathbf{x}+\mathbf{e}_1}$, while $E_{\mathbf{x},0} = \overline{V}_{\mathbf{x}}V_{\mathbf{x}+\mathbf{e}_0}$. Moreover, $\Phi = \{\overline{H}_{\mathbf{x}}, H_{\mathbf{x}}, \overline{V}_{\mathbf{x}}, V_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$ denotes the collection of all of these Grassmann variables and $\mathcal{D}\Phi$ is a shorthand for $\prod_{\mathbf{x}} d\overline{H}_{\mathbf{x}}dH_{\mathbf{x}}d\overline{V}_{\mathbf{x}}dV_{\mathbf{x}}$. The label $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, with $\alpha_1, \alpha_2 \in \{\pm\}$, refers to the boundary conditions, which are periodic or antiperiodic in the horizontal (resp. vertical) direction. $Z = \sum_{\boldsymbol{\alpha}\in\{\pm\}^2}\tau_{\boldsymbol{\alpha}}Z_{\boldsymbol{\alpha}}$ with $Z_{\boldsymbol{\alpha}} = Z_{\boldsymbol{\alpha}}(0)$ is the partition function. We also omitted the subscript i in Λ_i to avoid heavier notation.

The Grassmann representation is particularly convenient for computing energy-energy correlation functions: for instance,

$$\langle E_{\mathbf{x},j}; E_{\mathbf{y},j'}\rangle^T = \sum_{\boldsymbol{\alpha}}\frac{\tau_{\boldsymbol{\alpha}}Z_{\boldsymbol{\alpha}}}{2Z}\langle E_{\mathbf{x},j}; E_{\mathbf{y},j'}\rangle^T_{\boldsymbol{\alpha},i}\,, \tag{2.5}$$

where $\langle\cdot\rangle_{\boldsymbol{\alpha},i}$ is the average with respect to the Grassmann "measure" $\mathcal{D}\Phi e^{S(\Phi,\mathbf{0})}/Z_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha}$ boundary conditions; we shall also indicate by $\langle\cdot\rangle_{\boldsymbol{\alpha}}$ the $i\to\infty$ limit of $\langle\cdot\rangle_{\boldsymbol{\alpha},i}$. It is also convenient to write

$$J^{(j)}_{\mathbf{x}} = \sum_{n_0=-[L_{0,i}/2]}^{[(L_{0,i}-1)/2]}\ \sum_{n_1=-[L_{1,i}/2]}^{[(L_{1,i}-1)/2]}\hat{J}^{(j)}_{\mathbf{n}}e^{\mathrm{i}2\pi(\omega_{0,i}n_0x_0+\omega_{1,i}n_1x_1)}e^{\mathrm{i}2\pi(n_0\theta_{j,0}+n_1\theta_{j,1})}\,, \tag{2.6}$$

with $\mathbf{n} = (n_0, n_1)$.

We start the analysis from the partition function Z. We perform now a change of variables, see [36, 30, 21], $H, \overline{H}, V, \overline{V} \to \psi, \overline{\psi}, \xi, \overline{\xi}$ and we get, if $\hat{Z}_{\boldsymbol{\alpha}} = \frac{Z_{\boldsymbol{\alpha}}}{Z^0_{\boldsymbol{\alpha}}}$ with $Z^0_{\boldsymbol{\alpha}} = Z^0_{\boldsymbol{\alpha}}|_{\lambda=0}$

$$\hat{Z}_{\boldsymbol{\alpha}} = \int P(d\xi)\int P(d\psi)e^{V(\psi,\xi)} \tag{2.7}$$

where denoting with $t^{(j)}_{\mathbf{x}} := \tanh\beta J^{(j)}_{\mathbf{x}}$ by $t^{(j)} := \tanh\beta J^{(j)}$ and by $|\Lambda| = L_0L_1$,

(1) $P(d\xi)$ is the Gaussian Grassmann integration with propagator,

$$g_\xi(\mathbf{x},\mathbf{y}) = \frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}}e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}g_\xi(\mathbf{k})\,, \tag{2.8}$$

with $\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}$ and $\mathcal{D}_{\boldsymbol{\alpha}} = \left\{\mathbf{k} = (k_0,k_1)\in\mathbb{R}^2 \,\middle|\, \begin{array}{c} k_j = \frac{\pi}{L_j}(2\kappa_j + 1 - \alpha_j 1) \\ \kappa_j \in \{-L_j+1,\dots,0,1,\dots,L_j\}\end{array}\right\}$ and

$$g_\xi(\mathbf{k}) = \begin{pmatrix} -\mathrm{i}t^{(1)}\sin k_1 + t^{(0)}\sin k_0 & \mathrm{i}m_\xi(\mathbf{k}) \\ -\mathrm{i}m_\xi(\mathbf{k}) & -\mathrm{i}t^{(1)}\sin k_1 - t^{(0)}\sin k_0\end{pmatrix}^{-1} \tag{2.9}$$

and $m_\xi(\mathbf{k}) := t^{(1)}\cos k_1 + t^{(0)}\cos k_0 + 2(\sqrt{2}-1)$.

(2) $P(d\psi)$ is the Gaussian Grassmann integration with propagator

$$g_\psi(\mathbf{x},\mathbf{y}) = \frac{1}{|\Lambda|}\sum_{\mathbf{k}}e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\big(\widetilde{C}_\psi(\mathbf{k})\big)^{-1} \tag{2.10}$$

with $\widetilde{C}_\psi(\mathbf{k}) = C_\psi(\mathbf{k}) - Q(\mathbf{k})g_\xi(\mathbf{k})Q(\mathbf{k})$,

$$C_\psi(\mathbf{k}) := \begin{pmatrix} -\mathrm{i}t^{(1)}\sin k_1 + t^{(0)}\sin k_0 & \mathrm{i}m_\psi(\mathbf{k}) \\ -\mathrm{i}m_\psi(\mathbf{k}) & -\mathrm{i}t^{(1)}\sin k_1 - t^{(0)}\sin k_0 \end{pmatrix}, \tag{2.11}$$

$$Q(\mathbf{k}) := \begin{pmatrix} -\mathrm{i}t^{(1)}\sin k_1 - t^{(0)}\sin k_0 & \mathrm{i}\big(t^{(1)}\cos k_1 - t^{(0)}\cos k_0\big) \\ -\mathrm{i}\big(t^{(1)}\cos k_1 - t^{(0)}\cos k_0\big) & -\mathrm{i}t^{(1)}\sin k_1 + t^{(0)}\sin k_0 \end{pmatrix}, \tag{2.12}$$

and

$$Q(\mathbf{k})g_\xi(\mathbf{k})Q(\mathbf{k}) = O(|\mathbf{k}|^2) \qquad \text{as } \mathbf{k}\to 0\,. \tag{2.13}$$

(3) $V(\psi,\xi) = S^{(\xi)}_{\rm int} + S^{(\psi)}_{\rm int} + Q^{(\psi,\xi)}_{\rm int}$ where, if

$$P^{(1)} = \begin{pmatrix} -\mathrm{i}\sin\big(k_1 - \pi\omega_1 n_1\big) & \mathrm{i}\cos\big(k_1 - \pi\omega_1 n_1\big) \\ -\mathrm{i}\cos\big(k_1 - \pi\omega_1 n_1\big) & -\mathrm{i}\sin\big(k_1 - \pi\omega_1 n_1\big) \end{pmatrix}(1-\delta_{n_1,0})\,, \tag{2.14}$$

$$P^{(0)}(\mathbf{k},\mathbf{n}) = \begin{pmatrix} \sin\big(k_0 - \pi\omega_0 n_0\big) & \mathrm{i}\cos\big(k_0 - \pi\omega_0 n_0\big) \\ -\mathrm{i}\cos\big(k_0 - \pi\omega_0 n_0\big) & -\sin\big(k_0 - \pi\omega_0 n_0\big) \end{pmatrix}(1-\delta_{n_0,0})\,, \tag{2.15}$$

and

$$Q^{(1)}(\mathbf{k},\mathbf{n}) = P^{(1)}(\mathbf{k},\mathbf{n})\,, \qquad Q^{(0)}(\mathbf{k},\mathbf{n}) = -P^{(0)}(\mathbf{k},\mathbf{n})\,, \tag{2.16}$$

then, denoting by $\Omega = \begin{pmatrix}\omega_0 & 0\\ 0 & \omega_1\end{pmatrix}$, we have

$$S^{(\xi)}_{\rm int} = -\frac{1}{4\pi|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}}\sum_{j=0,1}\widehat{W}^{(j)}_{\mathbf{n}}\hat{\xi}_{-\mathbf{k}}\cdot P^{(j)}(\mathbf{k},\mathbf{n})\hat{\xi}_{\mathbf{k}-2\pi\Omega\mathbf{n}}\,, \tag{2.17}$$

$$S^{(\psi)}_{\rm int} = -\frac{1}{4\pi|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}}\hat{\boldsymbol{\psi}}_{-\mathbf{k}}\cdot\left(\sum_{j=0,1}\widehat{W}^{(j)}_{\mathbf{n}}P^{(j)}_\psi(\mathbf{k},\mathbf{n})\right)\hat{\boldsymbol{\psi}}_{\mathbf{k}-2\pi\Omega\mathbf{n}}\,, \tag{2.18}$$

and

$$Q^{(\psi,\xi)}_{\rm int} = \frac{1}{4\pi|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}}\sum_{j=0,1}\widehat{W}^{(j)}_{\mathbf{n}}\hat{\boldsymbol{\psi}}_{-\mathbf{k}}\cdot Q^{(j)}_\psi(\mathbf{k},\mathbf{n})\hat{\boldsymbol{\chi}}_{\mathbf{k}-2\pi\Omega\mathbf{n}} + (\psi\leftrightarrow\chi)\,, \tag{2.19}$$

with

$$Q^{(j)}_\psi = Q^{(j)}(\mathbf{k},\mathbf{n}) - Q(\mathbf{k})g_\xi(\mathbf{k})P^{(j)}(\mathbf{k},\mathbf{n})\,, \tag{2.20}$$

and

$$\begin{aligned} P^{(j)}_\psi(\mathbf{k},\mathbf{n}) = {} & P^{(j)}(\mathbf{k},\mathbf{n}) - Q^{(j)}(\mathbf{k},\mathbf{n})g_\xi(\mathbf{k}-2\pi\Omega\mathbf{n})Q(\mathbf{k}-2\pi\Omega\mathbf{n}) - \\ & Q(\mathbf{k})g_\xi(\mathbf{k})Q^{(j)}(\mathbf{k},\mathbf{n}) + Q(\mathbf{k})g_\xi(\mathbf{k})P^{(j)}(\mathbf{k},\mathbf{n})g_\xi(\mathbf{k}-2\pi\Omega\mathbf{n})Q(\mathbf{k}-2\pi\Omega\mathbf{n}) \end{aligned} \tag{2.21}$$

and finally $\widehat{W}^{(j)}_{\mathbf{n}} := \widehat{V}^{(j)}_{\mathbf{n}}e^{-\pi\mathrm{i}\omega_j n_j}e^{2\pi\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}_j}$ with $\widehat{W}^{(j)}_{-\mathbf{n}} = (\widehat{W}^{(j)}_{\mathbf{n}})^*$, $V^{(j)}_{\mathbf{x}} := \tanh\beta J^{(j)}_{\mathbf{x}} - \tanh\beta J^{(j)}$ and $V^{(j)}_{\Omega\mathbf{x}+\boldsymbol{\theta}_j} = \sum_{\mathbf{n}\in\mathbb{Z}^2}\widehat{V}^{(j)}_{\mathbf{n}}e^{2\pi\mathrm{i}\mathbf{n}\cdot\boldsymbol{\theta}_j}e^{2\pi\mathrm{i}\Omega\mathbf{n}\cdot\mathbf{x}}$.

Note hat $m_\psi(0)$ vanishes at $\beta=\beta_c$ defined by the condition $t^{(1)}(\beta_c)+t^{(0)}(\beta_c)-2(\sqrt{2}-1)=0$ while $m_\chi(0)$ is an $O(1)$ constant for any β. The χ variables then represent *non critical* modes and the second are the *critical* ones. The quasi-periodic potential produces extra terms still quadratic in the Grassmann variables but coupling different momenta.

3. Integration of Non Critical Modes

3.1. **Analyticity.** We define

$$
\begin{aligned}
e^{E^\chi+\widetilde{\mathcal{V}}(\psi)} &= \int P(d\xi)e^{V(\psi,\xi)} = e^{\sum_q \frac{1}{q!}\mathbb{E}^T_\xi(V(\psi,\cdot);q)} \\
&= \exp\left(E^\chi + \frac{1}{4\pi|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}} \boldsymbol{\psi}_{-\mathbf{k}}\cdot\widehat{\mathcal{V}}_{\mathbf{n}}(\mathbf{k})\boldsymbol{\psi}_{\mathbf{k}-2\pi\Omega\mathbf{n}}\right)
\end{aligned}
\tag{3.1}
$$

where $\mathbb{E}^T_\xi(V(\psi,\cdot);q)$ are the truncated expectations with respect to $P(\mathrm{d}\xi)$ and E^χ is the correction to the vacuum energy (i.e. the contribution of diagrams where only $\xi\xi$ vertices appear). Defining $\mathcal{W}_{\mathbf{n},1}(\mathbf{k}) := \sum_{j=0,1}\widehat{W}^{(j)}_{\mathbf{n}}P^{(j)}_\psi(\mathbf{k},\mathbf{n})$, we have

$$
\widehat{\mathcal{V}}_{\mathbf{n}}(\mathbf{k}) = \sum_{q=1}^{+\infty}\mathcal{W}_{\mathbf{n},q}(\mathbf{k})\,. \tag{3.2}
$$

Definition 3.1. Given an ordered collection of q elements, $\{(\mathbf{n}_1,j_1),\dots,(\mathbf{n}_q,j_q)\}$, $\mathbf{n}_s\in\mathbb{Z}^2$ $j_s\in\{0,1\}$, we define the associated *Feynman graph*, denoted as $\Gamma_{\{\mathbf{n}_r\}}$ as the set of q ordered vertices, $q-1$ internal lines and 2 external lines constructed as follows. To the first external line one attaches a vertex to which one attaches an internal line that connects it with the second vertex. The second vertex is attached to an internal line that connects it with the third verdex and so on. The last vertex is connected with an internal line only to the previous one and then it is attached the second external line.

To each of the r-th vertices one associates the pair $(\mathbf{n}_r,j_r)$ and given $\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}$ one associates to the $s-$th line the momentum $\mathbf{k}_s = \mathbf{k}-2\pi\Omega\sum_{r\leqslant s}\mathbf{n}_r+2\pi\mathbf{m}$ ($\mathbf{m}\in\mathbb{Z}^2$ chosen such that $\mathbf{k}_s\in\mathcal{D}_{\boldsymbol{\alpha}}$). The first external line has momentum $\mathbf{k}$ and the second external line has momentum $\mathbf{k}-2\pi\Omega\mathbf{n}-2\pi\mathbf{m}$ (with $\mathbf{m}\in\mathbb{Z}^2$ such that $\mathbf{k}-2\pi\Omega\mathbf{n}\in\mathcal{D}_{\boldsymbol{\alpha}}$) . Given a graph $\Gamma_{\{\mathbf{n}_r\}}$, we denote by $E(\Gamma_{\{\mathbf{n}_r\}})$ the set of vertices of $\Gamma_{\{\mathbf{n}_r\}}$.

We denote with $\mathcal{G}^\xi_{\mathbf{n},q}$ the set of Feynman graphs with q vertices and $\sum_{r=1}^q\mathbf{n}_r=\mathbf{n}$.

Note that, the notation used for a graph, namely $\Gamma_{\{\mathbf{n}_r\}}$, does not show explicitly the dependence on $\{j_r\}$ which is understood. An example of Feynman graph is given in Figure 1. In the pictorial representation, we attach a wavy line to each vertex to represent the loss/gain of momentum in the interaction.

Definition 3.2. Let $\Gamma_{\{\mathbf{n}_j\}}$ be the graph associated to the collection $\{(\mathbf{n}_1,j_1),\dots,(\mathbf{n}_q,j_q)\}$, then for $\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}$, we define the value of $\Gamma_{\{\mathbf{n}_j\}}$ as

$$
\mathrm{Val}(\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k})) := F_{v_1}(\mathbf{k})\left(\prod_{r=1}^{q-1}g_\chi(\mathbf{k}_r)F_{v_{r+1}}(\mathbf{k}_r)\right) \tag{3.3}
$$

where $v_r=(\mathbf{n}_r,j_r)$ and

$$
F_v(\mathbf{k}) = \widehat{W}^{(j_v)}_{\mathbf{n}_v}\times\begin{cases} Q^{(j_v)}_\psi(\mathbf{k},\mathbf{n}_v)\,, & \text{if } v=1,q\\ P^{(j_v)}(\mathbf{k},\mathbf{n}_v)\,, & \text{if } v=2,3,\dots,q-1\,.\end{cases} \tag{3.4}
$$

$\mathcal{W}_{\mathbf{n},q}(\mathbf{k})$ admits an expansion in terms of connected Feynman graphs as

$$
\mathcal{W}_{\mathbf{n},q}(\mathbf{k}) = \sum_{\Gamma_{\{\mathbf{n}_j\}}\in\mathcal{G}_{q,\mathbf{n}}}\mathrm{Val}\big(\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k})\big)\,. \tag{3.5}
$$

Note that the choice of assigning a graph to an *ordered* collection cancels precisely the $1/q!$ in (3.1).

From the explicit expression (2.9) one sees that there exists a positive constant $G_\xi>0$ such that

$$
|g_\xi(\mathbf{k})| \leqslant G_\xi\,, \tag{3.6}
$$

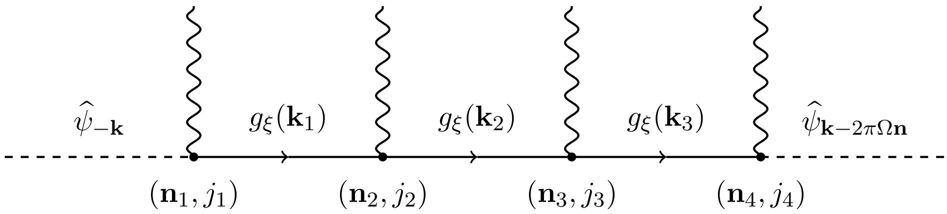

FIGURE 1. One of the graphs appearing in the expansion of $\mathcal{W}_{\mathbf{n},4}(\mathbf{k})$ in (3.5).

and from the analyticity of the potentials $\phi^{(j)}$ one has the existence of a positive constant U such that

$$|F_v(\mathbf{k}_{\text{in}})| \leqslant |\lambda| U\, e^{-\eta|\mathbf{n}_v|}\,. \tag{3.7}$$

Lemma 3.3. *If $|\lambda|$ is small enough then*

$$\|\widehat{\mathcal{V}}_{\mathbf{n}}(\mathbf{k})\| \leqslant V_0|\lambda|e^{-\frac{\eta}{2}|\mathbf{n}|} \tag{3.8}$$

with a constant V_0 which is independent of $\mathbf{k}$, Λ, λ and $\mathbf{n}$.

Proof. We start by bounding the value of a single graph. A graph $\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k}) \in \mathcal{G}^{\xi}_{q,\mathbf{n}}$ has $q-1$ lines and q vertices. Thus, using inequalities (3.6) and (3.7) we get

$$\|\mathrm{Val}\,\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k})\| \leqslant |\lambda|^q G_\xi^{q-1} U^q \prod_{v\in E(\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k}))} e^{-\eta|\mathbf{n}_v|} \leqslant G_\xi^{q-1}U^q e^{-\frac{\eta}{2}|\mathbf{n}|} \prod_{v\in E(\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k}))} e^{-\frac{\eta}{2}|\mathbf{n}_v|}$$

Now we have to sum over the graphs with q vertices, that is

$$\begin{aligned}
\|\mathcal{W}_{\mathbf{n},q}(\mathbf{k})\| &\leqslant \sum_{\Gamma_{\{\mathbf{n}_j\}}\in\mathcal{G}^{\xi}_{q,\mathbf{n}}} \|\mathrm{Val}(\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k}))\| \\
&\leqslant G_\xi^{q-1}U^q e^{-\frac{\eta}{2}|\mathbf{n}|} \sum_{\{\mathbf{n}_v\}_{v=1,\ldots,q}} \sum_{j\in\{0,1\}} \prod_{v\in E(\Gamma(\mathbf{k}))} e^{-\frac{\eta}{2}|\mathbf{n}_v|} \\
&\leqslant |\lambda|^q G_\xi^{q-1} U^q 4^q e^{-\frac{\eta}{2}|\mathbf{n}|} \sum_{r=1}^{q} \left(\sum_{n=0}^{+\infty} e^{-\frac{\eta}{2}n}\right)^2 \leqslant |\lambda|^q \frac{1}{G_\xi}\left(\frac{4UG_\xi}{(1-e^{-\frac{\eta}{2}})^2}\right)^q e^{-\frac{\eta}{2}|\mathbf{n}|}
\end{aligned}$$

The sum over q is convergent for $|\lambda| < \frac{(1-e^{-\frac{\eta}{2}})^2}{4UG_\xi}$, from where we read the existence of the constant V_0 with the desired properties. Analyticity follows from uniform convergence. ∎

3.2. **Symmetries.** To check the symmetries of the effective potential, one starts by writing explicitly the expression for the value of a Feynman graph (3.3):

$$\begin{aligned}
\mathrm{Val}(\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k})) = &\left(\prod_{v=1}^{q} \widehat{W}^{(j_v)}_{\mathbf{n}_v}\right) Q_\psi^{(j_1)}(\mathbf{k},\mathbf{n}_1) g_\xi(\mathbf{k}-2\pi\Omega\mathbf{n}_1)\times \\
&\times\left[\prod_{v=2}^{q-1} P^{(j_v)}\big(\mathbf{k}-2\pi\Omega\textstyle\sum_{v'=1}^{v-1}\mathbf{n}_{v'},\mathbf{n}_v\big) g_\chi\big(\mathbf{k}-2\pi\Omega\textstyle\sum_{v'=1}^{v}\mathbf{n}_{v'}\big)\right]\times \\
&\times Q_\psi^{(j_q)}\Big(\mathbf{k}-2\pi\Omega\textstyle\sum_{v'=1}^{q-1}\mathbf{n}_{v'},\mathbf{n}_q\Big)
\end{aligned} \tag{3.9}$$

with the observation that all matrices appearing in the previous expression (given by (2.9), (2.14), (2.15) and (2.16)), have the following structure:

$$\begin{pmatrix} a_{\mathbf{n}}(k) & b_{\mathbf{n}}(k) \\ b^*_{\mathbf{n}}(k) & -a^*_{\mathbf{n}}(k)\end{pmatrix} \tag{3.10}$$

with

- antisymmetry of the diagonal terms: $a_{\mathbf{n}}(\mathbf{k}) = -a_{-\mathbf{n}}(-\mathbf{k}) \in \mathbb{C}$;
- parity of the off-diagonal terms $b_{\mathbf{n}}(\mathbf{k}) = b_{-\mathbf{n}}(-\mathbf{k}) \in \mathrm{i}\mathbb{R}$.

In Lemma B.1 we prove that, writing the value of the graph as

$$\mathrm{Val}(\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k})) \;=\; \left(\prod_{v=1}^{q} \widehat{W}^{(j_v)}_{\mathbf{n}_v}\right) G_{\{\mathbf{n}_j\}}(\mathbf{k})\,, \tag{3.11}$$

the matrix $G_{\{\mathbf{n}_j\}}(\mathbf{k})$ has the form (3.10) with $a_{\mathbf{n}}(\mathbf{k}) = -a_{-\mathbf{n}}(-\mathbf{k}) \in \mathbb{C}$ and $b_{\mathbf{n}}(\mathbf{k}) = b_{-\mathbf{n}}(-\mathbf{k}) \in \mathbb{C}$. To derive the symmetry properties of $\widehat{\mathcal{V}}_0(\mathbf{k})$ we note that

$$\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}} \boldsymbol{\psi}_{-\mathbf{k}} \cdot \mathcal{W}_{0,q}(\mathbf{k})\boldsymbol{\psi}_{\mathbf{k}} = \frac{1}{4}\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}} \boldsymbol{\psi}_{-\mathbf{k}} \cdot \sum_{\Gamma_{\{\mathbf{n}_j\}}\in\mathcal{G}^{\xi}_{0,q}} \Big[\mathrm{Val}(\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k})) - \Big(\mathrm{Val}(\Gamma_{\{-\mathbf{n}_j\}}(-\mathbf{k}))\Big)^T + \\ +\mathrm{Val}(\Gamma_{\{-\mathbf{n}_j\}}(\mathbf{k})) - \Big(\mathrm{Val}(\Gamma_{\{\mathbf{n}_j\}}(-\mathbf{k}))\Big)^T\Big]\boldsymbol{\psi}_{\mathbf{k}} \tag{3.12}$$

where this equality is obtained using antisymmetry of Grassmann variables and the fact that $\sum_{r=0}^{q}\mathbf{n}_r = 0$. Taking into account this expression, following the procedure in Appendix C, one has the following structure for $\widehat{\mathcal{V}}_0(\mathbf{k})$:

$$\widehat{\mathcal{V}}_0(\mathbf{k}) \;=\; \begin{pmatrix} a(\mathbf{k}) & \mathrm{i}b(\mathbf{k}) \\ -\mathrm{i}b(\mathbf{k}) & -a^*(\mathbf{k}) \end{pmatrix} \tag{3.13}$$

with $a(\mathbf{k}) = -a(-\mathbf{k}) \in \mathbb{C}$ and $b(\mathbf{k}) = b(-\mathbf{k}) \in \mathbb{R}$.

The effective potential $\mathcal{V}(\psi)$ is now defined by removing the harmonics with $\mathbf{n} = 0$ from the expression of $\widetilde{\mathcal{V}}(\psi)$. If one wants an effective potential with zero average, one has to change the propagator of the ψ variables as

$$g_{\psi}^{-1}(\mathbf{k}) \;=\; C_{\psi}(\mathbf{k}) - Q(\mathbf{k})g_{\xi}(\mathbf{k})Q(\mathbf{k}) + \widehat{\mathcal{V}}_0(\mathbf{k})\,. \tag{3.14}$$

Lemma 3.4. *The propagator $g_{\psi}(\mathbf{k})$ has the following structure*

$$\begin{aligned} g_{\psi}(\mathbf{k}) \;&=\; \begin{pmatrix} g_{11}(\mathbf{k}) & g_{12}(\mathbf{k}) \\ g_{21}(\mathbf{k}) & g_{22}(\mathbf{k}) \end{pmatrix} \\ &=\; -\frac{1}{|a_{11}(\mathbf{k})|^2 + a_{12}(\mathbf{k})^2}\begin{pmatrix} -\left(a_{11}(\mathbf{k})\right)^* & -\mathrm{i}a_{12}(\mathbf{k}) \\ \mathrm{i}a_{12}(\mathbf{k}) & a_{11}(\mathbf{k}) \end{pmatrix} \end{aligned} \tag{3.15}$$

with $a_{11}(\mathbf{k}) = -a_{11}(-\mathbf{k}) \in \mathbb{C}$ and $a_{12}(\mathbf{k}) = a_{12}(-\mathbf{k}) \in \mathbb{R}$. If $|\lambda|$ is small enough, the denominator vanishes only for $\mathbf{k} = 0$ and $\mu_{\psi}(0) = 0$ with $\mu_{\psi}(0) := a_{12}(0) = m_{\psi}(0) + b(0) = m_{\psi}(0) + O(\lambda^2)$.

Proof. The structure of the denominator, eq. (3.15), follows from (C.1) and (2.10).

Since $\tilde{C}_{\psi}(\mathbf{k})$ (see eq. (2.10)) is bounded away from zero outside a neighborhood of $\mathbf{k} = 0$, and since for λ small enough $\widehat{\mathcal{V}}_0(\mathbf{k})$ is analytic and of order λ^2, then there is a treshold λ_* for which, for any $|\lambda| \leqslant \lambda_*$, $\|\tilde{C}_{\psi}(\mathbf{k}) + \widehat{\mathcal{V}}_0(\mathbf{k})\| \geqslant G$ for $G > 0$.

This is enough to prove that, for λ small enough, there are no poles in the propagator for $\mathbf{k}$ outside a neighbourhood of the origin. Let us focus on the neigborhood of the origin.

Due to the structure of the propagator, the determinant of $g_{\psi}^{-1}(\mathbf{k})$ vanishes if and only if $a_{11}(\mathbf{k}) = 0$ and $a_{12}(\mathbf{k}) = 0$. Using the asymptotics as $\mathbf{k} \downarrow 0$ of $a_{11}(\mathbf{k})$ one has for some $\alpha_0, \alpha_1 \in \mathbb{C}$,

$$\begin{aligned} |a_{11}(\mathbf{k})|^2 &\geqslant C|a_1k_1 + a_0k_0|^2 \\ &= |(t^{(1)} + \lambda\alpha_1)k_1 + \mathrm{i}(t^{(0)} + \lambda\alpha_0)k_0|^2 \\ &= (t^{(1)}k_1)^2 + (t^{(0)}k_0)^2 + 2\lambda Re(\mathrm{i}t^{(0)}\alpha_1 + t^{(1)}\alpha_0)k_0k_1 + \lambda^2|\alpha_1|^2k_1^2 + |\alpha_0|^2\lambda^2k_0^2 \end{aligned}$$

which clearly, for λ small enough vanishes if and only if $\mathbf{k} = 0$ since the mixed product is controlled as

$$-k_0^2 - k_1^2 \leqslant 2k_0k_1 \leqslant k_0^2 + k_1^2\,.$$

This is enough to prove that, for λ small enough the only critical mode is $\mathbf{k} = 0$. Thus, for criticality, it is necessary that $\mathbf{k} = 0$. This implies that, in terms of the parameters of the model, the critical condition now reads $a_{12}(0) = 0$. ∎

Remark 3.5. In case disorder is one-dimensional, say on direction $\mathbf{e}_0$, then from the explicit expression of $P^{(1)}(\mathbf{k}, \mathbf{n})$ in (2.14) one has automatically the desired reality properties for $\widehat{\mathcal{V}}_0(\mathbf{k})$. Indeed, to the list of properties after (3.10), one has to add the property that $a_\mathbf{n} \in \mathrm{i}\mathbb{R}$. This additional property derives from the fact that in case of the vertex, we can have only matrices of type $P^{(1)}$ and in case of the propagator, we are interested in the case $k_0 = k_1 = 0$ with $\omega_1 \neq 0$. This implies that also the entries of the propagator are purely imaginary.

At this point, most of the discussion concerning the structure of the effective potential is simplified by saying that the product of an odd number of imaginary numbers is imaginary.

4. Integration of Critical Modes

4.1. **Multiscale Decomposition.** In this section we introduce a multiscale analysis to compute

$$\hat{Z}_{\boldsymbol{\alpha}} = \int P_{g_\psi}(\mathrm{d}\psi) e^{\mathcal{V}(\psi)} . \tag{4.1}$$

We define a function $\chi \in C^\infty(\mathbb{R}^+)$ such that $\chi'(|\mathbf{k}|_\mathbb{T}) \leqslant 0$ and

$$\chi(\mathbf{k}) \;=\; \chi(|\mathbf{k}|_\mathbb{T}) \;=\; \begin{cases} 1\,, & \text{if } |\mathbf{k}|_\mathbb{T} < \gamma^{-1} \\ 0\,, & \text{if } |\mathbf{k}|_\mathbb{T} \geqslant 1 \end{cases} \tag{4.2}$$

with $\mathbb{T}$ denoting the torus of length 2π, $|\mathbf{k}|_\mathbb{T} := \min_{\mathbf{m}\in\mathbb{Z}^2} |\mathbf{k} - 2\pi\mathbf{m}|$. We also define

$$\chi_h(\mathbf{k}) \;:=\; \chi(\gamma^{-h}\mathbf{k})\,. \tag{4.3}$$

The function χ and the functions f_h defined below are represented in Figure 2

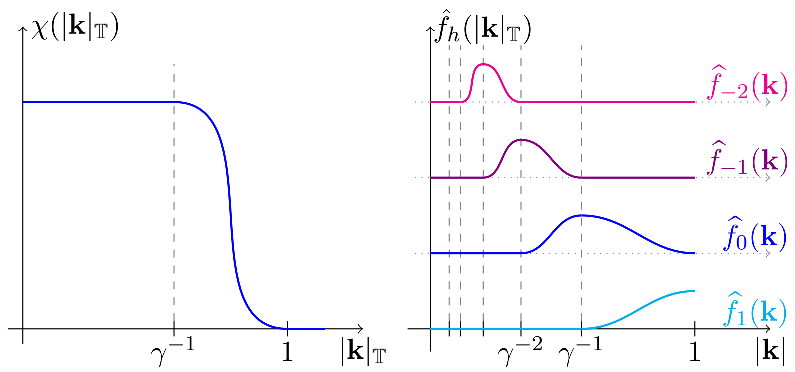

FIGURE 2. Plot of the function χ and some of the $\widehat{f}_h$.

First step. Let us proceed step by step to enlighten the ideas. First, we split the propagator

$$g_\psi(\mathbf{k}) \;=\; (1 - \chi(\mathbf{k}))g_\psi(\mathbf{k}) + \chi(\mathbf{k})g_\psi(\mathbf{k}) \tag{4.4}$$

which defines

$$g^{(1)}(\mathbf{k}) \;:=\; (1 - \chi(\mathbf{k}))g_\psi(\mathbf{k})\,. \tag{4.5}$$

Consequently, we split the Grassmann variables

$$\psi \;=\; \psi^{(\leqslant 0)} + \psi^{(1)} \tag{4.6}$$

and, as a consequence of the addition property of the Grassmann Gaussian measures [33, Section 2.4], one has

$$P(\mathrm{d}\psi) \;=\; \tilde{P}_{\leqslant 0}(\mathrm{d}\psi^{(\leqslant 0)})P_1(\mathrm{d}\psi^{(1)})\,. \tag{4.7}$$

Here, $\tilde{P}_{\leqslant 0}$ is the Gaussian Grassmann measure with covariance $\chi(\mathbf{k})g_\psi(\mathbf{k})$ and P_1 is the Gaussian Grassmann measure with covariance $g^{(1)}(\mathbf{k})$.

Note that, at this point, due to the support of $g^{(1)}(\mathbf{k})$ we have the following estimate for the propagator on scale 1:

$$\left\| g^{(1)}(\mathbf{k}) \right\| \leqslant G_\psi \gamma^{-1} \tag{4.8}$$

where G_ψ is a constant.

The integration over $\psi^{(1)}$ yields

$$\begin{aligned} \widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)}) + E_0 \; &:= \; \log \left(\int P_1(\mathrm{d}\psi^{(1)}) e^{\mathcal{V}(\psi^{(\leqslant 0)} + \psi^{(1)})} \right) \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \mathbb{E}_1^T(\mathcal{V}(\psi^{(\leqslant 0)} + \psi^{(1)}); j) \end{aligned} \tag{4.9}$$

where $\mathbb{E}_1^T$ denotes the truncated expectations with respect to P_1. $\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)})$ admits the following representation

$$\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)}) \; = \; \frac{1}{4\pi|\Lambda|} \sum_{\substack{\mathbf{k} \in \mathcal{D}_{\boldsymbol{\alpha}} \\ \mathbf{n} \in \mathbb{Z}^2}} \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{-\mathbf{k}} \cdot \widehat{\mathcal{V}}^{(0)}_{\mathbf{n}}(\mathbf{k}) \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{\mathbf{k}-2\pi\Omega\mathbf{n}} \, . \tag{4.10}$$

For a reason that will be clear later, it is convenient to define the localisation and renormalisation operators at scale $h = 0$. The first isolates the problem related to the resonances and its action is defined through the representation (4.10)

$$\mathcal{L}\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)}) \; := \; \frac{1}{4\pi|\Lambda|} \sum_{\mathbf{k} \in \mathcal{D}_{\boldsymbol{\alpha}}} \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{-\mathbf{k}} \cdot \big(\widehat{\mathcal{V}}^{(0)}_0(0) + \mathbf{k} \cdot \partial_{\mathbf{k}} \widehat{\mathcal{V}}^{(0)}_0(0) \big) \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{\mathbf{k}} \, , \tag{4.11}$$

and the second is defined as

$$\mathcal{R}\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)}) \; := \; (1 - \mathcal{L})\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)}) \, . \tag{4.12}$$

At the level of kernels, one has

$$\mathcal{L}\widehat{\mathcal{V}}^{(0)}_{\mathbf{n}}(\mathbf{k}) \; = \; \delta_{\mathbf{n},0} \left(\widehat{\mathcal{V}}^{(0)}_0(0) + \mathbf{k} \cdot \partial_{\mathbf{k}} \widehat{\mathcal{V}}^{(0)}_0(0) \right) \, . \tag{4.13}$$

It is also convenient to introduce the counterterms $\nu_0 \in \mathbb{R}$ and $\mathbf{c}_0 = (c_{0,0}, c_{0,1})$ which is a vector of 2×2 matrices. Considering the structure that emerges from the analysis of Appendix C, they are defined through the relations

$$\nu_0 \sigma_2 = \widehat{\mathcal{V}}^{(0)}_0(0) \, , \qquad \mathbf{c}_0 = \partial_{\mathbf{k}} \widehat{\mathcal{V}}^{(0)}_0(0) \, , \tag{4.14}$$

and they admit a representation in terms of graphs (it is sufficient to expand $\widehat{\mathcal{V}}^{(0)}_0(0)$).

Explicitly, the action of the renormalisation operator is

$$\begin{aligned} \mathcal{R}\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)}) \; &= \; (1 - \mathcal{L})\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)}) \\ &= \frac{1}{4\pi|\Lambda|} \sum_{\substack{\mathbf{k} \in \mathcal{D}_{\boldsymbol{\alpha}} \\ \mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}}} \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{-\mathbf{k}} \cdot \widehat{\mathcal{V}}^{(0)}_{\mathbf{n}}(\mathbf{k}) \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{\mathbf{k}-2\pi\Omega\mathbf{n}} + \frac{1}{4\pi|\Lambda|} \sum_{\mathbf{k} \in \mathcal{D}_{\boldsymbol{\alpha}}} \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{-\mathbf{k}} \cdot \mathcal{R}\widehat{\mathcal{V}}^{(0)}_0(\mathbf{k}) \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{\mathbf{k}} \, . \end{aligned} \tag{4.15}$$

For a reason that it will be clear in the following, it is convenient to write the action of the renormalisation operator in the following two equivalent ways

$$\mathcal{R}\widehat{\mathcal{V}}^{(0)}_0(\mathbf{k}) \; = \; \widehat{\mathcal{V}}^{(0)}_0(\mathbf{k}) - \widehat{\mathcal{V}}^{(0)}_0(0) - \mathbf{k} \cdot \partial_{\mathbf{k}} \widehat{\mathcal{V}}^{(0)}_0(0) = \int_0^1 \frac{\mathrm{d}^2}{\mathrm{d}t^2} \widehat{\mathcal{V}}^{(0)}_0(t\mathbf{k})(t-1) \, \mathrm{d}t \, . \tag{4.16}$$

We are ready to introduce the *effective potential on scale* 0 as

$$\mathcal{V}^{(0)}(\psi^{(\leqslant 0)}) \; := \; \frac{1}{4\pi|\Lambda|} \sum_{\mathbf{k} \in \mathcal{D}_{\boldsymbol{\alpha}}} \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{-\mathbf{k}} \cdot \nu_0 \sigma_2 \widehat{\boldsymbol{\psi}}^{(\leqslant 0)}_{\mathbf{k}} + \mathcal{R}\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)}) \, , \tag{4.17}$$

the propagator on scale $\leqslant 0$ is defined as

$$g^{(\leqslant 0)}(\mathbf{k}) \; := \; \left[(\chi(\mathbf{k}) g_\psi(\mathbf{k}))^{-1} + \mathbf{c}_0 \cdot \mathbf{k} \right]^{-1} = \; \chi(\mathbf{k}) \left[(g_\psi(\mathbf{k}))^{-1} + \chi(\mathbf{k}) \mathbf{c}_0 \cdot \mathbf{k} \right]^{-1} \, . \tag{4.18}$$

and, defining $f_0(\mathbf{k}) = (1-\chi_{-1}(\mathbf{k}))\chi(\mathbf{k})$, we define the propagator on scale 0 as

$$g^{(0)}(\mathbf{k}) = f_0(\mathbf{k}) g^{(\leqslant 0)}(\mathbf{k}) . \tag{4.19}$$

Changing the propagator results in a change of the normalisation constant:

$$\int P_{\chi g_\psi}(\mathrm{d}\psi^{(\leqslant 0)}) e^{\widetilde{\mathcal{V}}^{(0)}(\psi^{(\leqslant 0)})} = N_0 \int P_{g^{(\leqslant 0)}}(\mathrm{d}\psi^{(\leqslant 0)}) e^{\mathcal{V}^{(0)}(\psi^{(\leqslant 0)})} . \tag{4.20}$$

Due to the structure of the effective potential and the counterterms, adapting the argument used to obtain (3.15), we have the following asymptotic behaviour as $\mathbf{k} \to 0$ for the propagator on scale 0:

$$g^{(0)}(\mathbf{k}) = \frac{f_0(\mathbf{k})}{\left|a_1^{(0)} k_1 + a_0^{(0)} k_0\right|^2 + \mu_\psi^{(0)}(0)^2} \left[\begin{pmatrix} a_1^{(0)} k_1 + a_0^{(0)} k_0 & \mathrm{i}\mu_\psi^{(0)}(0) \\ -\mathrm{i}\mu_\psi^{(0)}(0) & -\left(a_1^{(0)}\right)^* k_1 - \left(a_0^{(0)}\right)^* k_0 \end{pmatrix} + R(\mathbf{k}) \right] \tag{4.21}$$

where

$$\mathbf{b}_0 := \mathbf{c}_0 , \qquad \begin{aligned} a_1^{(0)} &= (t^{(1)} + (\mathbf{b}_0)_1) , \\ a_0^{(0)} &= (\mathrm{i}t^{(0)} + (\mathbf{b}_0)_0) . \end{aligned} \tag{4.22}$$

and $\|R(\mathbf{k})\| \leqslant C|\mathbf{k}|^2$.

<u>The generic step</u>. Suppose we have just integrated the field on scale $h+1$, i.e. we just computed the integral

$$\int P_{\leqslant h+1}(\mathrm{d}\psi^{(\leqslant h+1)}) e^{\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)})} , \tag{4.23}$$

that is the same integral as (4.1) with the replacements $\mathcal{V}^{(0)} \mapsto \mathcal{V}^{(h+1)}$, $g^{(\leqslant 0)} \mapsto g^{(\leqslant h+1)}$ and $\psi^{(\leqslant 0)} \mapsto \psi^{(\leqslant h+1)}$. We define the propagator on lower scales as

$$g^{(\leqslant h+1)}(\mathbf{k}) = \chi_{h+1}(\mathbf{k}) \left[(g_\psi^{(\leqslant h+2)}(\mathbf{k}))^{-1} + \chi_{h+1}(\mathbf{k}) \mathbf{c}_{h+1} \cdot \mathbf{k} \right]^{-1} . \tag{4.24}$$

As a consequence, defining $f_{h+1}(\mathbf{k}) := (1-\chi_h(\mathbf{k}))\chi_{h+1}(\mathbf{k})$ and $g^{(h+1)}(\mathbf{k}) = f_{h+1}(\mathbf{k}) g^{(\leqslant h+1)}(\mathbf{k})$, we have

$$\begin{aligned} g^{(h+1)}(\mathbf{k}) =& f_{h+1}(\mathbf{k}) \frac{1}{\left|a_1^{(h+1)} k_1 + a_0^{(h+1)} k_0\right|^2 + \mu_\psi^{(h+1)}(0)^2} \times \\ & \times \left[\begin{pmatrix} a_1^{(h+1)} k_1 + a_0^{(h+1)} k_0 & \mathrm{i}\mu_\psi^{(h+1)}(0) \\ -\mathrm{i}\mu_\psi^{(h+1)}(0) & -(a_1^{(h+1)})^* k_1 - (a_0^{(h+1)})^* k_0 \end{pmatrix} + R(\mathbf{k}) \right] . \end{aligned} \tag{4.25}$$

with $\|R(\mathbf{k})\| \leqslant C|\mathbf{k}|^2$.

The result of the integration on scale $h+1$ is

$$\begin{aligned} \widetilde{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) + E_h &= \log \left(\int P_{h+1}(\mathrm{d}\psi^{(h+1)}) e^{\mathcal{V}^{(h+1)}(\psi^{(\leqslant h)} + \psi^{(h+1)})} \right) \\ &= \sum_{r=0}^{\infty} \frac{1}{r!} \mathbb{E}_{h+1}^T (\mathcal{V}^{(h+1)}(\psi^{(\leqslant h)} + \cdot); r) . \end{aligned} \tag{4.26}$$

Analogously to (4.10), one has

$$\widetilde{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) = \frac{1}{4\pi|\Lambda|} \sum_{\substack{\mathbf{k} \in \mathcal{D}_\alpha \\ \mathbf{n} \in \mathbb{Z}^2}} \boldsymbol{\psi}_{-\mathbf{k}}^{(\leqslant h)} \cdot \widehat{\mathcal{V}}_{\mathbf{n}}^{(h)}(\mathbf{k}) \boldsymbol{\psi}_{\mathbf{k}-2\pi\Omega\mathbf{n}}^{(\leqslant h)} . \tag{4.27}$$

One now localises and renormalises. We define the renormalisation and localisation operators on scale h as

$$\mathcal{L}\widetilde{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) := \frac{1}{4\pi|\Lambda|} \sum_{\mathbf{k} \in \mathcal{D}_\alpha} \widehat{\boldsymbol{\psi}}_{-\mathbf{k}}^{(\leqslant h)} \cdot \left(\widehat{\mathcal{V}}_0^{(h)}(0) + \mathbf{k} \cdot \partial_{\mathbf{k}} \widehat{\mathcal{V}}_0^{(h)}(0)\right) \widehat{\boldsymbol{\psi}}_{\mathbf{k}}^{(\leqslant h)} , \tag{4.28}$$

and

$$\mathcal{R}\widetilde{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) := (1-\mathcal{L})\widetilde{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) \tag{4.29}$$

that analogously to (4.16), can be written in the following equivalent forms

$$\begin{aligned} \mathcal{R}\widehat{\mathcal{V}}_0^{(h)}(\mathbf{k}) &= \widehat{\mathcal{V}}_0^{(h)}(\mathbf{k}) - \widehat{\mathcal{V}}_0^{(h)}(0) - \mathbf{k}\cdot\partial_{\mathbf{k}}\widehat{\mathcal{V}}_0^{(h)}(0) \\ &= \int_0^1 \frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{\mathcal{V}}_0^{(h)}(t\mathbf{k})(t-1)\,\mathrm{d}t\,. \end{aligned} \tag{4.30}$$

From a straightforward adaptation of the arguments in Appendices B and C, we obtain the following structure for the counterterms

$$\nu_h\sigma_2 = \gamma^{-2h}\widehat{\mathcal{V}}_0^{(h)}(0)\,, \tag{4.31}$$

$$\mathbf{c}_h = \partial_{\mathbf{k}}\widehat{\mathcal{V}}_0^{(h)}(0)\,, \tag{4.32}$$

and

$$\mathbf{b}_h = \mathbf{b}_{h+1} + \mathbf{c}_h\,. \tag{4.33}$$

We can now define the potential on scale h as

$$\mathcal{V}^{(h)}(\psi^{(\leqslant h)}) = \frac{1}{4\pi|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}}\widehat{\boldsymbol{\psi}}_{-\mathbf{k}}\cdot\gamma^{2h}\nu_h\sigma_2\widehat{\boldsymbol{\psi}}_{\mathbf{k}} + \mathcal{R}\widetilde{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) \tag{4.34}$$

and the propagator on scale $\leqslant h$

$$\begin{aligned} g^{(\leqslant h)}(\mathbf{k}) &= \left[(\chi_h(\mathbf{k})g^{(\leqslant h+1)}(\mathbf{k}))^{-1} + \mathbf{c}_h\cdot\mathbf{k}\right]^{-1} \\ &= \chi_h(\mathbf{k})\left[(g_\psi(\mathbf{k}))^{-1} + \mathbf{b}_{h+1}\cdot\mathbf{k} + \chi_h(\mathbf{k})\mathbf{c}_h\cdot\mathbf{k}\right]^{-1}\,. \end{aligned} \tag{4.35}$$

Changing the propagator results in a change of the renormalisation constant

$$\int P_{\chi_h g^{(\leqslant h+1)}}(\mathrm{d}\psi^{(\leqslant h)})e^{\widetilde{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)})} = N_h\int P_{g^{(\leqslant h)}}(\mathrm{d}\psi^{(\leqslant h)})e^{\mathcal{V}^{(h)}(\psi^{(\leqslant h)})} \tag{4.36}$$

To perform the integration on scale h we now decompose

$$g^{(\leqslant h)}(\mathbf{k}) = (1-\chi_{h-1}(\mathbf{k}))g^{(\leqslant h)}(\mathbf{k}) + \chi_{h-1}(\mathbf{k})g^{(\leqslant h)}(\mathbf{k}) \tag{4.37}$$

and thus we obtain

$$g^{(\leqslant h)}(\mathbf{k}) = g^{(h)}(\mathbf{k}) + \chi_{h-1}g^{(\leqslant h)}(\mathbf{k}) \tag{4.38}$$

with the explicit expression

$$g^{(h)}(\mathbf{k}) = f_h(\mathbf{k})\left[(g^{(h+1)}(\mathbf{k}))^{-1} + \chi_h(\mathbf{k})\mathbf{c}_h\cdot\mathbf{k}\right]^{-1} \tag{4.39}$$

Due to the structure of the effective potential and the counterterms we have the following asymptotic behaviour as $\mathbf{k}\to 0$ for the propagator on scale h:

$$g^{(h)}(\mathbf{k}) = f_h(\mathbf{k})\frac{1}{|a_1^{(h)}k_1 + a_0^{(h)}k_0|^2 + \mu_\psi^{(h)}(0)^2}\left[\begin{pmatrix} a_1^{(h)}k_1 + a_0^{(h)}k_0 & \mathrm{i}\mu_\psi^{(h)}(0) \\ -\mathrm{i}\mu_\psi^{(h)}(0) & -(a_1^{(h)})^*k_1 - (a_0^{(h)})^*k_0 \end{pmatrix} + R_h(\mathbf{k})\right]. \tag{4.40}$$

Here we have $\|R_h(\mathbf{k})\| \leqslant C|\mathbf{k}|^2$,

$$\begin{aligned} a_1^{(0)} &= (t^{(1)} + (\mathbf{b}_h)_1)\,, \\ a_0^{(0)} &= (\mathrm{i}t^{(0)} + (\mathbf{b}_h)_0)\,. \end{aligned} \tag{4.41}$$

We decompose the Grassmann variables as $\psi^{(\leqslant h)} = \psi^{(\leqslant h-1)} + \psi^{(h)}$ and we define

$$\begin{aligned} \widetilde{\mathcal{V}}^{(h-1)}(\psi^{(\leqslant h-1)}) + E_{h-1} &= \log\left(\int P_{g^{(\leqslant h-1)}}e^{\mathcal{V}^{(h)}(\psi^{(\leqslant h-1)}+\psi^{(h)})}\right) \\ &= \sum_{r=1}^{+\infty}\frac{1}{r!}\mathbb{E}_h^T(\mathcal{V}^{(h)}(\psi^{(\leqslant h-1)}+\cdot);r) \end{aligned} \tag{4.42}$$

and we can proceed iteratively.

4.2. **Renormalized Graphs.** To take into account the renormalisation process in the iterative procedure explained in the previous section, it is convenient to define renormalized Feynman graphs and *clusters*.

Definition 4.1. Given an ordered collection of q elements, $\{\mathbf{n}_1, \ldots, \mathbf{n}_q\}$, $\mathbf{n}_s \in \mathbb{Z}^2$ for any $s = 1, \ldots, q$ and a collection of $q+1$ elements, called *scales*, $\{h_0, \ldots, h_q\}$ with $h_s = 1, 0, -1, \ldots, -\infty$ and $h_0, h_q < \min\{h_1, \ldots, h_{q-1}\}$ we define the associated *Renormalized graph*, denoted as Γ, as the set of q ordered vertices, $q-1$ internal lines and 2 external lines constructed as follows. To the first external line on scale h_0 is attached to the first vertex. Then to the first vertex one attaches a line that connects it with the second vertices. The second vertex is attached to a line that connects it with the third verdex and so on. The last vertex is connected only to the previous one and to the last external line on scale h_q.

To each of the r-th vertices one associates the pair $\mathbf{n}_r$ and to the r-th internal line one associates the label h_r. Given a renormalized graph Γ, we denote by $E(\Gamma)$ the set of vertices of Γ. We say that a graph is on scale h_Γ if $h_\Gamma = \max\{h_0, h_1\}$.

Given $\mathbf{k} \in \mathcal{D}_{\boldsymbol{\alpha}}$ one associates to the s-th line the momentum $\mathbf{k}_s = \mathbf{k} - 2\pi\Omega \sum_{r \leqslant s} \mathbf{n}_r + 2\pi\mathbf{m}$, with $\mathbf{m} \in \mathbb{Z}^2$ such that $\mathbf{k}_s \in \mathcal{D}_{\boldsymbol{\alpha}}$.

Definition 4.2. Given a Renormalized graph Γ and a scale $h_T \in \{1, 0, -1, \ldots, -\infty\}$, we define a *cluster* (denoted by T) as an ordered and connected set of lines on scale $\geqslant h_T$ and their adjacency vertices v that lies between two lines on scale $< h_T$ (called *external lines of the cluster* T). Additionally, we require that at least one line of the cluster is on scale h_T. Each vertex in the cluster is called an *end-point* and it is said to be on scale 1 if it is non-resonant and on scale $h_{\tilde{T}}$ if it is resonant and its value is $\nu_{h_{\tilde{T}}}$. End-points are considered trivial clusters, and thus T can denote also an end-point.

Given a cluster T, whose external lines are on scale $h_{\rm in}$ and $h_{\rm out}$, we define its external scale $h_T^{\rm ext} = \max\{h_{\rm in}, h_{\rm out}\}$.

Note that, if Γ is a Renormalized graph, then trivially $T = \Gamma$ is a cluster (and the external lines of the cluster are the external lines of the graph) and so all the results below, stated for clusters, extend trivially to graphs.

To each vertex v of the cluster T it is associated a pair of numbers $\mathbf{n}_v \in \mathbb{Z}^2$. To a cluster T one associates $\mathbf{n}_T = \sum_{v \in E(T)} \mathbf{n}_v$, with $E(T)$ defined below.

To a cluster T we associate also:

- T_0 is the set of vertices and lines contained in T and not in any $\tilde{T} \subset T$; in particular, all the lines are at scale h_T;
- M_T is the number of the non-resonant clusters and non-resonant end-points contained in T and not in any smaller one (if T is an end-point, then $M_T = 0$);
- R_T is the number of the resonant clusters and resonant end-points contained in T (if T is an end-point, then $R_T = 0$);
- M_T^ν is the set of resonant end-points strictly contained in T and not in any smaller cluster;
- M_T^I is the set of non-resonant end-points strictly contained in T and not in any smaller cluster;
- $\mathbf{R}$ is the set of resonant clusters contained in T;
- $E(T)$ is the set of vertices contained in T.

To each line ℓ of a cluster T one associates a momentum $\mathbf{k}_\ell$. If ℓ' is the line preceeding ℓ, in the sense that $\ell' < \ell$ and ℓ' and ℓ are connected by a vertex v, then the following equality holds

$$\mathbf{k}_\ell - \mathbf{k}_{\ell'} \;=\; 2\pi\Omega\mathbf{n}_v + 2\pi\mathbf{m}\,, \tag{4.43}$$

for some $\mathbf{m} \in \mathbb{Z}^2$ such that $\mathbf{k}_\ell, \mathbf{k}_{\ell'} \in \mathcal{D}_{\boldsymbol{\alpha}}$. One also says that the line ℓ enters the vertex v and the line ℓ' exits the vertex v.

Definition 4.3. To a cluster T at scale h_T with entering momentum $\mathbf{k}$ one associates the value $W_T^{(h_T)}(\mathbf{k})$ defined by

$$W_T^{(h_T)}(\mathbf{k}) := \left[\prod_{j=1}^{M_T-1} \overline{W}_j^{(h_j)}(\mathbf{k}_{j-1}) g^{(h_T)}(\mathbf{k}_j) \right] \overline{W}_{M_T}^{(h_{M_T})}(\mathbf{k}_{L_T}) \tag{4.44}$$

where $\mathbf{k}_j = \mathbf{k}_{j-1} - 2\pi\Omega\mathbf{n}_j + 2\pi\mathbf{m}$ (for some $\mathbf{m} \in \mathbb{Z}^2$ such that $\mathbf{k}_j \in \mathcal{D}_{\boldsymbol{\alpha}}$), $\mathbf{k}_0 = \mathbf{k}$,

$$\overline{W}_j^{(h_j)}(\mathbf{k}) := \begin{cases} W_{\tilde{T}_j}^{(h_j)}(\mathbf{k}) & \text{if } M(j) = \tilde{T}_j \text{ and } \mathbf{n}_{\tilde{T}_j} \neq 0\,,\ h_j = h_{\tilde{T}_j} \\ \mathcal{R}W_{\tilde{T}_j}^{(h_j)}(\mathbf{k}) & \text{if } M(j) = \tilde{T}_j \text{ and } \mathbf{n}_{\tilde{T}_j} = 0\,,\ h_j = h_{\tilde{T}_j} \\ \gamma^{2h_T}\nu_{h_T}\sigma_2 & \text{if } M(j) = \nu \\ \widehat{\mathcal{V}}_{\mathbf{n}_j}(\mathbf{k}) & \text{if } M(j) = v_j \end{cases} . \tag{4.45}$$

In Figure 3 it is represented formula (4.44), and the nested structure of the cluster is represented in Figure 4.

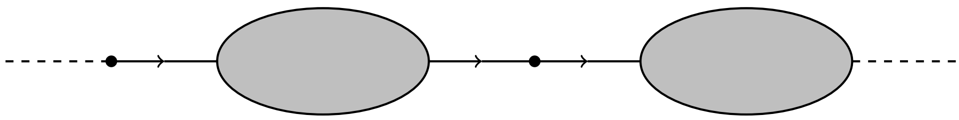

FIGURE 3. The structure of the cluster. In order we see: a vertex, a maximal cluster, a vertex and another maximal cluster.

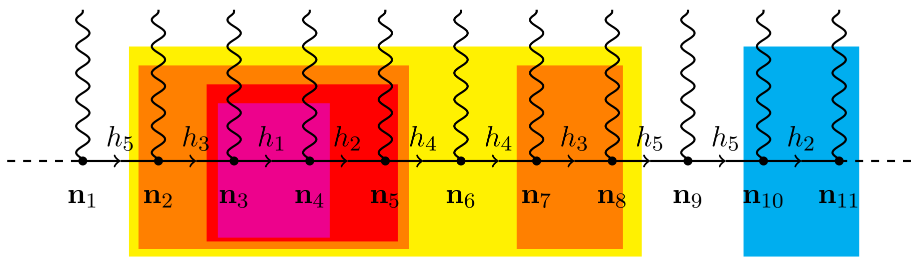

FIGURE 4. Graphical representation of a Renormalized graph with 11 vertices and 10 lines, that is the expansion of Figure 3. Esternal lines are dashed, while internal lines are solid. The arrows denote the ordering of the graph. The graph is on scale h_5 and it has two maximal clusters contained in it: the yellow one on scale h_4 (corresponding to the first grey bubble in Figure 3) and the blue one on scale h_2 (corresponding to the second bubble in Figure 3). The graph has two vertices (or, end-points) on scale h_5 with indices $\mathbf{n}_1$ and $\mathbf{n}_9$.

Note that, according to the definitions above, the value of the cluster T is zero if at least for one line ℓ, $\mathbf{k}_\ell \notin \text{supp}(f_{h_\ell})$.

The number of vertices inside a cluster T is defined as

$$q_T := \sum_{\tilde{T} \subseteq T} \left(M_{\tilde{T}}^I + M_{\tilde{T}}^\nu \right) . \tag{4.46}$$

We define the value of a renormalized graph Γ with q vertices on scale h_Γ for which $\sum_{r=1}^q \mathbf{n}_r = \mathbf{n}_\Gamma$ as

$$\text{Val}(\Gamma(\mathbf{k})) = W_T^{(h_T)}(\mathbf{k}) \tag{4.47}$$

where $T = \Gamma$ is a cluster with q vertices, $\mathbf{n}_T = \mathbf{n}_\Gamma$ and $h_T^{\text{ext}} = h_\Gamma$. We also denote by $\mathcal{G}_{\mathbf{n},q}^{R,h}$ the set of renormalized graphs with q vertices, $\mathbf{n}_\Gamma = \mathbf{n}$ and $h_\Gamma = h$.

Then, we define

$$\mathcal{W}_{q,\mathbf{n}}^{(h)}(\mathbf{k}) := \sum_{\Gamma \in \mathcal{G}_{\mathbf{n},q}^{R,h}} \text{Val}(\Gamma(\mathbf{k})) . \tag{4.48}$$

It is then obviously possible to compute the kernel of the effective potential on scale h (see (4.34) and (4.27)),

$$\begin{aligned} \widehat{\mathcal{V}}^{(h)}_{\mathbf{n}}(\mathbf{k}) &= \sum_{q=1}^{\infty} \mathcal{W}^{(h)}_{q,\mathbf{n}}(\mathbf{k}) \qquad\qquad \text{if } \mathbf{n} \neq 0\,, \\ \widehat{\mathcal{V}}^{(h)}_{0}(\mathbf{k}) &= \gamma^{2h}\nu_h\sigma_2 + \sum_{q=2}^{\infty} \mathcal{R}\mathcal{W}^{(h)}_{q,0}(\mathbf{k})\,. \end{aligned} \tag{4.49}$$

A convenient graphical representation for the labelled graphs is given in terms of labelled Feynman graphs, like the ones drawn for the integration of the heavy fermions, with, in addition, scale indices on the lines, see Figure 4.

4.3. **Analysis of Renormalized Graph.**

We start from the following lemma.

Lemma 4.4. *If $\mathbf{n}_T = (n_0, n_1) \neq 0$ is the pair of numbers associated to a non-resonant vertex or to a non-resonant cluster T, then*

$$|\mathbf{n}_T| \;\geqslant\; C_4\gamma^{-\frac{h_T^{ext}}{\tau}}\,. \tag{4.50}$$

Here, $\tau := \min\{\rho_1, \rho_0\}$.

Proof. Let $\mathbf{k}_{\text{in}}$ and $\mathbf{k}_{\text{out}}$ be the momenta associated to the lines lines that enter and exit from the vertex or the cluster

$$|\mathbf{k}_{\text{in}} - \mathbf{k}_{\text{out}}|_{\mathbb{T}} \;\leqslant\; 2\gamma^{h+1}\gamma^{-1} \;=\; 2\gamma^{h}\,.$$

Recalling that $\mathbf{n}$ is Diophantine, we can use conservation of momentum (4.43) to get

$$\begin{aligned} |\mathbf{k}_{\text{in}} - \mathbf{k}_{\text{out}}|_{\mathbb{T}} &= 2\pi \min_{m_0,m_1\in\mathbb{Z}} \big||\omega_1 n_1 - m_1| + |\omega_0 n_0 - m_0|\big| \;\geqslant\; C_1|n_1|^{-\rho_1} + C_0|n_0|^{-\rho_0} \\ &\geqslant\; C_1\,\big||n_1| + |n_0|\big|^{-\rho_1} + C_0\,\big||n_1| + |n_0|\big|^{-\rho_0} \\ &\geqslant\; \max(C_1, C_0)|\mathbf{n}_T|^{-\min(\rho_1,\rho_0)} \end{aligned}$$

and this last inequality implies in particular,

$$|\mathbf{n}_T| \;\geqslant\; \left(\frac{\max(C_1, C_0)}{2}\right)^{\frac{1}{\tau}} \gamma^{-\frac{h}{\tau}}\,.$$

∎

From now on, in this section we assume the following conditions to hold on scale h. These relations will be proved in the next section.

- There exists a constant G_ψ such that, for any h, one has

$$\|g^{(h)}(\mathbf{k})\| \;\leqslant\; G_\psi\gamma^{-h}\,, \qquad \forall \mathbf{k} \in \mathcal{D}_{\boldsymbol{\alpha}} \tag{4.51}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}g^{(h)}(t\mathbf{k} + \mathbf{q})\right\| \;\leqslant\; G_\psi\gamma^{-2h+h_{\text{ext}}} \qquad \forall \mathbf{k} \in \text{supp} f_{h_{\text{ext}}}\,, \forall \mathbf{q} \in [-\pi, \pi)^2 \tag{4.52}$$

$$\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}g^{(h)}(t\mathbf{k} + \mathbf{q})\right\| \;\leqslant\; G_\psi\gamma^{-3h+2h_{\text{ext}}} \qquad \forall \mathbf{k} \in \text{supp} f_{h_{\text{ext}}}\,, \forall \mathbf{q} \in [-\pi, \pi)^2 \tag{4.53}$$

- There exists a constant C_ν such that

$$|\nu_h| \leqslant C_\nu|\lambda|^2 \tag{4.54}$$

- There exists a constant C such that $\|\mathbf{c}_h\| \leqslant C\gamma^h|\lambda|^2$.
- γ is sufficiently large with respect to τ: $\gamma > 4^\tau$.

We consider now the renormalized graphs in the $i \to \infty$ limit and we prove the following estimate on the value of clusters.

Proposition 4.5. *Let T be a cluster at scale h_T with q vertices and $\mathbf{n}_T$. Let $h_T^{ext} < h_T$ be the external scale of T. Then, if (4.51)-(4.54) hold for scales $h > h_T$, one has for $\mathbf{n}_T = 0$ and for any $\vartheta \in (0,1)$,*

$$\left\|\mathcal{R}W_T^{(h_T)}(\mathbf{k})\right\| \leqslant \gamma^{(1+\vartheta)h_T^{ext}}(C_{\eta,\tau,\gamma}|\lambda|)^q \left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|}\right)\prod_{\tilde{T}\subseteq T}\gamma^{(1-\vartheta)(h_{\tilde{T}}^{ext}-h_{\tilde{T}})}, \tag{4.55}$$

and for $\mathbf{n}_T \neq 0$

$$\left\|W_T^{(h_T)}(\mathbf{k})\right\| \leqslant e^{-\frac{\eta}{32}C\tilde{\gamma}^{-h_T^{ext}}}(C_{\eta,\tau,\gamma}|\lambda|)^q e^{-\frac{\eta}{4}|\mathbf{n}|}\left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|}\right)\prod_{\tilde{T}\subseteq T}\gamma^{(1-\vartheta)(h_{\tilde{T}}^{ext}-h_{\tilde{T}})}. \tag{4.56}$$

where $\tilde{\gamma} = \frac{\gamma^{1/\tau}}{2}$.

Proof. For compactness of notation, we introduce

$$F_{\mathbf{n}_v}(\mathbf{k}_v) := \begin{cases} \gamma^{2h}\nu_h\sigma_2 & \text{if } \mathbf{n}_v = 0 \\ \hat{\mathcal{V}}_{\mathbf{n}_v}(\mathbf{k}_v) & \text{if } \mathbf{n}_v \neq 0 \end{cases} \tag{4.57}$$

where h is the scale associated to the maximum between the line entering the vertex v and the line exiting from it.

In this part of the proof, for $j \in \mathbb{N}$, we denote by $\mathbf{R}_j$ the set of resonant clusters strictly contained in $\mathbf{R}_{j-1}$ and not in any other *resonant* cluster. (This means $\mathbf{R} = \bigcup_{j=1}^{+\infty}\mathbf{R}_j$.) Moreover, we denote by

$$\mathbf{W}_T(\mathbf{k}) = \begin{cases} W_T^{(h_T)}(\mathbf{k}) & \mathbf{n}_T \neq 0 \\ \mathcal{R}W_T^{(h_T)}(\mathbf{k}) & \mathbf{n}_T = 0 \end{cases}. \tag{4.58}$$

From (4.44), we get

$$\begin{aligned}\|\mathbf{W}_T(\mathbf{k})\| &\leqslant \left[\prod_{j=1}^{M_T-1}\|\overline{W}_j^{(h_j)}(\mathbf{k}_{j-1})\|\|g^{(h_T)}(\mathbf{k}_j)\|\right]\|\overline{W}_{M_T}^{(h_{M_T})}(\mathbf{k}_{L_T})\| \\ &\leqslant \left(\prod_{v\in E(T)\setminus\mathbf{R}_1}\|F_{\mathbf{n}_v}(\mathbf{k}_v)\|\right)\left(\prod_{\ell\cap\mathbf{R}_1=\varnothing}\|g^{(h_\ell)}(\mathbf{k}_\ell)\|\right)\left(\prod_{T\in\mathbf{R}_1}\|\mathcal{R}W_T^{(h_T)}(\mathbf{k}_T)\|\right),\end{aligned} \tag{4.59}$$

(note that if $\mathbf{n}_T = 0$, we have $T \in \mathbf{R}_1$). With $\ell\cap\mathbf{R}_1$ we mean the lines that are outside maximal resonances. In case $T \in \mathbf{R}_1$, the first two products take the value 1. Using (4.30), and denoting with $\mathbf{R}_2$ the set of maximal resonances contained in $\mathbf{R}_1$, the value of renormalized resonant cluster can now be estimated as

$$\begin{aligned}\left\|\mathcal{R}W_{\tilde{T}}^{(h_{\tilde{T}})}(\mathbf{k}_{\tilde{T}})\right\| &\leqslant \sup_{t\in[0,1]}\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}W_{\tilde{T}}^{(h_{\tilde{T}})}(t\mathbf{k}_T)\right\| \\ &\leqslant \sum_{\substack{s_\ell,r_v,r_{T'}\in\{0,1,2\}\\ \sum_\ell s_\ell+\sum_v r_v+\sum_{T'} r_{T'}=2}}\left(\prod_{\substack{v\in(\tilde{T}\cap\mathbf{R}_1)\\ v\cap\mathbf{R}_2=\varnothing}}\left\|\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{r_v}F_{\mathbf{n}_v}(\mathbf{k}_v)\right\|\right)\times \\ &\times\left(\prod_{\ell\cap\mathbf{R}_2=\varnothing}\left\|\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{s_\ell}g^{(h_\ell)}(\mathbf{k}_\ell)\right\|\right)\left(\prod_{T'\in(\mathbf{R}_2\cap\tilde{T})}\left\|\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{r_{T'}}\mathcal{R}W_{T'}^{(h_{T'})}(\mathbf{k}_{T'})\right\|\right).\end{aligned} \tag{4.60}$$

One has now to analyze what happens when a derivative acts on a renormalized cluster.

If two derivatives corresponding to a resonance $\tilde{T}$ acts on the value of some renormalized resonant cluster $\tilde{T}' \subset \tilde{T}$, recalling that $\mathbf{k}_{\tilde{T}'} = t\mathbf{k} + \mathbf{q}$ for suitable $\mathbf{q}$, one has

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{R}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(t\mathbf{k}+\mathbf{q}) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[W^{(h_{\tilde{T}'})}_{\tilde{T}'}(t\mathbf{k}+\mathbf{q}) - W^{(h_{\tilde{T}'})}_{\tilde{T}'}(0) - (t\mathbf{k}+\mathbf{q})\cdot\partial_{\mathbf{k}}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(0)\right] \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(t\mathbf{k}+\mathbf{q})\,. \end{aligned} \tag{4.61}$$

If one derivative acts on a renormalized cluster, we have instead

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(t\mathbf{k}+\mathbf{q}) &= \mathbf{k}\cdot\partial_{\mathbf{k}}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(t\mathbf{k}+\mathbf{q}) - \mathbf{k}\cdot\partial_{\mathbf{k}}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(0) \\ &= \mathbf{k}\cdot\partial_{\mathbf{k}}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(s(t\mathbf{k}+\mathbf{q}))\,\mathrm{d}s = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(s(t\mathbf{k}+\mathbf{q}))\,\mathrm{d}s\,. \end{aligned} \tag{4.62}$$

Whence we get the two bounds

$$\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{R}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(t\mathbf{k}+\mathbf{q})\right\| = \left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(t\mathbf{k}+\mathbf{q})\right\|\,, \tag{4.63}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{R}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(t\mathbf{k}+\mathbf{q})\right\| \leqslant \sup_{s,t\in[0,1]}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}W^{(h_{\tilde{T}'})}_{\tilde{T}'}(s(t\mathbf{k}+\mathbf{q}))\right\|\,. \tag{4.64}$$

So, summarising, for the estimate we have the following:

- if two derivatives corresponding to a resonance $\tilde{T}$ act on the value of some resonance $\tilde{T}' \subset \tilde{T}$, one can replace with $\mathbf{1}$ the $\mathcal{R}$ operator;
- if one derivative corresponding to a resonance $\tilde{T}$ acts on the value of some resonance $\tilde{T}' \subset \tilde{T}$, one can replace with $\frac{\mathrm{d}}{\mathrm{d}s}$ the $\mathcal{R}$ operator and take the supremum over $s \in [0,1]$;
- if no derivatives act on a resonance, one can replace $\mathcal{R}$ with $\frac{\mathrm{d}^2}{\mathrm{d}s^2}$ and take the supremum over $s \in [0,1]$.

These remarks permit us to iterate this procedure considering the action of derivatives on resonances inside resonances. Proceeding in this way, we see that the R.H.S. of (4.60) can be bounded in the following way. We denote by f either a line or a vertex and with $T_R \in \mathbf{R}$ a resonant cluster.

- There is one term for each ordered pair (f_1, f_2), with $f_1, f_2 \in T_R$, not necessarily different (i.e. it may happen that $f_1 = f_2$).
- If $f_1 \in \tilde{T}_0$ and $\tilde{T}$ is a cluster contained in T_R, then $\tilde{T} = T^{(r)} \subset T^{(r-1)} \subset \cdots \subset T^{(1)} = T_R$ is the chain of clusters associated to f_1 containing $\tilde{T}$ and contained in T_R. Similarly, if $f_2 \in \hat{T}_0$ and $\hat{T}$ is a cluster contained in T_R, one constructs the chain of clusters associated to f_2 containing $\hat{T}$ and contained in T_R.
- At this point we replaced the $\mathcal{R}$ operator acting on the cluster T_R with two derivatives.
- If a resonant cluster belongs to both the chain of clusters (the one associated with f_1 and the one associated with f_2), then its $\mathcal{R}$ operator is removed.
- If instead there is a cluster (say, T_V) belonging to only one of the chain of clusters, then there is one term for any $f_3 \in T_V$. If $f_3 \in (T'_V)_0 \subset T_V$, then one considers the chain of cluster associated to f_3, containing T'_V and contained in T_V. One replaced the $\mathcal{R}$ operator acting on T_V.
- This construction is repeated until all $\mathcal{R}$ operators are replaced. At this point each cluster inside a resonance belongs to two chains of vertices.
- From their explicit expression, it is also obvious that one can estimate the action of a derivative on a vertex with the action of a derivative on a propagator on the same scale.
- Last, the number of terms that are generated in this procedure is estimated by 9^q (that is the number of terms generated when each vertex or each line can be derived zero, one or two times without any constraint).

We thus obtain the bound

$$\|\mathbf{W}_T(\mathbf{k})\| \leqslant 9^q \left(\prod_{v\in E(T)} \|F_{\mathbf{n}_v}(\mathbf{k}_v)\| \right) \left(\prod_{\tilde T\subseteq T} (G_\psi \gamma^{-h_{\tilde T}})^{M_{\tilde T}+R_{\tilde T}-1} \right) \prod_{\tilde T\in\mathbf{R}} \gamma^{2(h^{\rm ext}_{\tilde T}-h_{\tilde T})} . \tag{4.65}$$

where the factor $\prod_{\tilde T\in\mathbf{R}} \gamma^{2(h^{\rm ext}_{\tilde T}-h_{\tilde T})}$ is the result of the application of the $\mathcal{R}$ operation described above. We can write

$$\prod_{v\in E(T)} e^{-\frac{\eta}{2}|\mathbf{n}_v|} \leqslant e^{-\frac{\eta}{4}|\mathbf{n}|} \left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|} \right) \left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|} \right) . \tag{4.66}$$

Then, using Lemma 4.4 for all the clusters $\tilde T$ with $\mathbf{n}_{\tilde T} \neq 0$, we get

$$\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|} \leqslant \prod_{\tilde T\subseteq T} e^{-\frac{\eta}{32}2^{h^{ext}_{\tilde T}}|\mathbf{n}_{\tilde T}|} \leqslant \begin{cases} e^{-\frac{\zeta}{16}\tilde\gamma^{-h^{ext}_T}} \prod_{\tilde T\subseteq T} e^{-\frac{\zeta}{16}M_{\tilde T}\tilde\gamma^{-h_{\tilde T}}} & \text{if } \mathbf{n}_T \neq 0 \\ \prod_{\tilde T\subseteq T} e^{-\frac{\zeta}{16}M_{\tilde T}\tilde\gamma^{-h_{\tilde T}}} & \text{if } \mathbf{n}_T = 0 \end{cases} . \tag{4.67}$$

with $\zeta := C\eta/2$ and we used the fact that external lines are on scale h^{ext}_T $\max\{|\mathbf{k}|, |\mathbf{k} - 2\pi\Omega\mathbf{n}_T|\} \in \mathrm{supp} f_{h^{ext}_T}$.

We get therefore

$$\begin{aligned} \|\mathbf{W}_T(\mathbf{k})\| \leqslant C^q|\lambda|^q f_{\rm ext} e^{-\frac{\eta}{4}|\mathbf{n}|} \left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|} \right) \left(\prod_{\tilde T\subseteq T} \gamma^{-h_{\tilde T}(M_{\tilde T}+R_{\tilde T}-1)} \right) \times \\ \times \left(\prod_{\tilde T\in\mathbf{R}} \gamma^{2(h^{\rm ext}_{\tilde T}-h_{\tilde T})} \right) \left(\prod_{\tilde T\subseteq T} e^{-\frac{\zeta}{16}M_{\tilde T}\tilde\gamma^{-h_{\tilde T}}} \right) \prod_{\tilde T_\nu\subseteq T} \gamma^{2h_{\tilde T}M^\nu_{\tilde T}} . \end{aligned} \tag{4.68}$$

where $\tilde T_\nu$ are the resonant minimal clusters (with scale h_T) and f_{ext} is the gain over the external scale, that is

$$f_{ext} := \begin{cases} e^{-\frac{\zeta}{16}\tilde\gamma^{-h^{ext}_T}} & \text{if } \mathbf{n}_T \neq 0 \\ 1 & \text{if } \mathbf{n}_T = 0 \end{cases} . \tag{4.69}$$

We can bound

$$\prod_{\tilde T\subseteq T} e^{-\frac{\zeta}{16}M_{\tilde T}\tilde\gamma^{-h_{\tilde T}}} \leqslant \left(\prod_{\tilde T\subseteq T} \gamma^{4h_{\tilde T}M_{\tilde T}} \right) \prod_{\tilde T\subseteq T} \gamma^{2h_{\tilde T}M^I_{\tilde T}} \tag{4.70}$$

where we used that, by Cauchy inequality, we can bound:

$$\gamma^{-h_{\tilde T}} e^{-\zeta C\tilde\gamma^{-h_{\tilde T}/\tau}} \leqslant C_\zeta \tilde\gamma^{Nh_{\tilde T}} \leqslant C_{\tau,\zeta,\gamma}\gamma^{4h_{\tilde T}} , \tag{4.71}$$

and in the last step we choose N large enough, that is $N \geqslant \frac{4\tau\log\gamma}{\log\gamma-\tau\log 2}$. Note that, in principle, N can be chosen to be large enough and so one can get any gain in power of $\gamma^{Nh}_{\tilde T}$. We get therefore

$$\left(\prod_{\tilde T\subseteq T} \gamma^{-h_{\tilde T}M_{\tilde T}} \right) \prod_{\tilde T\subseteq T} e^{-\frac{\zeta}{16}M_{\tilde T}\tilde\gamma^{-h_{\tilde T}}} \leqslant (C^2_{\tau,\zeta,\gamma})^{\sum_{\tilde T\subseteq T} M_{\tilde T}} \left(\prod_{\tilde T\subseteq T} \gamma^{2h_{\tilde T}M^I_{\tilde T}} \right) \prod_{\tilde T\subseteq T} \gamma^{3h_{\tilde T}M_{\tilde T}} . \tag{4.72}$$

We can now use that $\sum_{\tilde T\subseteq T} M_{\tilde T} \leqslant 4q$ to estimate the constant in front of the products.

At this point, we have obtained

$$\begin{aligned} \|\mathbf{W}_T(\mathbf{k})\| \leqslant C^q|\lambda|^q f_{ext} e^{-\frac{\eta}{4}|\mathbf{n}|} \left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|} \right) \times \\ \times \left(\prod_{\tilde T\subseteq T} \gamma^{-h_{\tilde T}(R_{\tilde T}-1)} \right) \left(\prod_{\tilde T\in\mathbf{R}} \gamma^{2(h^{\rm ext}_{\tilde T}-h_{\tilde T})} \right) \left(\prod_{\tilde T\subseteq T} \gamma^{3h_{\tilde T}M_{\tilde T}} \right) \prod_{\tilde T\subseteq T} \gamma^{2h_{\tilde T}(M^I_{\tilde T}+M^\nu_{\tilde T})} \end{aligned} \tag{4.73}$$

and finally

$$\prod_{\tilde T\subseteq T}\gamma^{-h_{\tilde T}(R_{\tilde T}-1)}\gamma^{(h^{\rm ext}_{\tilde T}-h_{\tilde T})}\leqslant\begin{cases}C^q\gamma^{h^{ext}_T} & \text{if } \mathbf{n}_T=0\\ C^q & \text{else}\end{cases}\tag{4.74}$$

At this point, defining

$$\tilde f_{ext}:=\begin{cases}e^{-\frac{\zeta}{16}\tilde\gamma^{-h^{ext}_T}} & \text{if } \mathbf{n}_T\neq 0\\ \gamma^{h^{ext}_T} & \text{if } \mathbf{n}_T=0\end{cases}.$$

we get

$$\begin{aligned}\|\mathbf{W}_T(\mathbf{k})\|\leqslant C^q|\lambda|^q\tilde f_{ext}e^{-\frac{\eta}{4}|\mathbf{n}|}&\left(\prod_{v\in E(T)}e^{-\frac{\eta}{8}|\mathbf{n}_v|}\right)\times\\ \times\left(\prod_{\tilde T\in\mathbf{R}}\gamma^{h^{\rm ext}_{\tilde T}-h_{\tilde T}}\right)&\left(\prod_{\tilde T\subseteq T}\gamma^{3h_{\tilde T}M_{\tilde T}}\right)\prod_{\tilde T\subseteq T}\gamma^{2h_{\tilde T}(M^I_{\tilde T}+M^\nu_{\tilde T})}.\end{aligned}\tag{4.75}$$

Using the observation that for any cluster $\tilde T$, $h^{ext}_{\tilde T}<h^{ext}_{\tilde T}-h_{\tilde T}$, we get

$$\prod_{\tilde T\in\mathbf{R}}\gamma^{h^{\rm ext}_{\tilde T}-h_{\tilde T}}\prod_{\tilde T\subseteq T}\gamma^{3h_{\tilde T}M_{\tilde T}}\leqslant\prod_{\tilde T\subseteq T}\gamma^{h^{\rm ext}_{\tilde T}-h_{\tilde T}}\tag{4.76}$$

and using that for any cluster $\tilde T\subseteq T$, $\sum_{\hat T\subseteq\tilde T}\left(M^I_{\hat T}+M^\nu_{\hat T}\right)\geqslant 1$ we finally obtain, for any $\vartheta\in(0,1)$,

$$\begin{aligned}\|\mathbf{W}_T(\mathbf{k})\|\leqslant&C^q|\lambda|^q\gamma^{\vartheta h^{ext}_T}\tilde f_{ext}e^{-\frac{\eta}{4}|\mathbf{n}|}\left(\prod_{v\in E(T)}e^{-\frac{\eta}{8}|\mathbf{n}_v|}\right)\times\\ &\times\left(\prod_{\tilde T\subseteq T}\gamma^{(1-\vartheta)h_{\tilde T}(M^I_{\tilde T}+M^\nu_{\tilde T})}\right)\prod_{\tilde T\subseteq T}\gamma^{(1-\vartheta)(h^{\rm ext}_{\tilde T}-h_{\tilde T})}\end{aligned}\tag{4.77}$$

In the last line we can estimate $\prod_{\tilde T\subseteq T}\gamma^{(1-\vartheta)h_{\tilde T}(M^I_{\tilde T}+M^\nu_{\tilde T})}\leqslant 1$, and obtain the thesis. ∎

Proposition 4.6. *Assuming* (3.8) *and* (4.51)-(4.54) *valid up to scale* $h+1$*, there exists* C *independent of* $\mathbf{k}$*,* $\mathbf{n}$ *and* h *such that, for* $\mathbf{n}\neq 0$ *one has*

$$\left\|\mathcal{W}^{(h)}_{\mathbf{n},q}(\mathbf{k})\right\|\leqslant(C|\lambda|)^q e^{-\frac{\eta}{4}|\mathbf{n}|}.\tag{4.78}$$

and for $\mathbf{n}=0$ *one has, for any* $\vartheta\in(0,1)$,

$$\left\|\mathcal{W}^{(h)}_{0,q}(\mathbf{k})\right\|\leqslant(C|\lambda|)^q\gamma^{h(1+\vartheta)}.\tag{4.79}$$

Finally

$$\left\|\mathcal{V}^{(h)}_{\mathbf{n}}(\mathbf{k})\right\|\leqslant C|\lambda|e^{-\frac{\eta}{4}|\mathbf{n}|}\tag{4.80}$$

Proof. We use definition (4.48), we recall that the sum over all graphs with q vertices is equal to the sum over all $\{\mathbf{n}_v\}_{v=1,\dots,q}$ and to all lines $\{h_\ell\}_{\ell=1,\dots,q-1}$ and then we use the fact that the sum over the scales h_ℓ is equal to the sum over all dispositions of clusters on the graph and to the sum over all possible scales of the clusters. That is,

$$\mathcal{W}^{(h)}_{q,\mathbf{n}}(\mathbf{k})=\sum_{\{\mathbf{n}_v\}}\sum_{h_T>h}\sum_{\{h_\ell\}}W^{(h_T)}_T(\mathbf{k})=\sum_{\{\mathbf{n}_v\}}\sum_{\text{dispositions}}\sum_{\{h_{\tilde T}\}}W^{(h_T)}_T(\mathbf{k}).$$

Last, for every chain of clusters, the end points are at fixed scale and thus we can sum over the differences of scales,that is Since the sum over scales is done using $\sum_{h_{\tilde T}<h^{ext}_{\tilde T}}\gamma^{((1-\theta)h^{ext}_{\tilde T}-h_{\tilde T})}\leqslant C$. Using Proposition 4.5 and bounding the number of dispositions with 4^q, one gets (4.78), (4.79), and finally summing over q one gets (4.80). ∎

4.4. Choice of the Counterterms.

Proposition 4.7. *Let T be a resonant cluster on scale h_T with q vertices. If* (4.51)-(4.54) *hold for $h > h_T$, then*

$$\begin{aligned}\|W_T^{(h_T)}(0)\| \leqslant (C_{\eta,\gamma,\tau}|\lambda|)^q e^{-\frac{\eta}{32}C\tilde{\gamma}^{-h_T}} &\left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|}\right)\times \\ &\times\left(\prod_{\tilde{T}\subseteq T}\gamma^{(1-\vartheta)h_{\tilde{T}}(M^I_{\tilde{T}}+M^{\nu}_{\tilde{T}})}\right)\prod_{\tilde{T}\subseteq T}\gamma^{(1-\vartheta)(h^{ext}_{\tilde{T}}-h_{\tilde{T}})},\end{aligned}\tag{4.81}$$

$$\begin{aligned}\|\partial_{\mathbf{k}}W_T^{(h_T)}(0)\| \leqslant \gamma^{-h_T}(C_{\eta,\gamma,\tau}|\lambda|)^q e^{-\frac{\eta}{32}C\tilde{\gamma}^{-h_T}} &\left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|}\right)\times \\ &\times\left(\prod_{\tilde{T}\subseteq T}\gamma^{(1-\vartheta)h_{\tilde{T}}(M^I_{\tilde{T}}+M^{\nu}_{\tilde{T}})}\right)\prod_{\tilde{T}\subseteq T}\gamma^{(1-\vartheta)(h^{ext}_{\tilde{T}}-h_{\tilde{T}})}.\end{aligned}\tag{4.82}$$

here, $\tilde{\gamma} := \frac{\gamma^{\frac{1}{\tau}}}{2}$.

Remark 4.8. Note that, comparing (4.56) and (4.55) with (4.78) and (4.79) there is a difference in the gain: the latter pair gains over the scale of the external line h_T^{ext} while the former on the lowest scale of the cluster h_T.

Proof. First, we note that if the first or the last vertex are resonant, then the value of the graph is zero and both bounds are trivially satisfied. Indeed, suppose the first vertex is resonant, then the momentum associated to the first line in the cluster would be 0 which is outside of the support of $g^{(h)}(\mathbf{k})$ for each h. The same argument holds also for the last vertex.

Then, one has to estimate

$$\|W_T^{(h_T)}(0)\| \leqslant \left(\prod_{v\in E(T\backslash \mathbf{R}_2)}\|F_{\mathbf{n}_v}(\mathbf{k}_v)\|\right)\left(\prod_{\ell\cap\mathbf{R}_2=\varnothing}\|g^{(h_\ell)}(\mathbf{k}_\ell)\|\right)\prod_{\tilde{T}\in\mathbf{R}_2}\|\mathcal{R}W_{\tilde{T}}^{(h_{\tilde{T}})}(\mathbf{k}_{\tilde{T}})\|,\tag{4.83}$$

where we used the same notation for $\mathbf{R}_j$ as in the proof of Proposition 4.5.

Then, a part for the $\mathcal{R}$ operator in front of the graph, one has to estimate the same clusters of Proposition 4.5. Thus, adapting the argument in the proof of Proposition 4.5, one then obtains (instead of (4.73)),

$$\begin{aligned}\left\|W_T^{(h_T)}(0)\right\| \leqslant (C_{\zeta,\tau,\gamma}|\lambda|)^q &\left(\prod_{v\in\Gamma}e^{-\frac{\zeta}{4}|\mathbf{n}_v|}\right)\\ &\times\left[\prod_{\tilde{T}\subseteq T}\left(\gamma^{-h_{\tilde{T}}}e^{-\zeta C\tilde{\gamma}^{-h_{\tilde{T}}}}\right)^{M_{\tilde{T}}}\gamma^{h_{\tilde{T}}M^{(\nu)}_{\tilde{T}}}\right]\prod_{\tilde{T}\in\mathbf{R}\backslash T}\gamma^{h^{ext}_{\tilde{T}}-h_{\tilde{T}}}.\end{aligned}$$

Since, by the initial discussion of this proof, $M_T \neq 0$, then we can extract an exponential on scale h_T:

$$\prod_{\tilde{T}\subseteq T}\gamma^{-h_{\tilde{T}}}e^{-\eta C\tilde{\gamma}^{-h_{\tilde{T}}}M_{\tilde{T}}} \leqslant e^{-\frac{\eta}{2}C\tilde{\gamma}^{-h_T}}\prod_{\tilde{T}\subseteq T}\left(\gamma^{-h_{\tilde{T}}}e^{-\frac{\eta}{2}C\tilde{\gamma}^{-h_{\tilde{T}}/\tau}}\right)^{M_{\tilde{T}}}.$$

Then, the proof proceed exactly as the proof of Proposition 4.5.

For the second estimate, we note that

$$\|\partial_{\mathbf{k}}W_T^{(h_T)}(0)\| \leqslant \frac{1}{|\mathbf{k}|}\left\|\frac{\mathrm{d}}{\mathrm{d}t}W(t\mathbf{k})\Big|_{t=0}\right\|.$$

and then we repeat almost verbatim the argument we used to prove (4.81); this yields (4.82). ∎

Proposition 4.9. *If* (4.51)-(4.54) *hold for scales* $h' > h$ *and , then*

$$\|\mathbf{c}_h\| \leqslant \gamma^{2h} C_c |\lambda|^2 \tag{4.84}$$

Proof. By construction (4.32), the explicit formula for $\mathbf{c}_h$ is

$$\mathbf{c}_h \;=\; \sum_{q=1}^{+\infty} \sum_{\{\mathbf{n}_v\}} \sum_{\{h_{\tilde{T}}\}_{\tilde{T}\neq T}} \partial_{\mathbf{k}} W_T^{(h_T)}(0)$$

Noting that since we absorbed all $\mathbf{c}_{h'}$ with $h' > h$ in the propagator at higher scales, all the cluster on the RHS are on scale $h+1$. Thus, summing exactly as in the proof of Proposition 4.6 but not on scale h, we get

$$\|\mathbf{c}_h\| \;\leqslant\; \tilde{C}\gamma^{-h} e^{-C\eta\tilde{\gamma}^{-h}}\,.$$

which, in particular, implies the thesis with the use of the Cauchy inequality (4.71). ∎

We are now ready to prove the estimate on the propagator on scale h.

Proposition 4.10. *If* (4.51)-(4.54) *hold for* $h' > h$*, then there exists a constant* G_ψ *such that*

$$\|g^{(h)}(\mathbf{k})\| \;\leqslant\; G_\psi \gamma^{-h}\,, \tag{4.85}$$

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t} g^{(h)}(\mathbf{k})\right\| \;\leqslant\; G_\psi |\mathbf{k}| \gamma^{-2h}\,, \tag{4.86}$$

$$\left\|\frac{\mathrm{d}^2}{\mathrm{d}t^2} g^{(h)}(\mathbf{k})\right\| \;\leqslant\; G_\psi |\mathbf{k}|^2 \gamma^{-3h}\,, \tag{4.87}$$

Proof. Using the asymptotics at zero for $g^{(\leqslant h)}(\mathbf{k})$, we have that for $\mu(0) = 0$ one has

$$\|g^{(h)}(\mathbf{k})\| \;\leqslant\; \frac{f_h(\mathbf{k})}{|a_0^{(h)} k_0 + a_1^{(h)} k_1|} \;\leqslant\; \frac{f_h(\mathbf{k})}{|t_0 k_0 + \mathrm{i} t_1 k_1|} \frac{t_0 k_0 + \mathrm{i} t_1 k_1}{|a_0^{(h)} k_0 + a_1^{(h)} k_1|} \;\leqslant\; \frac{f_h(\mathbf{k}) C}{|t_0 k_0 + \mathrm{i} t_1 k_1|}$$

where in the last inequality we used the explicit form of the $a_0^{(h)}$ and $a_1^{(h)}$ and, for λ small enough, $\frac{t_0 k_0 + \mathrm{i} t_1 k_1}{|a_0^{(h)} k_0 + a_1^{(h)} k_1|} \leqslant C$.

A similar argument holds to get (4.86) and (4.87). ∎

To control the ν counter-terms we need a slightly different argument. Indeed, for this analysis, it is convenient to look at the ν_h's as a collection $\underline{\nu} = \{\nu_0, \nu_{-1}, \nu_{-2}, \dots\} \in \ell^\infty(-\mathbb{N})$. Making explicit the dependence on $\underline{\nu}$ in (4.31) one has

$$\nu_h \;=\; \gamma^{-2h} \widehat{\mathcal{V}}_0^{(h)}(0, \underline{\nu}) \tag{4.88}$$

where on the RHS the dependence is only on $\nu_{h'}$ with $h' > h$.

Using the cluster representation (4.49) for $\widehat{\mathcal{V}}_{\mathbf{n}}^{(h)}(\mathbf{k})$, we get naturally

$$\nu_h \;=\; \gamma^2 \nu_{h+1} + \gamma^{-2h} \sum_{\substack{T \\ h_T = h+1}} W_T^{(h_T)}(0, \underline{\nu})\,. \tag{4.89}$$

If we define

$$\beta_h(\underline{\nu}) := \gamma^{-2h} \sum_{\substack{T \\ h_T = h+1}} W_T^{(h_T)}(0, \underline{\nu}) \tag{4.90}$$

then, if $\nu_h \to 0$ as $h \to -\infty$ then, a solution to (4.88) is

$$\nu_h \;=\; -\gamma^{-2h} \sum_{j=-\infty}^{h-1} \gamma^{2j} \beta_j(\underline{\nu})\,. \tag{4.91}$$

It is convenient to name the R.H.S. of this equation as

$$\nu_h \;=\; B_h(\underline{\nu})\,. \tag{4.92}$$

The problem of finding the collection $\underline{\nu}$ is now equivalent to the problem of solving the equation $\underline{\nu} = \underline{B}(\underline{\nu})$ in $\ell^\infty(-\mathbb{N})$. This is done showing that if $\|\underline{\nu}\|_{\ell^\infty} \leqslant C|\lambda|^2$ then $\underline{B}$ is a contraction.

For this purpose, the key observation is that (4.81) implies the following estimate

$$\|\beta_h(\underline{\nu})\| \leqslant \gamma^{Nh} C_N |\lambda|^2 \tag{4.93}$$

holding for any positive N. In particular, for $N \geqslant 3$ and this gain justifies, through standard procedures, that for λ small enough, $\underline{B}$ is a contraction and also the estimate (4.54).

4.5. **Infinite Volume Limit.** The kernels $\widehat{\mathcal{V}}^{(h)}_{\mathbf{n}}(\mathbf{k})$ can be written as an expansion in renormalized graphs; we can divide them in two terms, a first one in which the graphs are computed in the $i \to \infty$ limit and a second which is the rest. The graphs contributing to the second term contain

a) A difference of propagators at finite and infinite i bounded by $1/L_i$ with $L_i := \min(L_{i,0}, L_{i,1})$ or
b) A difference between vertices bounded by $|n||\omega_i - \omega|e^{-\xi|n|}$ with $|\omega_i - \omega| \leqslant C/L_i$

The best approximants are defined starting from the continuous fraction representation $\omega = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$, with p_i/q_i defined as $\frac{p_1}{q_1} = a_0 + \frac{1}{a_1}$, $\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}$ and so on; moreover (see *e.g.* [15]),

$$\left|\omega - \frac{p_i}{q_i}\right| \leqslant \frac{C}{q_i^2} \,. \tag{4.94}$$

and $q_i = L_i$. The series of graphs for the rest is convergent at $\beta \neq \beta_c$ and, by an immediate adaptaion of the proof of Lemma 3.3, is $O(1/L_i)$ for $\beta \neq \beta_c$, and the dominant term is expressed by a convergent series, as proven above; this proves the existence of the limit.

5. Energy-energy Correlations

We consider now $\langle E_{\mathbf{x},j}; E_{\mathbf{y},j'}\rangle^T_{\alpha,i}$; as the action is quadratic, it is sufficient to compute the two point function defined as

$$S_{\omega,\omega'}(\mathbf{x},\mathbf{y}) \;=\; \langle \psi_{\omega,\mathbf{x}}\psi_{\omega',\mathbf{y}}\rangle = \int P(\mathrm{d}\psi)\psi_{\omega,\mathbf{x}}\psi_{\omega',\mathbf{y}} e^{\mathcal{V}(\psi)} \tag{5.1}$$

for $\omega, \omega' = \pm$. That is equivalently rewritten as

$$I_{\phi_{-\omega,\mathbf{x}},\phi_{-\omega',\mathbf{y}}} \;=\; \frac{\partial^2}{\partial\phi_{-\omega,\mathbf{x}}\partial\phi_{-\omega',\mathbf{y}}} \int P(\mathrm{d}\psi) e^{\mathcal{V}(\psi)+\sum_{\mathbf{x}\in\Lambda}\left(\phi_{+,\mathbf{x}}\psi_{-,\mathbf{x}}+\psi_{+,\mathbf{x}}\phi_{-,\mathbf{x}}\right)}\Big|_{\phi_{\pm,\mathbf{x}}=0} \tag{5.2}$$

The multiscale integration is perfomed as before, the only difference in the graph expansion is that there is also a vertex $\phi\psi$. We define the $\mathcal{L}$ operation exactly as for the $\psi\psi$ term; note however that the are no local terms of the form $\phi\psi$; indeed we note that the local part and its derivative is zero, as as the first line of the graph is on a certain scale h_{η_1} and thus $g^{(h_{\eta_1})}(0) = 0$.

The iterative integration on scales produces, for any $h \leqslant 0$,

$$I_{\phi_{-\omega,\mathbf{x}},\phi_{-\omega',\mathbf{y}}} \;=\; \sum_{h'=h}^{0} V^{(h')}_{\omega,\omega'}(\mathbf{x},\mathbf{y}) + I^{(h)}_{\phi_{-\omega,\mathbf{x}},\phi_{-\omega',\mathbf{y}}} \,, \tag{5.3}$$

where

$$\begin{aligned} I^{(h)}_{\phi_{-\omega,\mathbf{x}},\phi_{-\omega',\mathbf{y}}} \;&=\; \frac{\partial^2}{\partial\phi_{-\omega,\mathbf{x}}\partial\phi_{-\omega',\mathbf{y}}} \\ &\times \frac{1}{\mathcal{N}_h}\int P(\mathrm{d}\psi^{(\leqslant h)}) e^{\sum_{\mathbf{x}}\left(\phi_{+,\mathbf{x}}\psi^{(\leqslant h)}_{-,\mathbf{x}}+\psi^{(\leqslant h)}_{+,\mathbf{x}}\phi_{-,\mathbf{x}}\right)+\mathcal{V}^{(h)}(\psi^{(\leqslant h)})+W^{(h)}(\psi^{(\leqslant h)},\phi)}\Big|_{\phi=0} \,, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\mathscr{N}_h &= \int P(\mathrm{d}\psi^{(\leqslant h)})e^{\mathcal{V}^{(h)}(\psi^{(\leqslant h)})}, &(5.5)\\ W^{(h)}(\psi^{(\leqslant h)},\phi) &= \sum_{\mathbf{x},\mathbf{y}\in\Lambda}\left(\phi_{+,\mathbf{x}}K^{(h)}_{\phi,\psi}(\mathbf{x},\mathbf{y})\psi^{(\leqslant h)}_{-,\mathbf{y}}+\psi^{(\leqslant h)}_{+,\mathbf{x}}K^{(h)}_{\psi,\phi}(\mathbf{x},\mathbf{y})\phi_{-,\mathbf{y}}\right), &(5.6)\\ V^{(h)}_{\omega,\omega'}(\mathbf{x},\mathbf{y}) &= g^{(h+1)}_{\omega,\omega'}(\mathbf{x},\mathbf{y})+K^{(h)}_{\phi,\phi,\omega,\omega'}(\mathbf{x},\mathbf{y}) &(5.7)\end{aligned}$$

and

$$\begin{aligned}&\sum_{\mathbf{x},\mathbf{y}\in\Lambda}\eta_{1,\mathbf{x}}K^{(h)}_{\eta_1,\eta_2}(\mathbf{x},\mathbf{y})\eta_{2,\mathbf{y}} = \\ &\quad= \frac{1}{|\Lambda|}\sum_{\mathbf{n}}\sum_{\mathbf{k}}\widehat{K}^{(h)}_{\eta_1,\eta_2,\mathbf{n}}(\mathbf{k})\hat{\eta}_{1,-\mathbf{k}}\hat{\eta}_{2,\mathbf{k}-2\pi\Omega\mathbf{n}}\end{aligned} \tag{5.8}$$

The kernels $\widehat{K}^{(h)}_{\eta_1,\eta_2}(\mathbf{k})$ can be represented as sums of graphs of the same type of those appearing in the graph expansion of the effective potential $\mathcal{V}^{(h)}$. These new graphs differ only in the following aspects:

- if $\eta_2=\phi$, the external line is associated to the ϕ field and the graph ends with a vertex carrying a 1 factor;
- if $\eta_1=\phi$, the left external line is associated to the ϕ_+ field and the graph begins with a vertex carrying a 1 factor;
- for all graphs with $\eta_1=\eta_2=\phi$, $\min\{h_1,\dots,h_{q-1}\}=h+1$.

We have, in particular,

$$K^{(h)}_{\phi,\phi,\omega,\omega'}(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{n}\in\mathbb{Z}^2}\frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}}e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-2\mathrm{i}\pi\Omega\mathbf{n}\cdot\mathbf{y}}\widehat{K}^{(h)}_{\phi,\phi,\omega,\omega',\mathbf{n}}(\mathbf{k}) \tag{5.9}$$

Each of the $\widehat{K}^{(h)}_{\phi,\phi,\omega,\omega',\mathbf{n}}(\mathbf{k})$ can be written in terms of renormalized Feynman graphs (as the ones in Section 4.2) with the only difference that the first and the last vertex are vertices of type $\phi\psi$. Clusters containing one of those vertices are called *special* clusters. Therefore, it is convenient to define

$$\widetilde{W}^{(h_T)}_T(\mathbf{k}) := \mathscr{W}^{(h_1)}_{\mathbf{n}_1}(\mathbf{k})g^{(h_T)}(\mathbf{k}_1)\left[\prod_{j=2}^{M_T-1}\overline{W}^{(h_j)}_j(\mathbf{k}_{j-1})g^{(h_T)}(\mathbf{k}_j)\right]\mathscr{W}^{(h_{M_T})}_{\mathbf{n}_{M_T}}(\mathbf{k}_{L_T}), \tag{5.10}$$

where $\mathscr{W}$ denotes the value of special clusters which is defined analogously to (4.44), with the only difference that the vertex $\phi\psi$ has value 1. One has that

$$\widehat{K}^{(h)}_{\phi,\phi,\omega,\omega',\mathbf{n}}(\mathbf{k}) = \left[\sum_{q=3}^{+\infty}\sum_{\substack{T\\ \text{s.t.}q_T=q\\ \sum_j\mathbf{n}_j=\mathbf{n}}}\widetilde{W}^{(h+1)}_T(\mathbf{k})\right]_{\omega,\omega'}. \tag{5.11}$$

Again by the same argument as at the end of §4.2 the $\lim_{i\to\infty}$ of the 2-point function exists and in the following lemma we get a bound for the $i\to\infty$ limit .

Lemma 5.1. *The following estimate holds:*

$$|\widehat{K}^{(h)}_{\phi,\phi,\omega,\omega',\mathbf{n}}(\mathbf{k})| \leqslant C|\lambda|\gamma^{-h}e^{-\frac{\eta}{4}|\mathbf{n}|} \tag{5.12}$$

Proof. The difference with respect to the estimates in Propositions 4.5 and 4.6 is the presence of special clusters. Indeed, when the initial and the finial vertices are of $\phi\psi$ type, it is no longer always true that both external lines of a special cluster $\hat{T}$ are on scale $h^{ext}_{\hat{T}}<h_{\hat{T}}$ and thus the strategy used to prove (4.81) and (4.82) do not provide a gain.

Thus, let us consider a graph T whose minimal scale is h_T. We recall that M^n_T is the number of non resonant normal clusters (including non-resonant endpoints), R^n_T is the number of the resonant normal clusters (including resonant endpoints), M^s_T are the non resonant special clusters (i.e. the ones that contain a vertex $\phi\psi$), R^s_T are the resonant special clusters (included the end-points corresponding to the vertices $\phi\psi$). Taking into account these differences, instead of (4.68) we get

$$\begin{aligned}\left\|\widetilde{W}_T^{(h_T)}(\mathbf{k})\right\| \leqslant C^q|\lambda|^q e^{-\frac{\eta}{2}|\mathbf{n}|}\left(\prod_{v\in E(T)} e^{-\frac{\eta}{8}|\mathbf{n}_v|}\right)\left(\prod_{\tilde{T}\subseteq T}\gamma^{-h_{\tilde{T}}(M^n_{\tilde{T}}+R^n_{\tilde{T}}+M^s_{\tilde{T}}+R^s_{\tilde{T}}-1)}\right)\times \\ \times\left(\prod_{\tilde{T}\in\mathbf{R}}\gamma^{2(h^{\rm ext}_{\tilde{T}}-h_{\tilde{T}})}\right)\left(\prod_{\tilde{T}\subseteq T} e^{-\frac{\zeta}{16}M^n_{\tilde{T}}\tilde{\gamma}^{-h_{\tilde{T}}}}\right)\prod_{\tilde{T}\subseteq T}\gamma^{2h_{\tilde{T}}M^\nu_{\tilde{T}}}.\end{aligned} \tag{5.13}$$

We now use Cauchy inequality (4.71) and, from the remaining part of the proof we use the following notation: $A \lesssim B$ means $A \leqslant C^q|\lambda|^q e^{-\frac{\zeta}{2}|\mathbf{n}|}\left(\prod_{v\in E(T)} e^{-\frac{\zeta}{4}|\mathbf{n}_v|}\right)B$. We denote by $\mathbf{R}_n$ the set of resonant normal clusters, by $\mathbf{R}_s$ the set of resonant special clusters and by $\mathbf{NR}_s$ the set of non-resonant special clusters.

$$\begin{aligned}\|\widetilde{W}_{\mathbf{n}}^{(h_T)}(\mathbf{k})\| \lesssim& \left(\prod_{\tilde{T}\subseteq T}\gamma^{-h_{\tilde{T}}(M^n_{\tilde{T}}+R^n_{\tilde{T}}+M^s_{\tilde{T}}+R^s_{\tilde{T}}-1)}\right)\left(\prod_{\tilde{T}\subseteq T}\gamma^{4h_{\tilde{T}}M^n_{\tilde{T}}}\right)\left(\prod_{\tilde{T}\subseteq T}\gamma^{2h_{\tilde{T}}M^\nu_{\tilde{T}}}\right) \\ &\left(\prod_{\tilde{T}\in\mathbf{R}_n}\gamma^{2(h^{ext}_{\tilde{T}}-h_{\tilde{T}})}\right)\prod_{\tilde{T}\in\mathbf{R}_s}\gamma^{2(h^{ext}_{\tilde{T}}-h_{\tilde{T}})}.\end{aligned} \tag{5.14}$$

We denote by η_1 and η_2 the two endpoints and by h_{η_1} and h_{η_2} the scale of the lines attached to them. Writing $1=\gamma^{-h_T}\gamma^{h_T-h_{\tilde{T}_{\eta_1}}}\gamma^{h_{\tilde{T}_{\eta_1}}}$ and $1=\gamma^{-h_T}\gamma^{h_T-h_{\tilde{T}_{\eta_2}}}\gamma^{h_{\tilde{T}_{\eta_2}}}$ we get

$$\begin{aligned}\|\widetilde{W}_{\mathbf{n}}^{(h_T)}(\mathbf{k})\| \lesssim& \gamma^{-2h_T}\left(\prod_{\tilde{T}\subseteq T}\gamma^{-h_{\tilde{T}}(M^n_{\tilde{T}}+R^n_{\tilde{T}}+M^s_{\tilde{T}}+R^s_{\tilde{T}}-1)}\right)\left(\prod_{\tilde{T}\subseteq T}\gamma^{4h_{\tilde{T}}M^n_{\tilde{T}}}\right)\left(\prod_{\tilde{T}_\nu\subseteq T}\gamma^{2h_{\tilde{T}}}\right)\times \\ &\times\left(\prod_{\tilde{T}\,|\,\eta_j\in\tilde{T}_0}\gamma^{h_{\tilde{T}}}\right)\left(\prod_{\tilde{T}\in\mathbf{R}_n}\gamma^{2(h^{ext}_{\tilde{T}}-h_{\tilde{T}})}\right)\left(\prod_{\tilde{T}\in\mathbf{R}_s}\gamma^{2(h^{ext}_{\tilde{T}}-h_{\tilde{T}})}\right)\prod_{\tilde{T}\in\mathbf{NR}_s}\gamma^{h^{ext}_{\tilde{T}}-h_{\tilde{T}}}.\end{aligned} \tag{5.15}$$

At this point, proceeding as in the proof of Proposition 4.5, one gets

$$\|\widetilde{W}_{\mathbf{n}}^{(h_T)}(\mathbf{k})\| \lesssim \gamma^{-h_T}\left(\prod_{\tilde{T}\subseteq T}\gamma^{3h_{\tilde{T}}M^n_{\tilde{T}}}\right)\left(\prod_{\tilde{T}_\nu\subseteq T}\gamma^{h_{\tilde{T}}}\right)\left(\prod_{\tilde{T}\in\mathbf{R}_n}\gamma^{h^{ext}_{\tilde{T}}-h_{\tilde{T}}}\right)\left(\prod_{\tilde{T}\in\mathbf{R}_s}\gamma^{h^{ext}_{\tilde{T}}-h_{\tilde{T}}}\right). \tag{5.16}$$

We have now to perform the sum over all $h_{\tilde{T}}$; we can distinguish the following cases

(1) If $\tilde{T}$ is resonant, both special and normal, we can use $\sum_{h_{\tilde{T}}<h^{ext}_{\tilde{T}}}\gamma^{h^{ext}_{\tilde{T}}-h_{\tilde{T}}} \leqslant C$
(2) if $\tilde{T}$ is non-resonant and normal then as $M^n_{\tilde{T}} \geqslant 1$ (otherwise it would be a resonance) hence $\sum_{h_{\tilde{T}}}\gamma^{h_{\tilde{T}}} \leqslant C$;
(3) if $\tilde{T}$ is non-resonant and special, then $M^s_{\tilde{T}}+R^s_{\tilde{T}}=1$; if $R^s_{\tilde{T}}=1$ then $M^n_{\tilde{T}} \geqslant 1$ so that we use $\sum_{h_{\tilde{T}}}\gamma^{h_{\tilde{T}}} \leqslant C$; if $M^s_{\tilde{T}}=1$ then or $M^n_{\tilde{T}} \geqslant 1$ or $M^n_{\tilde{T}}=0$; in this last case there is a resonant normal cluster with a line external to $\tilde{T}$ and the other internal with the same momentum, so that $h^{ext}_{\tilde{T}}=h_{\tilde{T}}-1$ and $\sum_{h_T} \leqslant 2$

Therefore we can can sum over all the scales and $\mathbf{n}, q$ and this yields (5.12). ■

To get an estimate of the kernel in the position variables one has

$$K^{(h)}_{\phi,\phi}(\mathbf{x},\mathbf{y}) = \frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}}\sum_{\mathbf{n}\in\mathbb{Z}^2} e^{-\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})-2\pi\mathrm{i}\Omega\mathbf{n}\cdot\mathbf{y}}\widehat{K}^{(h)}_{\phi,\phi,\mathbf{n}}(\mathbf{k}) \tag{5.17}$$

We note that the $\mathbf{k}$ on which the sum gives a non-zero factor are actually of order $\gamma^{2h}|\Lambda|$. Indeed, in each of the graph that gives a non-zero contribution to $\widehat{K}^{(h)}_{\phi,\phi,\omega,\omega',\mathbf{n}}$, one has that the sum runs over $\mathbf{k}$ such that, on a certain line, the value $\mathbf{k}_\ell \in \mathrm{supp} f_h$. Thus, we can estimate $\frac{1}{|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}} \leqslant C\gamma^{2h}$ and we get

$$|K^{(h)}_{\phi,\phi,\omega,\omega'}(\mathbf{x},\mathbf{y})| \leqslant C|\lambda|\gamma^h\,. \tag{5.18}$$

Repeating the analysis of [6], one has

$$|\mathbf{x}-\mathbf{y}|^N|K^{(h)}_{\phi,\phi,\omega,\omega'}(\mathbf{x},\mathbf{y})| \;\leqslant\; C_N|\lambda|\gamma^{-hN}|\lambda|\gamma^h \tag{5.19}$$

which in turns imply

$$|K^{(h)}_{\phi,\phi,\omega,\omega'}(\mathbf{x},\mathbf{y})| \;\leqslant\; |\lambda|\frac{\gamma^h C_N}{1+\gamma^{hN}|\mathbf{x}-\mathbf{y}|^N}\,. \tag{5.20}$$

Back to (5.3), we get at $\beta=\beta_c$

$$S_{\omega,\omega'}(\mathbf{x},\mathbf{y}) \;:=\; I_{\phi_{-\omega,\mathbf{x}},\phi_{-\omega',\mathbf{y}}} \;=\; \sum_{h=-\infty}^{1} g^{(h)}_{\omega,\omega'}(\mathbf{x},\mathbf{y}) + \sum_{h=-\infty}^{1} K^{(h)}_{\phi,\phi,\omega,\omega'}(\mathbf{x},\mathbf{y}) \tag{5.21}$$

and

$$|K_{\phi,\phi,\omega,\omega'}(\mathbf{x},\mathbf{y})| \leqslant \frac{\tilde C_N|\lambda|}{1+|\mathbf{x}-\mathbf{y}|}\,. \tag{5.22}$$

A similar analysis can be repeated at $\beta\neq\beta_c$ providing the faster than any power decay and an argument essentially identical to the one presented inf §s4.5 shows that the limit $i\to\infty$ exists.

We have finally to compute (2.5). First we note that

$$\langle E_{\mathbf{x},j};E_{\mathbf{y},j'}\rangle^T_{\boldsymbol{\alpha},i} = \sum_{\omega_1,\omega_2=\pm}[S_{\omega_1,\omega_2,i}(\mathbf{x},\mathbf{y})S_{-\omega_2,-\omega_1,i}(\mathbf{y},\mathbf{x})+S_{\omega_1,-\omega_2,i}(\mathbf{x},\mathbf{y},i)S_{-\omega_2,\omega_1,i}(\mathbf{y},\mathbf{x})]+R_i(\mathbf{x},\mathbf{y}) \tag{5.23}$$

where $R_i(\mathbf{x},\mathbf{y})$ are faster decaying terms. Moreover

$$\sum_{\boldsymbol{\alpha}}\frac{\tau_{\boldsymbol{\alpha}}Z_{\boldsymbol{\alpha}}}{2Z}\langle E_{\mathbf{x},j};E_{\mathbf{y},j'}\rangle^T_{\boldsymbol{\alpha},i} = \langle E_{\mathbf{x},j};E_{\mathbf{y},j'}\rangle^T_{--,i} + \tilde R_i(\mathbf{x},\mathbf{y})\;, \tag{5.24}$$

where we use that Z is non vanishing; indeed

$$Z = \hat Z_{--}Z^0 + \hat Z_{--}\sum_{\boldsymbol{\alpha}}\tau_{\boldsymbol{\alpha}}Z^0_{\boldsymbol{\alpha}}\left(\frac{\hat Z_{\boldsymbol{\alpha}}}{\hat Z_{--}}-1\right) \tag{5.25}$$

and by Appendix G of [30] (ω_i does not depend on $\boldsymbol{\alpha}$) $|\log \hat Z_\alpha/\hat Z_{--}| \leqslant L_{0,i}L_{1,i}\lambda e^{-|\beta-\beta_c|\bar L_i}$; as Z^0 is non vanishing then also Z is non vanishing. Finally $R(\mathbf{x},\mathbf{y})$ is vanishing by the same analysis as in Section 4.5 as $i\to\infty$ and this completes the proof of the main Theorem.

Appendix A. Derivation of (2.7)

From (2.4) one changes Grassmann variables into critical eigenmodes, called χ,ψ (see e.g. Section II.B in [20]). This transforms the action as $S=S^{(\chi)}+S^{(\psi)}+Q^{(\psi,\chi)}$ with

$$\begin{aligned} S^{(\chi)} \;:=\; &-\frac{1}{4\pi}\sum_{\mathbf{x}\in\Lambda} t^{(1)}_{\mathbf{x}}\begin{pmatrix}\chi_{+,\mathbf{x}}\\ \chi_{-,\mathbf{x}}\end{pmatrix}\cdot\begin{pmatrix}-1 & +\mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix}\begin{pmatrix}\chi_{+,\mathbf{x}+\mathbf{e}_1}\\ \chi_{-,\mathbf{x}+\mathbf{e}_1}\end{pmatrix}+\\ &-\frac{1}{4\pi}\sum_{\mathbf{x}\in\Lambda} t^{(0)}_{\mathbf{x}}\begin{pmatrix}\chi_{+,\mathbf{x}}\\ \chi_{-,\mathbf{x}}\end{pmatrix}\cdot\begin{pmatrix}-\mathrm{i} & +\mathrm{i}\\ -\mathrm{i} & +\mathrm{i}\end{pmatrix}\begin{pmatrix}\chi_{+,\mathbf{x}+\mathbf{e}_0}\\ \chi_{-,\mathbf{x}+\mathbf{e}_0}\end{pmatrix}\\ &-\frac{1}{4\pi}\sum_{\mathbf{x}\in\Lambda}\mathrm{i}\big[t^{(0)}_{\mathbf{x}}+t^{(1)}_{\mathbf{x}}+2(\sqrt{2}-1)\big]\big(\chi_{+,\mathbf{x}}\chi_{-,\mathbf{x}}-\chi_{-,\mathbf{x}}\chi_{+,\mathbf{x}}\big)\,. \end{aligned} \tag{A.1}$$

$$
\begin{aligned}
S^{(\psi)} \;:=\; & -\frac{1}{4\pi}\sum_{\mathbf{x}\in\Lambda} t^{(1)}_{\mathbf{x}} \begin{pmatrix}\psi_{+,\mathbf{x}}\\ \psi_{-,\mathbf{x}}\end{pmatrix}\cdot\begin{pmatrix}-1 & +\mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix}\begin{pmatrix}\psi_{+,\mathbf{x}+\mathbf{e}_1}\\ \psi_{-,\mathbf{x}+\mathbf{e}_1}\end{pmatrix} + \\
& -\frac{1}{4\pi}\sum_{\mathbf{x}\in\Lambda} t^{(0)}_{\mathbf{x}} \begin{pmatrix}\psi_{+,\mathbf{x}}\\ \psi_{-,\mathbf{x}}\end{pmatrix}\cdot\begin{pmatrix}-\mathrm{i} & +\mathrm{i}\\ -\mathrm{i} & +\mathrm{i}\end{pmatrix}\begin{pmatrix}\psi_{+,\mathbf{x}+\mathbf{e}_0}\\ \psi_{-,\mathbf{x}+\mathbf{e}_0}\end{pmatrix} \\
& -\frac{1}{4\pi}\sum_{\mathbf{x}\in\Lambda} \mathrm{i}\big[t^{(0)}_{\mathbf{x}} + t^{(1)}_{\mathbf{x}} - 2(\sqrt{2}-1)\big]\big(\psi_{+,\mathbf{x}}\psi_{-,\mathbf{x}} - \psi_{-,\mathbf{x}}\psi_{+,\mathbf{x}}\big)\,.
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
Q^{(\psi,\chi)} \;:=\; & \frac{1}{4\pi}\sum_{\mathbf{x}\in\Lambda} t^{(1)}_{\mathbf{x}} \begin{pmatrix}\psi_{+,\mathbf{x}}\\ \psi_{-,\mathbf{x}}\end{pmatrix}\cdot\begin{pmatrix}-1 & \mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix}\begin{pmatrix}\chi_{+,\mathbf{x}+\mathbf{e}_1}\\ \chi_{-,\mathbf{x}+\mathbf{e}_1}\end{pmatrix} + \\
& \frac{1}{4\pi}\sum_{\mathbf{x}\in\Lambda} t^{(0)}_{\mathbf{x}} \begin{pmatrix}\psi_{+,\mathbf{x}}\\ \psi_{-,\mathbf{x}}\end{pmatrix}\cdot\begin{pmatrix}\mathrm{i} & -\mathrm{i}\\ \mathrm{i} & -\mathrm{i}\end{pmatrix}\begin{pmatrix}\chi_{+,\mathbf{x}+\mathbf{e}_0}\\ \chi_{-,\mathbf{x}+\mathbf{e}_0}\end{pmatrix} + (\psi\leftrightarrow\chi)\,.
\end{aligned}
\tag{A.3}
$$

Splitting $t^{(j)}_{\mathbf{x}} = t^{(j)} + V^{(j)}_{\mathbf{x}}$, one has

$$S \;=\; S_{\text{free}} + S_{\text{int}}\,, \tag{A.4}$$

with

$$S_{\#} \;=\; S^{(\chi)}_{\#} + S^{(\psi)}_{\#} + Q^{(\psi,\chi)}_{\#} \tag{A.5}$$

and $\#$ = int, free.

$$S^{(\eta)}_{\text{free}} \;=\; -\frac{1}{4\pi|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}} \widehat{\boldsymbol{\eta}}_{-\mathbf{k}}\cdot C_\eta(\mathbf{k})\widehat{\boldsymbol{\eta}}_{\mathbf{k}}\,, \tag{A.6}$$

In the derivation of the Grassmann representation, the only point we would like to highlight is the writing of the interaction part. The computation for one of the pieces of the intarection part of the action goes as follows:

$$
\begin{aligned}
&\sum_{\mathbf{x}\in\Lambda} V^{(1)}_{\mathbf{x}}\begin{pmatrix}\widehat{\chi}_{+,\mathbf{x}}\\ \widehat{\chi}_{-,\mathbf{x}}\end{pmatrix}\cdot\begin{pmatrix}-1 & \mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix}\begin{pmatrix}\widehat{\chi}_{+,\mathbf{x}+\mathbf{e}_1}\\ \widehat{\chi}_{-,\mathbf{x}+\mathbf{e}_1}\end{pmatrix}\\
&\quad= \frac{1}{2|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}} \widehat{V}^{(1)}_{\mathbf{n}}\begin{pmatrix}\widehat{\chi}_{+,-\mathbf{k}}\\ \widehat{\chi}_{-,-\mathbf{k}}\end{pmatrix}\cdot e^{\mathrm{i}(k_1-2\pi\omega_1 n_1)}\begin{pmatrix}-1 & \mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix}\begin{pmatrix}\widehat{\chi}_{+,\mathbf{k}-2\pi\Omega\mathbf{n}}\\ \widehat{\chi}_{-,\mathbf{k}-2\pi\Omega\mathbf{n}}\end{pmatrix}\\
&\qquad+\frac{1}{2|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}} \widehat{V}^{(1)}_{\mathbf{n}}\begin{pmatrix}\widehat{\chi}_{+,\mathbf{k}-2\pi\Omega\mathbf{n}}\\ \widehat{\chi}_{-,\mathbf{k}-2\pi\Omega\mathbf{n}}\end{pmatrix}\cdot e^{-\mathrm{i}k_1}\begin{pmatrix}-1 & \mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix}\begin{pmatrix}\widehat{\chi}_{+,-\mathbf{k}}\\ \widehat{\chi}_{-,-\mathbf{k}}\end{pmatrix}\\
&\quad= \frac{1}{2|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}} \widehat{V}^{(1)}_{\mathbf{n}}\begin{pmatrix}\widehat{\chi}_{+,-\mathbf{k}}\\ \widehat{\chi}_{-,-\mathbf{k}}\end{pmatrix}\cdot\left[e^{\mathrm{i}(k_1-2\pi\omega_1 n_1)}\begin{pmatrix}-1 & \mathrm{i}\\ -\mathrm{i} & -1\end{pmatrix} - e^{-\mathrm{i}k_1}\begin{pmatrix}-1 & -\mathrm{i}\\ \mathrm{i} & -1\end{pmatrix}\right]\begin{pmatrix}\widehat{\chi}_{+,\mathbf{k}-2\pi\Omega\mathbf{n}}\\ \widehat{\chi}_{-,\mathbf{k}-2\pi\Omega\mathbf{n}}\end{pmatrix}\\
&\quad= \frac{1}{|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}} \widehat{V}^{(1)}_{\mathbf{n}}e^{-\pi\mathrm{i}\omega_1 n_1}\begin{pmatrix}\widehat{\chi}_{+,-\mathbf{k}}\\ \widehat{\chi}_{-,-\mathbf{k}}\end{pmatrix}\cdot\begin{pmatrix}-\mathrm{i}\sin\big(k_1-\pi\omega_1 n_1\big) & \mathrm{i}\cos\big(k_1-\pi\omega_1 n_1\big)\\ -\mathrm{i}\cos\big(k_1-\pi\omega_1 n_1\big) & -\mathrm{i}\sin\big(k_1-\pi\omega_1 n_1\big)\end{pmatrix}\begin{pmatrix}\widehat{\chi}_{+,\mathbf{k}-2\pi\Omega\mathbf{n}}\\ \widehat{\chi}_{-,\mathbf{k}-2\pi\Omega\mathbf{n}}\end{pmatrix}.
\end{aligned}
$$

To write compactly the interaction part it is convenient to define

$$\widehat{W}^{(j)}_{\mathbf{n}} \;:=\; \widehat{V}^{(j)}_{\mathbf{n}}e^{-\pi\mathrm{i}\omega_j n_j}e^{2\pi\mathrm{i}\mathbf{n}\cdot\boldsymbol{\vartheta}_j} \qquad \text{for } j=0,1\,. \tag{A.7}$$

At this point it is important to note that the dependence on the angles $\boldsymbol{\vartheta}_j$ enters only through a phase in the Fourier representation of the interaction. It is therefore clear that their value is not important in the study of the (absolute) convergence of the perturbative series and thus we drop the dependence of $\widehat{W}^{(j)}_{\mathbf{n}}$ on $\boldsymbol{\vartheta}_j$. Let us notice that, as a consequence of reality of ϕ we have $\widehat{W}^{(j)}_{-\mathbf{n}} = (\widehat{W}^{(j)}_{\mathbf{n}})^*$. For the interacting part we have a structure that is similar to the unperturbed part:

$$S^{(\eta)}_{\text{int}} \;=\; -\frac{1}{4\pi|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}}\sum_{j=0,1}\widehat{W}^{(j)}_{\mathbf{n}}\widehat{\boldsymbol{\eta}}_{-\mathbf{k}}\cdot P^{(j)}(\mathbf{k},\mathbf{n})\widehat{\boldsymbol{\eta}}_{\mathbf{k}-2\pi\Omega\mathbf{n}}\,, \tag{A.8}$$

with Q defined above. Finally, we introduce new Grassmann variables $\widehat{\boldsymbol{\xi}}_{\mathbf{k}}$

$$\widehat{\boldsymbol{\chi}}_{\mathbf{k}} = \widehat{\boldsymbol{\xi}}_{\mathbf{k}} + g_\chi(\mathbf{k})Q(\mathbf{k})\widehat{\boldsymbol{\psi}}_{\mathbf{k}} \tag{A.9}$$

and with a straightforward computation we get

$$S_{\text{free}} \;=\; S^{(\xi)}_{\text{free}} + S^{(\psi)}_{\text{free}}\,, \qquad S_{\text{int}} \;=\; S^{(\xi)}_{\text{int}} + S^{(\psi)}_{\text{int}} + Q^{(\psi,\xi)}_{\text{int}}\,. \tag{A.10}$$

After this change of variables, we obtain $S^{(\xi)}_{\text{free}} = S^{(\chi)}_{\text{free}}\big|_{\chi=\xi}$ and $S^{(\psi)}_{\text{free}} = -\frac{1}{4\pi|\Lambda|}\sum_{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}}\widehat{\boldsymbol{\psi}}_{-\mathbf{k}}\cdot(C_\psi(\mathbf{k}) - Q(\mathbf{k})g_\chi(\mathbf{k})Q(\mathbf{k}))\,\widehat{\boldsymbol{\psi}}_{\mathbf{k}}$ which implies (2.10). For the integration of the ψ variables it is important to notice that

$$Q(\mathbf{k})g_\chi(\mathbf{k})Q(\mathbf{k}) \;=\; O(|\mathbf{k}|^2)\,, \qquad \text{as } \beta \to \beta_c\,. \tag{A.11}$$

Concerning the interacting part of the action, with (A.9) we have

$$S^{(\xi)}_{\text{int}} \;=\; S^{(\chi)}_{\text{int}}\big|_{\chi=\xi}\,, \tag{A.12}$$

$$S^{(\psi)}_{\text{int}} \;=\; -\frac{1}{4\pi|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}}\widehat{\boldsymbol{\psi}}_{-\mathbf{k}}\cdot\left(\sum_{j=0,1}\widehat{W}^{(j)}_{\mathbf{n}}P^{(j)}_\psi(\mathbf{k},\mathbf{n})\right)\widehat{\boldsymbol{\psi}}_{\mathbf{k}-2\pi\Omega\mathbf{n}}\,, \tag{A.13}$$

with $P^{(j)}_\psi(\mathbf{k},\mathbf{n})$ as in (2.15) and (2.14), and

$$Q^{(\psi,\xi)}_{\text{int}} \;=\; -\frac{1}{4\pi|\Lambda|}\sum_{\substack{\mathbf{k}\in\mathcal{D}_{\boldsymbol{\alpha}}\\ \mathbf{n}\in\mathbb{Z}^2}}\widehat{\boldsymbol{\psi}}_{-\mathbf{k}}\cdot\left(\sum_{j=0,1}\widehat{W}^{(j)}_{\mathbf{n}}Q^{(j)}_\psi(\mathbf{k},\mathbf{n})\right)\widehat{\boldsymbol{\psi}}_{\mathbf{k}-2\pi\Omega\mathbf{n}} \tag{A.14}$$

with $Q^{(j)}_\psi(\mathbf{k},\mathbf{n})$ as in (2.16). As a final remark, to avoid heavier notation, we removed the hats on Grassmann variables in the mail part of the text.

Appendix B. Lemma B.1

Lemma B.1. *The matrix $G_{\{\mathbf{n}_j\}}(\mathbf{k})$ has the form* (3.10) *with $a_{\mathbf{n}}(\mathbf{k}) = -a_{-\mathbf{n}}(-\mathbf{k}) \in \mathbb{C}$ and $b_{\mathbf{n}}(\mathbf{k}) = b_{-\mathbf{n}}(-\mathbf{k}) \in \mathbb{C}$.*

Proof. The proof is done by induction in the number of vertices of the graph. We prove that a graph with 2 vertices (for which the matrix $G_{\{\mathbf{n}_j\}}(\mathbf{k})$ is the product of three matrices).

For the graph with three vertices we have to consider the product of three matrices $A_{\mathbf{n}_A}(\mathbf{k}), g_\chi(\mathbf{k} - 2\pi\Omega\mathbf{n}_A), B_{\mathbf{n}_B}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A)$. Here A and B can be either $P^{(j)}$ or $Q^{(j)}$. The following explicit computation yields

$$\begin{aligned}A_{\mathbf{n}_A}(\mathbf{k})g_\chi(\mathbf{k} - 2\pi\Omega\mathbf{n}_A) &= \begin{pmatrix} a^{(A)}_{\mathbf{n}_A}(\mathbf{k}) & b^{(A)}_{\mathbf{n}_A}(\mathbf{k}) \\ (b^{(A)}_{\mathbf{n}_A}(\mathbf{k}))^* & -(a^{(A)}_{\mathbf{n}_A})^* \end{pmatrix}\begin{pmatrix} a^{(\xi)}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A) & b^{(\xi)}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A) \\ (b^{(\xi)}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A))^* & -(a^{(\xi)}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A))^* \end{pmatrix} \\ &= \begin{pmatrix} \beta(\mathbf{k},\mathbf{n}_A) & \alpha(\mathbf{k},\mathbf{n}_A) \\ -\alpha^*(\mathbf{k},\mathbf{n}_A) & \beta^*(\mathbf{k},\mathbf{n}_A) \end{pmatrix}\end{aligned}$$

Where, explicitly,

$$\begin{aligned}\alpha(\mathbf{k},\mathbf{n}_A) &:= a^{(A)}_{\mathbf{n}_A}(\mathbf{k})b^{(\xi)}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A) - b^{(A)}_{\mathbf{n}_A}(\mathbf{k})(a^{(\xi)}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A))^*\,, \\ \beta(\mathbf{k},\mathbf{n}_B) &:= a^{(A)}_{\mathbf{n}_A}(\mathbf{k})a^{(\xi)}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A) + b^{(A)}_{\mathbf{n}_A}(\mathbf{k})(b^{(\xi)}(\mathbf{k} - 2\pi\Omega\mathbf{n}_A))^*\,.\end{aligned}$$

It follows now from the symmetry properties of a and b that $\alpha(\mathbf{k},\mathbf{n}_A) = -\alpha(-\mathbf{k},-\mathbf{n}_A)$ and $\beta(\mathbf{k},\mathbf{n}_A) = \beta(-\mathbf{k},\mathbf{n}_A)$.

Computing the value of the graph, one has

$$
\begin{aligned}
&A_{\mathbf{n}_A}(\mathbf{k})g_\chi(\mathbf{k}-2\pi\Omega\mathbf{n}_A)B_{\mathbf{n}_B}(\mathbf{k}-2\pi\Omega\mathbf{n}_A) =\\
&\quad= \begin{pmatrix} \beta(\mathbf{k},\mathbf{n}_A) & \alpha(\mathbf{k},\mathbf{n}_A) \\ -\alpha^*(\mathbf{k},\mathbf{n}_A) & \beta^*(\mathbf{k},\mathbf{n}_A)\end{pmatrix} \begin{pmatrix} a^{(B)}_{\mathbf{n}_A}(\mathbf{k}-2\pi\Omega\mathbf{n}_A) & b^{(B)}_{\mathbf{n}_B}(\mathbf{k}-2\pi\Omega\mathbf{n}_A) \\ (b^{(B)}_{\mathbf{n}_B}(\mathbf{k}-2\pi\Omega\mathbf{n}_A))^* & -(a^{(B)}_{\mathbf{n}_B}(\mathbf{k}-2\pi\Omega\mathbf{n}_A))^*\end{pmatrix}\\
&\quad= \begin{pmatrix} a_{\mathbf{n}}(\mathbf{k}) & b_{\mathbf{n}}(\mathbf{k}) \\ b^*_{\mathbf{n}}(\mathbf{k}) & -a^*_{\mathbf{n}}(\mathbf{k})\end{pmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
a_{\mathbf{n}}(\mathbf{k}) &= \beta(\mathbf{k},\mathbf{n}_A)a^{(B)}_{\mathbf{n}_A}(\mathbf{k}-2\pi\Omega\mathbf{n}_A) + \alpha(\mathbf{k},\mathbf{n}_A)(b^{(B)}_{\mathbf{n}_B}(\mathbf{k}-2\pi\Omega\mathbf{n}_A))^*\,,\\
b_{\mathbf{n}}(\mathbf{k}) &= \beta(\mathbf{k},\mathbf{n}_A)b^{(B)}_{\mathbf{n}_B}(\mathbf{k}-2\pi\Omega\mathbf{n}_A) - \alpha(\mathbf{k},\mathbf{n}_A)(a^{(B)}_{\mathbf{n}_B}(\mathbf{k}-2\pi\Omega\mathbf{n}_A))^*\,.
\end{aligned}
$$

From the symmetry properties of a, b, α and β it follows that under the exchange $\{\mathbf{n}_A, \mathbf{n}_B, \mathbf{k}\} \mapsto \{-\mathbf{n}_A, -\mathbf{n}_B, \mathbf{k}\}$ one has that $a_{\mathbf{n}}(\mathbf{k}) = -a_{-\mathbf{n}}(-\mathbf{k})$ and $b_{\mathbf{n}}(\mathbf{k}) = b_{-\mathbf{n}}(-\mathbf{k})$. This completes the proof for the graph with two vertices.

In case the graph has q vertices, by inductive hypotesis we assume that the product of the matrices associated to the sub-graph of the first $q-1$ points has the structure (3.10). Then the exact same computation above prove the thesis. ∎

Appendix C. Symmetries of $\widehat{\mathcal{V}}_0(\mathbf{k})$: proof of (3.13)

Lemma C.1. *For $\widehat{\mathcal{V}}_0(\mathbf{k})$ we have the following matricial structure*

$$
\widehat{\mathcal{V}}_0(\mathbf{k}) \;=\; \begin{pmatrix} a(\mathbf{k}) & \mathrm{i}b(\mathbf{k}) \\ -\mathrm{i}b(\mathbf{k}) & -a^*(\mathbf{k})\end{pmatrix} \tag{C.1}
$$

with $a(\mathbf{k}) = -a(-\mathbf{k}) \in \mathbb{C}$ and $b(\mathbf{k}) = b(-\mathbf{k}) \in \mathbb{R}$, with $\widehat{\mathcal{V}}_0(\mathbf{k}) = O(\lambda^2)$ as $\lambda \downarrow 0$.

Expanded in taylor series around $\mathbf{k} = 0$, one has

$$
\widehat{\mathcal{V}}_0(\mathbf{k}) = \begin{pmatrix} a_1k_1 + a_0k_0 & \mathrm{i}\mu \\ -\mathrm{i}\mu & -a_1^*k_1 - a_0^*k_0\end{pmatrix} + O(|\mathbf{k}|^2)\,. \tag{C.2}
$$

Proof. The proof consists in proving the structure (C.1) for each of the summands in (3.12). We start by computing

$$
\mathrm{Val}\,\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k}) - \mathrm{Val}\,\Gamma_{\{-\mathbf{n}\}}(-\mathbf{k})^T \;=\; \Big(\prod_{v\in\Gamma}\widehat{W}^{(j_v)}_{\mathbf{n}_v}\Big)G_{\{\mathbf{n}_j\}}(\mathbf{k}) - \Big(\prod_{v\in\Gamma}\widehat{W}^{(j_v)}_{-\mathbf{n}_v}\Big)\Big(G_{\{-\mathbf{n}_j\}}(-\mathbf{k})\Big)^T \tag{C.3}
$$

It is convenient to call $f_{\mathbf{n}} := \prod_{v\in\Gamma}\widehat{W}^{(j_v)}_{\mathbf{n}_v}$ and to use (3.10) for $G_{\{\mathbf{n}_j\}}$. Then, using that $f_{-\mathbf{n}} = f^*_{\mathbf{n}}$, we have

$$
\begin{aligned}
&\mathrm{Val}\,\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k}) - \Big(\mathrm{Val}\,\Gamma_{\{-\mathbf{n}_j\}}(-\mathbf{k})\Big)^T\\
&\quad= f_{\mathbf{n}}\begin{pmatrix} a_{\mathbf{n}}(\mathbf{k}) & b_{\mathbf{n}}(\mathbf{k}) \\ b^*_{\mathbf{n}}(\mathbf{k}) & -a^*_{\mathbf{n}}(\mathbf{k})\end{pmatrix} - f^*_{\mathbf{n}}\begin{pmatrix} a_{-\mathbf{n}}(-\mathbf{k}) & b^*_{-\mathbf{n}}(-\mathbf{k}) \\ b_{-\mathbf{n}}(-\mathbf{k}) & -a^*_{-\mathbf{n}}(-\mathbf{k})\end{pmatrix}\\
&\quad\overset{\text{Lemma B.1}}{=} f_{\mathbf{n}}\begin{pmatrix} a_{\mathbf{n}}(\mathbf{k}) & b_{\mathbf{n}}(\mathbf{k}) \\ b^*_{\mathbf{n}}(\mathbf{k}) & -a^*_{\mathbf{n}}(\mathbf{k})\end{pmatrix} - f^*_{\mathbf{n}}\begin{pmatrix} -a_{\mathbf{n}}(\mathbf{k}) & b^*_{\mathbf{n}}(\mathbf{k}) \\ b_{\mathbf{n}}(\mathbf{k}) & a^*_{\mathbf{n}}(\mathbf{k})\end{pmatrix}\\
&\quad= \begin{pmatrix} (f_{\mathbf{n}} + f^*_{\mathbf{n}})a_{\mathbf{n}}(\mathbf{k}) & f_{\mathbf{n}}b_{\mathbf{n}}(\mathbf{k}) - f^*_{\mathbf{n}}b^*_{\mathbf{n}}(\mathbf{k}) \\ f_{\mathbf{n}}b^*_{\mathbf{n}}(\mathbf{k}) - f^*_{\mathbf{n}}b_{\mathbf{n}}(\mathbf{k}) & -(f_{\mathbf{n}}(\mathbf{k}) + f^*_{\mathbf{n}})a^*_{\mathbf{n}}(\mathbf{k})\end{pmatrix}\\
&\quad= \begin{pmatrix} \alpha_{\mathbf{n}}(\mathbf{k}) & \beta_{\mathbf{n}}(\mathbf{k}) \\ \beta^*_{\mathbf{n}}(\mathbf{k}) & -\alpha^*_{\mathbf{n}}(\mathbf{k})\end{pmatrix}
\end{aligned} \tag{C.4}
$$

with

$$
\begin{aligned}
\alpha_{\mathbf{n}}(\mathbf{k}) &= (f_{\mathbf{n}} + f^*_{\mathbf{n}})a_{\mathbf{n}}(\mathbf{k})\,,\\
\beta_{\mathbf{n}}(\mathbf{k}) &= f_{\mathbf{n}}b_{\mathbf{n}}(\mathbf{k}) - f^*_{\mathbf{n}}b^*_{\mathbf{n}}(\mathbf{k})\,.
\end{aligned} \tag{C.5}
$$

With those explicit expressions at hand, it is clear that $\beta_{\mathbf{n}}(\mathbf{k}) \in \mathrm{i}\mathbb{R}$ and $\alpha_{\mathbf{n}}(\mathbf{k}) \in \mathbb{C}$, in general.

To see explicitly the symmetries under the exchange $\mathbf{k} \mapsto -\mathbf{k}$ and $\{\mathbf{n}_j\} \mapsto \{-\mathbf{n}_j\}$ we compute the summand in (3.12)

$$\begin{aligned}
&\mathrm{Val}\,\Gamma_{\{\mathbf{n}_j\}}(\mathbf{k}) - \mathrm{Val}\,\Gamma_{\{-\mathbf{n}_j\}}(-\mathbf{k})^T + \mathrm{Val}\,\Gamma_{\{-\mathbf{n}_j\}}(\mathbf{k}) - \mathrm{Val}\,\Gamma_{\{-\mathbf{n}_j\}}(-\mathbf{k})^T \\
&= \begin{pmatrix} \alpha_{\mathbf{n}}(\mathbf{k}) & \beta_{\mathbf{n}}(\mathbf{k}) \\ \beta^*_{\mathbf{n}}(\mathbf{k}) & -\alpha^*_{\mathbf{n}}(\mathbf{k}) \end{pmatrix} + \begin{pmatrix} \alpha_{-\mathbf{n}}(\mathbf{k}) & \beta_{-\mathbf{n}}(\mathbf{k}) \\ \beta^*_{-\mathbf{n}}(\mathbf{k}) & -\alpha^*_{-\mathbf{n}}(\mathbf{k}) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{\mathbf{n}}(\mathbf{k}) + \alpha_{-\mathbf{n}}(\mathbf{k}) & \beta_{\mathbf{n}}(\mathbf{k}) + \beta_{-\mathbf{n}}(\mathbf{k}) \\ \beta^*_{\mathbf{n}}(\mathbf{k}) + \beta^*_{-\mathbf{n}}(\mathbf{k}) & -\alpha^*_{\mathbf{n}}(\mathbf{k}) - \alpha^*_{-\mathbf{n}}(\mathbf{k}) \end{pmatrix} \\
&= \begin{pmatrix} \alpha_{\mathbf{n}}(\mathbf{k}) - \alpha_{\mathbf{n}}(-\mathbf{k}) & \beta_{\mathbf{n}}(\mathbf{k}) + \beta_{\mathbf{n}}(-\mathbf{k}) \\ \beta^*_{\mathbf{n}}(\mathbf{k}) + \beta^*_{\mathbf{n}}(-\mathbf{k}) & -\alpha^*_{\mathbf{n}}(\mathbf{k}) + \alpha^*_{\mathbf{n}}(-\mathbf{k}) \end{pmatrix} .
\end{aligned} \tag{C.6}$$

This concludes the proof. ∎

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