Existence of bound states for Schrödinger-Poisson type systems

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Abstract

In this paper we consider the following elliptic system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + u - K(x)\phi(x)u = a(x)|u|u & x \in \mathbb{R}^3 \\ -\Delta \phi = \dfrac{K(x)}{2}u^2 & x \in \mathbb{R}^3 \end{cases}$$

where $K(x), a(x)$ are non-negative real functions defined on $\mathbb{R}^3$ so that $\lim\limits_{|x|\to+\infty} K(x) = K_\infty > 0$ and $\lim\limits_{|x|\to+\infty} a(x) = a_\infty > 0$. We prove that, when $K(x) \equiv K_\infty$ and $a(x) \equiv a_\infty$, under suitable assumptions, the above system has a unique radial solution which is non-degenerate. Using these results we further prove, under additional assumptions on $K(x)$ and $a(x)$, but not requiring any symmetry property on them, the existence of positive solutions for the system.

Keywords. Non autonomous Schrödinger-Poisson system; Lack of compactness; Variational methods.

1 Introduction and main results

In [20], the author considers the following elliptic system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + u - K(x)\phi(x)u = a(x)|u|^{p-1}u & x \in \mathbb{R}^3 \\ \\ -\Delta \phi = \dfrac{K(x)}{2}u^2 & x \in \mathbb{R}^3 \end{cases} \tag{SN}$$

where $K(x)$ and $a(x)$ are two positive real functions defined on $\mathbb{R}^3$ and $p \in (1,5)$. The system (SN) describes the interaction of a particle with its own gravitational field. For physical motivations see [20].
Since $-\Delta\phi = \frac{K(x)}{2}u^2$ has a unique positive solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$, the problem (SN) can be transformed into a single equation. Indeed, substituting ϕ_u into the first equation of (SN) we have to deal with the equivalent problem

$$-\Delta u + u - K(x)\phi_u u = a(x)|u|^{p-1}u, \tag{SN$'$}$$

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which is variational in nature, that is its solutions are the critical points of the C^2 functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as follows

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right)\, dx - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1}\, dx. \tag{1.1}$$

Let us assume that $a(x)$ and $K(x)$ verify, respectively

(a1) $\lim_{|x|\to+\infty} a(x) = a_\infty > 0, \quad \alpha(x) := a(x) - a_\infty \in L^{\frac{6}{5-p}}(\mathbb{R}^3);$

$\mathcal{A} := \inf_{\mathbb{R}^3} a(x) > 0;$

(K1) $\lim_{|x|\to+\infty} K(x) = K_\infty > 0, \quad \eta(x) := K(x) - K_\infty \in L^3(\mathbb{R}^3);$

$\mathcal{K} := \inf_{\mathbb{R}^3} K(x) > 0.$

Then it is natural to associate to I its "functional at infinity" which is given by

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right)\, dx - \frac{1}{4}\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_u u^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a_\infty|u|^{p+1}\, dx \tag{1.2}$$

whose critical points are the solutions of the following elliptic system:

$$\begin{cases} -\Delta u + u - K_\infty\tilde{\phi}(x)u = a_\infty|u|^{p-1}u & x \in \mathbb{R}^3 \\ -\Delta\tilde{\phi} = \dfrac{K_\infty}{2}u^2. & x \in \mathbb{R}^3 \end{cases} \tag{SN$_\infty$}$$

In [20] it was proved the existence of a positive, radial ground state solution of $(\mathrm{SN})_\infty$. Since any symmetry property on $K(x)$ and $a(x)$ is given, it is useful a deep understanding of the compactness situation for the functional I. In [20] it was studied the behavior of the Palais-Smale sequences for I proving that the "bad" levels for the compactness can be located by the energy of the solutions of $(\mathrm{SN})_\infty$. However, in striking contrast with the scalar case, very few is known on the ground states of $(\mathrm{SN})_\infty$. This might depend on the fact that the study of $(\mathrm{SN})_\infty$ requires some hard work which is far from trivial.
In the present paper we deal with the case $p = 2$.
When we consider the interaction of a particle with an electrostatic field, the system that one deals with is of the same type of $(\mathrm{SN})_\infty$ with the only difference that, now, the nonlocal term has a positive sign. For such a system the case $p = 2$ turns out to be a critical one. Indeed, it was proved (see [16]) that it does not admit any non trivial solution. The situation is completely different when we consider a Newton potential. In this case the minus sign before the Poisson term allows us to find solutions.
In particular we prove, under additional assumptions, the existence of a unique radial solution of $(\mathrm{SN})_\infty$ which is also non-degenerate. These results are of independent interest and we hope one may use them in other contexts.
In [20], through a minimization argument on the Nehari manifold related to I,

it was possible to conclude the existence of positive ground states solution for (SN) under some particular assumptions on $K(x)$ and $a(x)$. Here we prove that if

(H) $K(x) \leq K_\infty$; $a(x) \leq a_\infty$ for all $x \in \mathbb{R}^3$ and $a_\infty - a(x) > 0$ on a positive measure set;

holds, then the infimum of I on its Nehari manifold cannot be achieved. However, the uniqueness of the positive solution of $(\mathrm{SN})_\infty$ with other simple energy estimates, allow us to deduce that the compactness condition is recovered not only below a certain threshold, but also that, above the first level in which the Palais-Smale condition fails, some other energy interval exists where the compactness hold. Hence it is natural to look for solutions of (SN') at higher levels. More precisely we prove the following result:

Theorem 1.1 *Let (a1)-(K1)-(H) holds. Then there exists (at least) one positive bound state solution of* (SN) *provided* $\dfrac{\max\{K_\infty^2, a_\infty\}}{\min\{\mathcal{K}^2, \mathcal{A}\}}$ *is sufficiently small.*

The paper is organized as follows: in Section 2 we discuss some preliminary results useful in the rest of the paper and we fix some notations. In Section 3 we deal with the problem at infinity $(\mathrm{SN})_\infty$. We prove the exponential decay of all positive radial solution of $(\mathrm{SN})_\infty$, then we establish the radial symmetry of all its positive solutions. Finally we prove that the radial solution of $(\mathrm{SN})_\infty$ is unique and moreover non-degenerate. At the end, in Section 4, we prove that the infimum of I on its Nehari manifold cannot be attained and we find a bound state solution at a linking level by using the notion of *barycenter* (see [2], [6]).

2 Notations and Preliminaries

Hereafter we use the following notation:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} [\nabla u \nabla v + uv] dx; \qquad \|u\|^2 = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2\right] dx.$$

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

- H^1_r and $D^{1,2}_r$ denote, respectively, the space of functions in $H^1(\mathbb{R}^3)$ and in $D^{1,2}(\mathbb{R}^3)$ which are radially symmetric.

- $L^q(\Omega)$, $1 \leq q \leq +\infty$, $\Omega \subseteq \mathbb{R}^3$, denotes a Lebesgue space, the norm in L^q is denoted by $|u|_{q,\Omega}$ when Ω is a proper subset of $\mathbb{R}^3$, by $|\cdot|_p$ when $\Omega = \mathbb{R}^3$.

- For any $\rho > 0$ and for any $z \in \mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z and $|B_\rho(z)|$ denotes its Lebesgue measure. We write only B_ρ if $z = 0$.

- C, C', C_i are various positive constants.
- S_q is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ in $L^q(\mathbb{R}^3)$, $q \in (2,6)$, that is
$$S_q = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|}{|u|_q}.$$
- $\bar{S}$ is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, that is
$$\bar{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_{D^{1,2}}}{|u|_6}.$$
- In what follows $'$ means $\frac{d}{dr}$, $r = |x|$.

In the following we recall some useful results proved in [20].

2.1 The Poisson equations

The Poisson equation
$$-\Delta\phi = \frac{K(x)}{2}u^2$$
has a unique solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$ which is positive and the following representation formula holds
$$\phi_u(x) = \frac{1}{2}\int_{\mathbb{R}^3} \frac{K(y)}{|x-y|}u^2(y)dy = \frac{1}{|x|} * \frac{K}{2}u^2. \tag{2.1}$$
Moreover
$$\|\phi_u\|_{D^{1,2}} \leq M_1 \cdot \|u\|^2; \tag{2.2}$$
where
$$M_1 := \frac{\bar{S}^{-1}}{2}\left[S_4^{-2}|\eta|_3 + K_\infty S_{12/5}^{-2}\right],$$
and
$$\int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx \leq 2M_1^2\|u\|^4. \tag{2.3}$$

Analogously, the equations
$$\text{a)} \quad -\Delta\tilde{\phi} = \frac{K_\infty}{2}u^2; \qquad \text{b)} \quad -\Delta\bar{\phi} = \frac{\eta(x)}{2}u^2. \tag{2.4}$$
admit, respectively, a unique positive solution $\tilde{\phi}_u \in D^{1,2}(\mathbb{R}^3)$ and $\bar{\phi}_u \in D^{1,2}(\mathbb{R}^3)$ and the following estimates hold:
$$\text{a)} \quad \|\tilde{\phi}_u\|_{D^{1,2}} \leq M_2\|u\|^2; \qquad \text{b)} \quad \|\bar{\phi}_u\|_{D^{1,2}} \leq M_3\|u\|^2 \tag{2.5}$$
with $M_2 := \frac{K_\infty}{2}\bar{S}^{-1} \cdot S_{12/5}^{-2}$, $M_3 := \frac{\bar{S}^{-1}}{2} \cdot S_4^{-2}|\eta|_3$ and
$$\text{a)} \quad \int_{\mathbb{R}^3} K_\infty\tilde{\phi}_u u^2\, dx \leq 2M_2^2\|u\|^4; \qquad \text{b)} \quad \int_{\mathbb{R}^3} \eta(x)\bar{\phi}_u u^2\, dx \leq 2M_3^2\|u\|^4. \tag{2.6}$$

Let us now define the operators

$$\Phi, \bar{\Phi}, \tilde{\Phi} : H^1(\mathbb{R}^3) \longrightarrow D^{1,2}(\mathbb{R}^3)$$

as

$$\Phi[u] = \phi_u, \qquad \bar{\Phi}[u] = \bar{\phi}_u, \qquad \tilde{\Phi}[u] = \tilde{\phi}_u.$$

In the following lemma we summarize some properties of Φ, $\bar{\Phi}$, $\tilde{\Phi}$ useful to study our problem (see [20] for the proof).

Lemma 2.1 *1)* $\Phi, \bar{\Phi}, \tilde{\Phi}$ *are continuous;*

2) $\Phi, \bar{\Phi}, \tilde{\Phi}$ *map bounded sets into bounded sets;*

3) If $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$ *then*

- *i)* $\Phi[u_n] \rightharpoonup \Phi[u]$ *in* $D^{1,2}(\mathbb{R}^3)$*;*
- *ii)* $\bar{\Phi}[u_n] \rightharpoonup \bar{\Phi}[u]$ *in* $D^{1,2}(\mathbb{R}^3)$*;*
- *iii)* $\tilde{\Phi}[u_n] \rightharpoonup \tilde{\Phi}[u]$ *in* $D^{1,2}(\mathbb{R}^3)$*;*

2.2 Variational setting

It is convenient to consider I restricted to a natural constraint, the Nehari manifold, that contains all the critical points of I and on which I turns out to be bounded from below. We set

$$\mathcal{N} := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : G(u) = 0 \right\}$$

where

$$G(u) = I'(u)[u] = \|u\|^2 - \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 dx - \int_{\mathbb{R}^3} a(x)|u|^3 dx.$$

We remark that there holds

$$\begin{aligned} I_{|\mathcal{N}}(u) &= \frac{1}{6}\|u\|^2 + \frac{1}{12}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\, dx && (2.7)\\ &= \frac{1}{4}\|u\|^2 - \frac{1}{12}\int_{\mathbb{R}^3} a(x)|u|^3\, dx && (2.8)\\ &= \frac{1}{6}\int_{\mathbb{R}^3} a(x)|u|^3\, dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\, dx && (2.9)\end{aligned}$$

It is possible to show (see [20]) that the Nehari manifold $\mathcal{N}$ is a C^1 regular manifold diffeomorphic to the sphere of $H^1(\mathbb{R}^3)$ on which the functional I is bounded from below by a positive constant. Hence, if we set

$$m := \inf_{u \in \mathcal{N}} I(u),$$

then m turns out to be a positive number. Moreover u is a free critical point of I if and only if u is a critical point of I constrained on $\mathcal{N}$. From these properties of $\mathcal{N}$ it follows that to any $u \in H^1(\mathbb{R}^3)$ there corresponds a (unique) function $t(u) \in \mathcal{N}$ such that

$$I(t(u)u) = \max_{t \geq 0} I(tu). \qquad (2.10)$$

Finally, let us recall the classical Hardy-Littlewood-Sobolev inequality which will be useful in the sequel

$$\left| \int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|^\gamma}\,dx\,dy \right| \le C_{s,\gamma,3}|f|_q|g|_s, \qquad \forall\, f \in L^q(\mathbb{R}^3),\ \forall\, g \in L^s(\mathbb{R}^3), \tag{2.11}$$

where $0 < \gamma < 3$, $1 < q, s < \infty$ and $\frac{1}{q} + \frac{1}{s} + \frac{\gamma}{3} = 2$.

3 The problem at infinity

Since $K(x) \xrightarrow{|x|\to\infty} K_\infty$ and $a(x) \xrightarrow{|x|\to\infty} a_\infty$, it can be possible to prove that the problem at infinity related to (SN') turns out to be the following problem

$$-\Delta u + u - K_\infty \tilde{\phi}_u u = a_\infty |u|u. \tag{SN$'$)$_\infty$}$$

The solutions of (SN'_∞) are the critical points of the functional $\mathcal{J} \in C^2(H^1(\mathbb{R}^3), \mathbb{R})$ defined in (1.2). Let now

$$\mathcal{M} := \left\{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : H(u) = 0 \right\}$$

where

$$H(u) = \mathcal{J}'(u)[u] = \|u\|^2 - \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u(x) u^2 dx - \int_{\mathbb{R}^3} a_\infty |u|^3\,dx,$$

the Nehari manifold related to $\mathcal{J}$ and set

$$c_\infty := \inf\left\{ \mathcal{J}(u) \ : \ u \in \mathcal{M} \right\}.$$

It is easy to prove that $\mathcal{M}$ is a C^1 regular manifold on which $\mathcal{J}$ is bounded from below by a positive constant. Hence c_∞ is a positive number.
Moreover $\mathcal{M}$ contains all critical points of $\mathcal{J}$, that is u is a free critical point of $\mathcal{J}$ if and only if u is a critical point of $\mathcal{J}$ on $\mathcal{M}$. Moreover to any $u \in H^1(\mathbb{R}^3)$ there corresponds a (unique) $\xi(u) > 0$ called the *projection* of u on $\mathcal{M}$ such that

$$\mathcal{J}(\xi(u)u) = \max_{\xi\ge 0} \mathcal{J}(\xi u). \tag{2.12}$$

In [20], it was proved the following result:

Proposition 3.1 *The problem* $(\mathrm{SN}')_\infty$ *has a positive radial ground state* $w \in \mathcal{M}$ *such that* $\mathcal{J}(w) = c_\infty$.

In the sequel we want to analyze some properties of the solutions of $(\mathrm{SN}')_\infty$. Let us point out explicitly that if u is a radial function then also $\tilde{\phi}_u$ is radial.

1. Exponential Decay of Radial solutions

Lemma 3.2 *Let u be a positive radial solution of* $(\mathrm{SN}')_\infty$. *Then there exists $\delta > 0$ such that*

$$|D^\alpha u(x)| \le Ce^{-\delta|x|}, \qquad \forall\, x \in \mathbb{R}^3 \tag{2.13}$$

for $|\alpha| \le 2$.

Proof Since $u = u(r)$ is radially symmetric then $(\mathrm{SN}')_\infty$ can be written as

$$u'' + \frac{2}{r}u' = u - K_\infty \tilde{\phi}_u u - a_\infty u^2. \tag{2.14}$$

Let $Y = ru$. Then Y satisfies

$$Y'' = \left[1 - K_\infty \tilde{\phi}_u - a_\infty u\right] Y = q(r)Y$$

with $q(r) := 1 - K_\infty \tilde{\phi}_u - a_\infty u$. Since u and $\tilde{\phi}_u$ are radially symmetric, $u, \tilde{\phi}_u \to 0$ as $r \to +\infty$. Then there is $R > 0$ such that $q(r) \geq \frac{1}{2}$ for all $r \geq R$. Set $X = Y^2$. Then X satisfies

$$\frac{1}{2}X'' = (Y')^2 + q(r)X \geq \frac{1}{2}X, \qquad \text{for } r \geq R.$$

Hence $X'' - X \geq 0$ for $r \geq R$. Let $Z = e^{-r}(X' + X)$. Then

$$Z' = e^{-r}(X'' - X) \geq 0.$$

Hence Z is a non-decreasing function. If, by contradiction, there is $R_1 > R$ such that $Z(R_1) > 0$, then $Z(R) \geq Z(R_1)$ for all $R \geq R_1$. However

$$X' + X = e^r Z(r) \geq e^r Z(R_1) > 0$$

and this implies that $X' + X$ is not integrable for $r \geq R$ and this is a contradiction since X', X are integrable functions. Hence $Z(r) < 0$ for all $r \geq R$. Then

$$(e^r X)' = e^r (X' + X) = Z(r)e^{2r} \leq 0$$

and this implies $X \leq C \cdot e^{-r}$, namely $Y \leq C \cdot e^{-r/2}$ and, since u is positive

$$u(r) \leq C\frac{e^{-r/2}}{r}.$$

This proves (2.13) with $\alpha = 0$. Next we estimate the derivatives of u. By (2.14), it holds

$$(r^2 u')' = -r^2 \left[(K_\infty \tilde{\phi}_u - 1)u + a_\infty u^2\right]. \tag{2.15}$$

Then

$$|r^2 u'(r)| \leq \int_0^r s^2 \left|(K_\infty \tilde{\phi}_u - 1)u + a_\infty u^2\right| ds. \tag{2.16}$$

Since the integrand of right-hand side of (2.16) has an exponential decay, we conclude that u' has also an exponential decay. Finally, the estimate on u'' can be easily deduced by (2.14). □

2. Radial symmetry of all positive solutions of $(\mathrm{SN}')_\infty$

In the next we prove that all positive solutions of $(\mathrm{SN}')_\infty$ are radially symmetric. The presence of the nonlocal term in $(\mathrm{SN}')_\infty$ does not allow us to use the standard method of moving planes (based on the maximum principle) to get the radial symmetry of the solutions. Our method for proving the radial symmetry of positive solutions to (SN'_∞) is motivated by the works of Lieb [11], Chen et al. [8] and Li and Ma [10], see also [15], [13], [14] for related results.

Theorem 3.3 *The positive solutions of* $(\mathrm{SN}')_\infty$ *must be radially symmetric and monotone decreasing about some fixed point.*

The key step to prove Theorem 3.3 is to transform the differential equation $(\mathrm{SN}')_\infty$ into an integral system by virtue of the Bessel potentials (see [13],[17]). First we prepare some basic properties of the Bessel potentials.

Remark 3.3.1 The Bessel potential $\mathcal{G}_\beta$, denoted by

$$\mathcal{G}_\beta = (I-\Delta)^{-\beta/2}, \qquad \beta > 0,$$

is defined by the Fourier transform $\mathcal{F}$ in the following way

$$\mathcal{F}(\mathcal{G}_\beta(f))(x) = (1+4\pi^2|x|^2)^{-\beta/2}\mathcal{F}(f)(x), \quad \forall\ f \in H^1(\mathbb{R}^3).$$

For convenience, the Bessel potential is usually expressed in the convolution form

$$\mathcal{G}_\beta(f) = g_\beta * f,$$

in which the Bessel kernel g_β can be determined explicitly by

$$g_\beta(x) = \frac{1}{(4\pi)^{\beta/2}\Gamma(\beta/2)}\int_0^\infty e^{-\pi|x|^2/\delta}e^{-\delta/4\pi}\delta^{-(3-\beta)/2}\frac{d\delta}{\delta}.$$

Hereafter we only concentrate on the special case $\beta = 2$. When $\beta = 2$, $\mathcal{G}_2 = (I-\Delta)^{-1}$ can be seen as the inverse operator of the positive operator $I-\Delta$ in the Sobolev space $H^1(\mathbb{R}^3)$. Hence we can transform the differential equation $(\mathrm{SN}')_\infty$ into an integral equation involving the Bessel potential $\mathcal{G}_2$. Indeed,

$$\begin{aligned} u &= (-\Delta+1)^{-1}\left(K_\infty\tilde{\phi}_u u + a_\infty u^2\right) = (-\Delta+1)^{-1}\left[K_\infty^2\left(\frac{1}{|x|}*u^2\right)u + a_\infty u^2\right] \\ &= g_2 * \left[K_\infty^2\left(\frac{1}{|x|}*u^2\right)u + a_\infty u^2\right] \end{aligned}$$

or equivalently

$$\begin{cases} u = g_2 * \left(K_\infty^2 vu + a_\infty u^2\right) \\ v = \dfrac{1}{|x|} * u^2. \end{cases} \tag{2.17}$$

The most useful fact concerning Bessel potentials is that it can be employed to characterize the Sobolev space $W^{k,p}(\mathbb{R}^3)$. If $\beta = 2$ we have for all $p \in (1,+\infty)$ that

$$\mathcal{G}_2(f) = g_2 * f \in W^{2,p}(\mathbb{R}^3), \qquad \forall\ f \in L^p(\mathbb{R}^3).$$

By the Sobolev embedding, we obtain the estimate

$$|\mathcal{G}_2(f)|_q \le C_{r,s,3}|f|_s, \quad \forall\ f \in L^s(\mathbb{R}^3), \tag{2.18}$$

in which $0 \le \frac{1}{s} - \frac{2}{3} \le \frac{1}{q} \le \frac{1}{s}$. The estimate (2.18) will be very useful in our arguments below.

Proof[Theorem 3.3] For a given real number λ, let us define

$$\Sigma_\lambda := \left\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 \geq \lambda\right\},$$
$$\Sigma_\lambda^u := \{x \in \Sigma_\lambda \ : \ u_\lambda(x) > u(x)\},$$
$$\Sigma_\lambda^v := \{x \in \Sigma_\lambda \ : \ v_\lambda(x) > v(x)\},$$

and we denote by $x^\lambda = (2\lambda - x_1, x_2, x_3)$ the reflected point with respect to the plane $\{x_1 = \lambda\}$ and denote $u_\lambda(x) = u(x^\lambda)$ and $v_\lambda(x) = v(x^\lambda)$.

Claim 1:
For any positive solution of $(\mathrm{SN}')_\infty$*, we have for all* $x \in \mathbb{R}^3$ *that*

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} \left(g_2(x-y) - g_2(x^\lambda - y)\right)\left[K_\infty^2(v_\lambda u_\lambda - vu) + a_\infty(u_\lambda^2 - u^2)\right] dy. \tag{2.19}$$

Indeed, it is easy to see that

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^3} g_2(x-y)\left(K_\infty^2 v(y)u(y) + a_\infty u^2(y)\right) dy \\ &= \int_{\Sigma_\lambda} g_2(x-y)\left(K_\infty^2 v(y)u(y) + a_\infty u^2(y)\right) dy + \\ &\quad + \int_{\mathbb{R}^3 \setminus \Sigma_\lambda} g_2(x-y)\left(K_\infty^2 v(y)u(y) + a_\infty u^2(y)\right) dy \\ &= \int_{\Sigma_\lambda} g_2(x-y)\left(K_\infty^2 v(y)u(y) + a_\infty u^2(y)\right) dy + \\ &\quad + \int_{\Sigma_\lambda} g_2(x^\lambda - y)\left(K_\infty^2 v_\lambda(y)u_\lambda(y) + a_\infty u_\lambda^2(y)\right) dy. \end{aligned}$$

Substituting x by x^λ, we get that

$$\begin{aligned} u_\lambda(x) &= \int_{\Sigma_\lambda} g_2(x^\lambda - y)\left(K_\infty^2 v(y)u(y) + a_\infty u^2(y)\right) dy + \\ &\quad + \int_{\Sigma_\lambda} g_2(x-y)\left(K_\infty^2 v_\lambda(y)u_\lambda(y) + a_\infty u_\lambda^2(y)\right) dy. \end{aligned}$$

Thus a straightforward computation yields the claim.
In a similar way, we obtain also the decomposition of $v_\lambda - v$, namely

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|} - \frac{1}{|x^\lambda - y|}\right)(u_\lambda^2(y) - u^2(y))dy. \tag{2.20}$$

Step 1: By the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$, one has $u \in L^2(\mathbb{R}^3)$. Moreover $v \in L^6(\mathbb{R}^3)$.
We remark that $|x-y| \leq |x^\lambda - y|$ for all $x, y \in \Sigma_\lambda$. Hence

$$g_2(x-y) - g_2(x^\lambda - y) \geq 0.$$

From (2.19), we have for all $x \in \Sigma_\lambda$, that

$$\begin{aligned}
u_\lambda(x) - u(x) &= \int_{\Sigma_\lambda} \big(g_2(x-y) - g_2(x^\lambda - y)\big)\left[K_\infty^2(v_\lambda u_\lambda - vu) + a_\infty(u_\lambda^2 - u^2)\right] dy \\
&\le \int_{\Sigma_\lambda \cap \{u_\lambda v_\lambda > uv\}} \big(g_2(x-y) - g_2(x^\lambda - y)\big)\left[K_\infty^2(v_\lambda u_\lambda - vu)\right] dy \\
&\quad + \int_{\Sigma_\lambda \cap \{u_\lambda > u\}} \big(g_2(x-y) - g_2(x^\lambda - y)\big)\left[a_\infty(u_\lambda^2 - u^2)\right] dy \\
&\le K_\infty^2 \int_{\Sigma_\lambda \cap \{u_\lambda v_\lambda > uv\}} g_2(x-y)(v_\lambda u_\lambda - vu) dy + a_\infty \int_{\Sigma_\lambda^u} g_2(x-y)(u_\lambda^2 - u^2) dy \\
&\le K_\infty^2 \int_{\Sigma_\lambda^u} g_2(x-y)(u_\lambda - u) v_\lambda dy + K_\infty^2 \int_{\Sigma_\lambda^v} g_2(x-y)(v_\lambda - v) u \, dy \\
&\quad + 2a_\infty \int_{\Sigma_\lambda^u} g_2(x-y)(u_\lambda - u) u_\lambda dy
\end{aligned}$$

By (2.18) it follows that

$$\begin{aligned}
|u_\lambda - u|_{2,\Sigma_\lambda^u} &\le C_1 \cdot K_\infty^2 |v_\lambda(u_\lambda - u)|_{\frac{3}{2},\Sigma_\lambda^u} + C_2 \cdot K_\infty^2 |u(v_\lambda - v)|_{\frac{3}{2},\Sigma_\lambda^v} \\
&\quad + C_3 \cdot a_\infty |u_\lambda(u_\lambda - u)|_{\frac{3}{2},\Sigma_\lambda^u} \\
&\le \bar{C}_1 \cdot |v_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u} + \bar{C}_2 \cdot |u|_{2,\Sigma_\lambda^v} \cdot |v_\lambda - v|_{6,\Sigma_\lambda^v} \\
&\quad + \bar{C}_3 \cdot |u_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u}. \qquad (2.21)
\end{aligned}$$

Similarly, from (2.20), we obtain, for all $x \in \Sigma_\lambda$, that

$$v_\lambda(x) - v(x) \le 2 \int_{\Sigma_\lambda^u} \frac{1}{|x-y|} u_\lambda(y)(u_\lambda(y) - u(y)) dy. \qquad (2.22)$$

By the Hardy-Littlewood-Sobolev inequality (2.11), we deduce from (2.22) that

$$|v_\lambda - v|_{6,\Sigma_\lambda^v} \le C_4 |u_\lambda(u_\lambda - u)|_{\frac{6}{5},\Sigma_\lambda^v} \le \bar{C}_4 |u_\lambda|_{3,\Sigma_\lambda^v} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^v}. \qquad (2.23)$$

The estimates (2.21) and (2.23) are the beginning of the moving plane method.

Step 2: We show that for sufficient negative values of λ, the set Σ_λ^u and Σ_λ^v must be empty. In fact, (2.21) and (2.23) imply

$$\begin{aligned}
|u_\lambda - u|_{2,\Sigma_\lambda^u} &\le \bar{C}_1 \cdot |v_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u} + \bar{C}_2 \cdot |u|_{2,\Sigma_\lambda^v} \cdot |v_\lambda - v|_{6,\Sigma_\lambda^v} \\
&\quad + \bar{C}_3 \cdot |u_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u} \\
&\le \bar{C}_1 \cdot |v_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u} + \bar{C}_5 \cdot |u|_{2,\Sigma_\lambda^v} \cdot |u_\lambda|_{3,\Sigma_\lambda^v} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^v} \\
&\quad + \bar{C}_3 \cdot |u_\lambda|_{6,\Sigma_\lambda^u} \cdot |u_\lambda - u|_{2,\Sigma_\lambda^u}.
\end{aligned}$$

We can choose N sufficiently large such that for $\lambda \le -N$, we have

$$\bar{C}_1 |v_\lambda|_{6,\Sigma_\lambda^u} \le \frac{1}{6}, \quad \bar{C}_5 \cdot |u|_{2,\Sigma_\lambda^v} \cdot |u_\lambda|_{3,\Sigma_\lambda^v} \le \frac{1}{6}, \quad \bar{C}_3 |u_\lambda|_{6,\Sigma_\lambda^u} \le \frac{1}{6},$$

which implies that

$$|u_\lambda - u|_{2,\Sigma_\lambda^u} = 0,$$

and therefore Σ_λ^u must be measure zero and hence empty. From (2.23) it follows that also $\Sigma_\lambda^v = \emptyset$.

Step 3: Now we have that for $\lambda \le -N$

$$u(x) \ge u_\lambda(x), \qquad \forall\, x \in \Sigma_\lambda. \tag{2.24}$$

Thus we can start moving the plane $\{x_1 = \lambda\}$ continuously from $\lambda \le -N$ to the right as long as (2.24) holds. Suppose that at a λ_0 we have $u \ge u_{\lambda_0}$ on Σ_{λ_0}, but $u \not\equiv u_{\lambda_0}$ on Σ_{λ_0}, we will show that the plane can be moved further to the right. More precisely, we prove that there exists an ϵ depending on the solution u itself such that $u \ge u_\lambda$ on Σ_λ for all λ in $[\lambda_0, \lambda_0 + \epsilon)$.
By (2.19), we have that $u > u_{\lambda_0}$ in the interior of Σ_{λ_0}. Let

$$\overline{\Sigma_{\lambda_0}^-} = \{x \in \Sigma_{\lambda_0} \ : \ u(x) \le u_{\lambda_0}(x)\}\,.$$

Then it is easy to see that $\overline{\Sigma_{\lambda_0}^-}$ has measure zero and $\lim\limits_{\lambda\to\lambda_0} \Sigma_{\lambda_0}^- \subset \overline{\Sigma_{\lambda_0}^-}$.
Let $(\Sigma_{\lambda_0}^-)^*$ be the reflection of $\Sigma_{\lambda_0}^-$ about the plane $\{x_1 = \lambda\}$. Since $u \in L^{r_i}(\mathbb{R}^3)$, $r_i = 2, 3, 6$ and $v \in L^6(\mathbb{R}^3)$, one can choose ϵ small enough so that for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$ we have

$$\bar{C}_1 |v_\lambda|_{6,\Sigma_\lambda^u} \le \frac{1}{6}, \quad \bar{C}_5 \cdot |u|_{2,\Sigma_\lambda^v} \cdot |u_\lambda|_{3,\Sigma_\lambda^v} \le \frac{1}{6}, \quad \bar{C}_3 |u_\lambda|_{6,\Sigma_\lambda^u} \le \frac{1}{6},$$

from which it follows again that $\Sigma_\lambda^u = \emptyset$ for λ close to λ_0 and it contradicts with the choice of λ_0. Thus we have proved that when moving plane process stops, we must have $u \equiv u_{\lambda_0}$, and $u_\lambda \le u$ on Σ_λ when $\lambda < \lambda_0$.
By a translation, we may assume that $u(0) = \max_{x\in\mathbb{R}^3} u(x)$. Then it follows that the moving plane process from any direction must stop at the origin. Hence u must be radially symmetric and monotone decreasing in the radial direction. □

3. Uniqueness of the radial solution of $(\mathrm{SN}')_\infty$

In the next we show the uniqueness of the positive radial solution of $(\mathrm{SN}')_\infty$. Let us first recall the following theorem known as Newton's Theorem (see [12], Thm 9.7).

Theorem 3.4 *For any radial function $\rho = \rho(|x|) \in L^1(\mathbb{R}^3, (1+|x|)^{-1}dx)$, we have*

$$\left(|x|^{-1} * \rho\right)(r) = V(\rho) - F_\rho(r)$$

where

$$V(\rho) = \int_{\mathbb{R}^3} \frac{\rho(|x|)}{|x|} dx, \qquad F_\rho(r) = 4\pi \int_0^r s\left(1 - \frac{s}{r}\right) \rho(s)\, ds.$$

Since all positive solutions of $(\mathrm{SN}')_\infty$ are radial, we have to show the uniqueness of the radial solution of $(\mathrm{SN}')_\infty$. By using Theorem 3.4 then $(\mathrm{SN}')_\infty$ can be transformed into

$$-\Delta u + K_\infty^2 F_{u^2} u - a_\infty u^2 = \mu u, \tag{2.25}$$

where $\mu := K_\infty^2 V(u^2) - 1$. It is possible to show that $\mu > 0$. We set

$$A(u) := K_\infty^2 F_{u^2} - a_\infty u,$$

then (2.25) becomes

$$-\Delta u + A(u)u = \mu u. \tag{2.26}$$

In the next we show the following result:

Proposition 3.5 *The problem (2.26) has a unique radial positive solution.*

To prove the above Proposition, we shall prove the following uniqueness result:

Proposition 3.6 *The problem*

$$-\Delta u + A(u)u = u \tag{2.27}$$

has at most one solution.

Proof[Equivalence of Prop. 3.5 and Prop. 3.6] Let u_1, u_2 two positive solutions of (2.26) and let us define

$$v_1(x) := \mu_1^{-1} u_1(\mu_1^{-1/2} x); \qquad v_2(x) := \mu_2^{-1} u_2(\mu_2^{-1/2} x),$$

where $\mu_i := K_\infty^2 V(u_i) - 1$, $i = 1, 2$. Then one can prove that v_1, v_2 are solutions of (2.27). Then $v_1 = v_2$, namely

$$u_1(x) = \beta u_2(\beta^{1/2} x), \qquad \beta := \frac{\mu_1}{\mu_2}.$$

Hence

$$\begin{aligned} -u_1(x) &= -\Delta u_1 - K_\infty^2 \left(\int_{\mathbb{R}^3} \frac{1}{|x-y|} u_1^2(y)\, dy \right) u_1 - a_\infty u_1^2 \\ &= -\beta^2 \left[-\Delta u_2(\beta^{1/2} x) - K_\infty^2 \left(\int_{\mathbb{R}^3} \frac{1}{|\beta^{1/2} x - y|} u_2^2(y)\, dy \right) u_2 - a_\infty u_2^2(\beta^{1/2} x) \right] \\ &= -\beta^2 u_2(\beta^{1/2} x) = -\beta u_1(x) \end{aligned}$$

from which it follows that $\beta = 1$ and hence $u_1 = u_2$. The converse follows in a similar way. $\square$

Proof[Proof of Proposition 3.6] By contradiction, let us suppose the existence of two positive radial solutions of (2.27), u_1, u_2 and let us assume, without loss of generality, that $u_2(0) > u_1(0)$. First we show that $u_2(r) > u_1(r)$ for all $r \geq 0$, namely we show that two positive solutions of (2.27) cannot intersect.
We consider the Wronskian of u_1, u_2, namely

$$W(r) := u_2' u_1 - u_1' u_2.$$

Thus $W(r)$ satisfies

$$W' + \frac{2}{r} W = (A(u_2) - A(u_1)) u_1 u_2. \tag{2.28}$$

If, by contradiction, there exists some $r_* > 0$ such that $u_2(r) > u_1(r)$ on $[0, r_*)$ and $u_2(r_*) = u_1(r_*)$ then it is easy to see

$$W(r_*) = (u_2'(r_*) - u_1(r_*))\, u_1(r_*) \le 0.$$

Let us evaluate the sign of $A(u_2) - A(u_1)$ in $[0, r_*)$ on which $u_2(r) > u_1(r)$. The functions u_i, $i = 1, 2$ are both solutions of (2.27). Let $z := u_2 - u_1$. Then z is a solution of the problem

$$-\Delta z + (A(u_2) - A(u_1))u_2 + (A(u_1) - 1)z = 0. \tag{2.29}$$

We multiply by z (2.29) and we integrate on $[0, r_*)$ obtaining:

$$\int_0^{r_*} |z'|^2 + \int_0^{r_*} (A(u_2) - A(u_1))u_2 z + \int_0^{r_*} (A(u_1) - 1)z = 0.$$

Recalling the definition of A we find

$$\underbrace{\int_0^{r_*} |z'|^2}_{(I)} + \underbrace{\int_0^{r_*} K_\infty^2 (F_{u_2^2} - F_{u_1^2})u_2 z}_{(II)} + \underbrace{\int_0^{r_*} (-a_\infty u_2 + A(u_1) - 1)\, z^2}_{(III)} = 0.$$

Since $(I), (II)$ are positive in $[0, r_*)$ then (III) is negative. Hence

$$A(u_1) < 1 + a_\infty u_2(r).$$

Now, a simple computation shows that

$$K_\infty^2 F_{u_2^2} \ge \gamma \tilde{\phi}_{u_2}(r), \qquad \gamma > 0, \gamma \ constant.$$

Then

$$\begin{aligned} A(u_2) - A(u_1) &> \gamma \tilde{\phi}_{u_2}(r) - 2a_\infty u_2(r) - 1 \\ &= \gamma(\tilde{\phi}_{u_2}(r) - 1) + (\gamma - 1) - 2a_\infty u_2(r) \\ &\ge (\gamma - 1) - 2a_\infty u_2(0) \end{aligned}$$

where, in last inequality we have used the fact that u_2 is a decreasing function. Hence $A(u_2) - A(u_1) > 0$ provided $\gamma > 1 + 2a_\infty u_2(0)$.
Hence $r^2 W$ is an increasing function on $(0, r_*)$. However $W(0) = 0$ and hence $W(r_*) > 0$ and this yields a contradiction. Hence $u_2(r) > u_1(r)$ for all $r \ge 0$. A similar arguments as before shows that $A(u_2) - A(u_1) > 0$ for all $r \ge 0$. Then from (2.28) it follows that $r^2 W$ is increasing for all $r \ge 0$ and, moreover $W(0) = 0$. Since u_1, u_2 are radial solutions of (2.27), it is easy to see that they have an exponential decay at infinity together their derivatives (by using Lemma 3.2). Hence

$$\lim_{r \to +\infty} r^2 W(r) = 0,$$

and this is a contradiction. □

4. Non-degeneracy condition

In the next we show that the unique positive radial solution of $(\mathrm{SN}')_\infty$ is also non-degenerate, namely the linearized operator around w has a kernel which is given by

$$\left\{ \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_3} \right\}.$$

Theorem 3.7 *Let $(v,\psi) \in H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ be a solution of*

$$\begin{cases} \Delta v - v + K_\infty \tilde{\phi}_w v + K_\infty w\psi + 2a_\infty wv = 0 \\ \\ \Delta\psi + K_\infty vw = 0. \end{cases} \tag{2.30}$$

where $(w, \tilde{\phi}_w)$ is the solution of $(\mathrm{SN})_\infty$. *Then*

$$(v,\psi) \in \mathrm{span}\left\{ \frac{\partial(w,\tilde{\phi}_w)}{\partial x_j} \; ; \; j = 1,2,3 \right\}.$$

Remark 3.7.1 Suppose that $v \in H^2(\mathbb{R}^3)$ satisfies the following problem

$$\Delta v - v + K_\infty \tilde{\phi}_w v + K_\infty w \int_{\mathbb{R}^3} \frac{K_\infty v(y) w(y)}{|x-y|}\, dy + 2a_\infty wv = 0,$$

then by Theorem 3.7 it follows that

$$v \in \mathrm{span}\left\{ \frac{\partial w}{\partial x_j} \; : \; j = 1,2,3. \right\}.$$

Proof First let us fix some notations. Let $r = |x|$ and $\theta = \frac{x}{|x|} \in S^2$, the unit sphere in $\mathbb{R}^3$. Let be Δ_r the Laplacian operator in radial coordinates and Δ_{S^2} the Laplacian-Beltrami operator. Let $e_k(\theta)$ be solution of

$$-\Delta_{S^2} e_k(\theta) = \mu_k e_k(\theta), \qquad k = 1,2,\ldots$$

The eigenvalues of μ_k are given by

$$\mu_1 = 0,\ \mu_2 = \mu_3 = \mu_4 = 2,\ \mu_4 < \mu_5, \mu_6, \mu_7 \ldots$$

e_k' s are normalized so that they form a complete orthonormal basis of $L^2(S^2)$. Then, any solution (v,ψ) of (2.30) can be written as

$$v(x) = \sum_{k=0}^{\infty} v_r(r) e_k(\theta), \qquad \psi(x) = \sum_{k=0}^{\infty} \psi_k(r) e_k(\theta)$$

where we set

$$v_k(r) = \int_{S^2} v(r,\theta) e_k(\theta)\, d\theta \in H^1, \qquad \psi_k(r) = \int_{S^2} \psi(r,\theta) e_k(\theta)\, d\theta \in H^1.$$

We recall that

$$\Delta v = \Delta_r v + \frac{1}{r^2} \Delta_{S^2} v.$$

Then

$$\begin{aligned} \Delta_r v_k &= \int_{S^2} \Delta_r v \cdot e_k(\theta)\, d\theta = \int_{S^2} \left(\Delta v - \frac{1}{r^2} \Delta_{S^2} v \right) e_k(\theta)\, d\theta \\ &= v_k - K_\infty \tilde{\phi}_w v_k - K_\infty w \psi_k - 2a_\infty w v_k + \frac{\mu_k}{r^2} v_k. \end{aligned}$$

In the same way we obtain

$$\Delta_r \psi_k = -K_\infty w v_k + \frac{\mu_k}{r^2} \psi_k.$$

Hence (v_k, ψ_k) satisfies the problem

$$\begin{cases} \Delta_r v_k - v_k + K_\infty \tilde{\phi}_w v_k + K_\infty w \psi_k + 2a_\infty w v_k - \dfrac{\mu_k}{r^2} v_k = 0 \\ \\ \Delta_r \psi_k + K_\infty w v_k - \dfrac{\mu_k}{r^2} \psi_k = 0. \end{cases} \tag{2.31}$$

We remark that $(w', \tilde{\phi}'_w)$ solves the system

$$\begin{cases} \Delta_r w' - w' + K_\infty \tilde{\phi}_w w' + K_\infty w \tilde{\phi}'_w + 2a_\infty w w' - \dfrac{2}{r^2} w' = 0 \\ \\ \Delta_r \tilde{\phi}'_w + K_\infty w w' - \dfrac{2}{r^2} \tilde{\phi}'_w = 0. \end{cases} \tag{2.32}$$

Moreover $\tilde{\phi}'_w, w' < 0$. The proof will be divided into three steps.

STEP 1: If $k \geq 5$ then $v_k = \psi_k \equiv 0$.

By contradiction, we multiply by w' the first equation of (2.31) and we integrate on B_r with $r > 0$. By making an integration by parts we find

$$\begin{aligned} 0 &= \int_{B_r} \Big(\Delta_r v_k - v_k + K_\infty \tilde{\phi}_w v_k + K_\infty w \psi_k + 2a_\infty w v_k - \frac{\mu_k}{r^2} v_k \Big) w' \\ &= \int_{\partial B_r} (v'_k w' - v_k w'') + \int_{B_r} \Big(\Delta_r w' - w' + K_\infty \tilde{\phi}_w w' + 2a_\infty w w' \Big) v_k \\ &\quad + \int_{B_r} K_\infty w w' \psi_k - \int_{B_r} \frac{\mu_k}{r^2} v_k w' \\ &= \int_{\partial B_r} (v'_k w' - v_k w'') - \int_{B_r} K_\infty w \tilde{\phi}'_w v_k + \int_{B_r} K_\infty w w' \psi_k + \\ &\quad + \int_{B_r} \frac{2 - \mu_k}{r^2} v_k w' \end{aligned} \tag{2.33}$$

We now multiply the second equation of (2.31) by $\tilde{\phi}'_w$ and we integrate over B_r. As before, an integration by parts yields

$$0 = \int_{\partial B_r} \Big(\psi'_k \tilde{\phi}'_w - \psi_k \tilde{\phi}''_w \Big) + \int_{B_r} K_\infty w \tilde{\phi}'_w v_k - \int_{B_r} K_\infty w w' \psi_k + \int_{B_r} \frac{2 - \mu_k}{r^2} \psi_k \tilde{\phi}'_w. \tag{2.34}$$

Adding (2.33) and (2.34) we get

$$\begin{aligned} 0 &= \int_{\partial B_r} (v'_k w' - v_k w'') + \int_{\partial B_r} \Big(\psi'_k \tilde{\phi}'_w - \psi_k \tilde{\phi}''_w \Big) + \\ &\quad + \int_{B_r} \frac{2 - \mu_k}{r^2} \Big(v_k w' + \psi_k \tilde{\phi}'_w \Big) := I_1(r) + I_2(r) + I_3(r) \end{aligned} \tag{2.35}$$

where $I_j(r)$, $j = 1, 2, 3$ are defined by the last inequality. We now choose an appropriate r and estimate each of the terms $I_j(r)$. By definition we have

$$v'_k(0) = \psi'_k(0) = 0.$$

Without loss of generality we assume that there is some $r_1 > 0$ such that $v_k(r) < 0$ for $r \in (0, r_1)$ and $v_k(r_1) = 0$. (We choose $r_1 = \infty$ if $v_k(r) < 0$ for all $r > 0$). By standard ODE theory it follows that $v_k'(r_1) > 0$.

Claim: $\psi_k(r) < 0$ for r small.

Suppose that this is not the case. Then

$$\Delta_r \psi_k = \frac{\mu_k}{r^2}\psi_k - K_\infty w v_k > 0.$$

Hence $\psi_k(r)$ cannot have a local maximum. This implies

$$\psi_k(r) > \psi_k(0) > 0 \qquad \text{for } r \in (0, r_1).$$

Then

$$\begin{aligned}
0 &< \int_{B_{r_1}} \left(\frac{\mu_k - 2}{r^2} v_k - K_\infty w \psi_k \right) w' \\
&= \int_{B_{r_1}} \left(\Delta_r v_k - v_k + K_\infty \tilde{\phi}_w v_k + 2a_\infty w v_k \right) w' - \int_{B_{r_1}} \frac{2}{r^2} v_k w' \\
&= \int_{B_{r_1}} (v_k' w' - v_k w'') + \int_{B_{r_1}} \left(\Delta_r w' - w' + K_\infty \tilde{\phi}_w w' + 2a_\infty w w' \right) v_k \\
&\quad - \int_{B_{r_1}} \frac{2}{r^2} v_k w' \\
&= 4\pi r_1^2 v_k'(r_1) w'(r_1) - \int_{B_{r_1}} K_\infty v_k \tilde{\phi}_w' w < 0
\end{aligned}$$

and this gives a contradiction. Therefore $\psi_k(r) < 0$ for r small. Then there exists some $r_2 > 0$ such that $\psi_k(r) < 0$ for $r \in (0, r_2)$ and $\psi_k(r_2) = 0$ (we choose $r_2 = \infty$ if $\psi_k(r) < 0$ for all $r > 0$). By standard ODE theory it follows that $\psi_k'(r_2) > 0$. We distinguish now three different cases that can arise.

Case 1: $r_1 = r_2$.
Set $r := r_1 = r_2$ then an easy computation shows that $I_j(r) < 0$ for $j = 1, 2, 3$. By (2.35), this gives a contradiction.

Case 2: $r_2 < r_1$.
We easily calculate $I_3(r_2) < 0$ and $I_2(r_2) < 0$. It is more difficult now evaluate $I_1(r_2)$. We define

$$V(r) = r^2 v_k' w' - r^2 w'' v_k.$$

Then for $r \in (r_2, r_1)$

$$V'(r) = (r^2 v_k')' w' - (r^2 w'')' v_k.$$

Now we use

$$\begin{aligned}
\frac{1}{r^2}(r^2 v_k')' &= v_k - K_\infty \tilde{\phi}_w v_k - K_\infty w \psi_k - 2a_\infty w v_k + \frac{\mu_k}{r^2} v_k \\
\frac{1}{r^2}(r^2 w'')' &= w' - K_\infty \tilde{\phi}_w w' - K_\infty w \tilde{\phi}_w' - 2a_\infty w w' + \frac{2}{r^2} w'.
\end{aligned}$$

Then

$$V'(r) = (\mu_k - 2)v_k w' + r^2 K_\infty w \tilde{\phi}'_w v_k - r^2 K_\infty w \psi_k w' > 0.$$

Here we have used that for $r \in (r_2, r_1)$, $\psi_k(r)$ has no local maximum since $\Delta \psi_k > 0$.
Then V is an increasing function. Hence for $r_2 < r_1$ we get

$$\frac{1}{4\pi} I_1(r_2) = V(r_2) < V(r_1) = \frac{1}{4\pi} I_1(r_1) < 0.$$

By (2.35) this gives a contradiction.

Case 3: $r_1 < r_2$.
We easily calculate $I_3(r_1) < 0$ and $I_1(r_1) < 0$. It is more difficult now evaluate $I_2(r_1)$. We define

$$\Psi(r) = r^2 \psi'_k \tilde{\phi}'_w - r^2 \tilde{\phi}''_w \psi_k.$$

Then for $r \in (r_1, r_2)$

$$\Psi'(r) = (r^2 \psi'_k) \tilde{\phi}'_w - (r^2 \tilde{\phi}''_w)' \psi_k.$$

Now we use

$$\frac{1}{r^2}(r^2 \psi'_k)' = \frac{\mu_k}{r^2} \psi_k - K_\infty w v_k$$
$$\frac{1}{r^2}(r^2 \tilde{\phi}''_w)' = \frac{2}{r^2} \tilde{\phi}'_w - K_\infty w w'.$$

Then

$$\Psi'(r) = (\mu_k - 2)\tilde{\phi}'_w \psi_k + r^2 K_\infty w w' \psi_k - r^2 K_\infty w v_k \tilde{\phi}'_w.$$

The function $\Psi'(r) > 0$ if we prove that $v_k(r) > 0$ in all (r_1, r_2). Suppose, by contradiction, that $v_k(r) > 0$ does not hold in all (r_1, r_2). Then there exists some $r_3 \in (r_1, r_2)$ such that $v_k(r) > 0$ in (r_1, r_3), $v_k(r_3) = 0$ and $v_k(r) < 0$ in (r_3, r_2). By standard ODE theory $v'_k(r_3) < 0$. Moreover we must have $\Delta_r v_k(r_3) \leq 0$. However

$$\begin{aligned} 0 &> \int_{B_{r_3} \setminus B_{r_1}} \left(\frac{\mu_k - 2}{r^2} v_k - K_\infty w \psi_k \right) w' \\ &= \int_{B_{r_3} \setminus B_{r_1}} \left(\Delta_r v_k - v_k + K_\infty \tilde{\phi}_w v_k + 2a_\infty w v_k \right) w' - \int_{B_{r_3} \setminus B_{r_1}} \frac{2}{r^2} v_k w' \\ &= 4\pi r_3^2 v'_k(r_3) w'(r_3) - 4\pi r_1^2 v'_k(r_1) w'(r_1) - \int_{B_{r_3} \setminus B_{r_1}} K_\infty \tilde{\phi}'_w w v_k > 0 \end{aligned}$$

and this is a contradiction. Hence Ψ is an increasing function. Then for $r_1 < r_2$ we get

$$\frac{1}{4\pi} I_2(r_1) = \Psi(r_1) < \Psi(r_2) = \frac{1}{4\pi} I_2(r_2) < 0.$$

By (2.35) this gives a contradiction.

STEP 2: For $k = 2, 3, 4$ (v_k, ψ_k) is a one-dimensional solution.

For $k = 2, 3, 4$ we recall that $\mu_k = 2$. Then (2.31) becomes

$$\begin{cases} \Delta_r v_k - v_k + K_\infty \tilde{\phi}_w v_k + K_\infty w \psi_k + 2a_\infty w v_k - \dfrac{2}{r^2} v_k = 0 \\ \\ \Delta_r \psi_k + K_\infty w v_k - \dfrac{2}{r^2} \psi_k = 0. \end{cases} \tag{2.36}$$

Since (v_k, ψ_k) solves (2.36) then $v_k(0) = \psi_k(0) = 0$. Let us assume, without loss of generality, that $\psi_k'(0) > 0$. We prove that $v_k'(0) > 0$. Hence by linearity this would imply that (v_k, ψ_k) is a 1-dimensional solution for $k = 2, 3, 4$.
By contradiction we suppose $v_k'(0) \leq 0$. Since $\psi_k(0) = 0$ and $\psi_k'(0) > 0$ then there exists some $\rho > 0$ such that $\psi_k(r) > 0$ in $(0, \rho)$ and $\psi_k(\rho) = 0$ (we choose $\rho = \infty$ if $\psi_k(r) > 0$ for all $r > 0$).
If $v_k'(0) < 0$ then, since $v_k(0) = 0$, there exists some $r_1 > 0$ such that $v_k(r) < 0$ for $r \in (0, r_1)$ and $v_k(r_1) = 0$. By multiplying the first equation in (2.36) by w' and integrating over B_r with $r = \min\{r_1, \rho\}$ we get

$$4\pi r^2 (w'')^2 \left(\frac{v_k}{w'}\right)' = \int_{B_r} K_\infty \tilde{\phi}_w' w v_k - \int_{B_r} K_\infty w w' \psi_k > 0.$$

Hence $\frac{v_k}{w'}$ is an increasing function. We now prove that $\rho < r_1$. If this is not the case, then $\rho \geq r_1$. Hence

$$\frac{v_k(\rho)}{w'(\rho)} \geq \frac{v_k(r_1)}{w'(r_1)} = 0.$$

Then, since $w' < 0$ we get $v_k(\rho) < 0$ and this is a contradiction.
Hence $\rho < r_1$ Then for $r \in (0, \rho)$ we get

$$\Delta \psi_k = \frac{2}{r^2} \psi_k - K_\infty w v_k > 0$$

and so ψ_k has no local maximum point in $(0, \rho)$. Hence $\psi_k(\rho) > \psi_k(0) = 0$ and this is a contradiction since $\psi_k(\rho) = 0$.
If $v_k'(0) = 0$ then from the first equation in (2.36) we get also $v_k''(0) = 0$. By making an expansion of v_k in $r = 0$ we get $v_k'''(0) < 0$ and as before we can suppose the existence of some $r_1 > 0$ such that $v_k(r) < 0$ in $(0, r_1)$. Reasoning as before we get $\rho < r_1$ and we reach again a contradiction.

STEP 3: If $k = 1$ then $v_1 = \psi_1 \equiv 0$.

Let $k = 1$. Then $\mu_1 = 0$. Hence (2.31) becomes

$$\begin{cases} \Delta_r v_1 - v_1 + K_\infty \tilde{\phi}_w v_1 + K_\infty w \psi_1 + 2a_\infty w v_1 = 0 \\ \\ \Delta_r \psi_1 + K_\infty w v_1 = 0 \end{cases} \tag{2.37}$$

where $v_1 = v_1(r)$, $\psi_1 = \psi_1(r)$, $(v_1, \psi_1) \in H^1(\mathbb{R}^3) \times W^{2,q}(\mathbb{R}^3)$, $q > 3$.
By definition

$$v_1'(0) = \psi_1'(0) = 0.$$

Let us suppose that $\psi_1(0) > 0$ and let $\rho > 0$ be such that $\psi_1(r) > 0$ for $r \in (0, \rho)$ and $\psi_1(\rho) = 0$. We now prove that also $v_1(0) < 0$. By contradiction, we multiply by w the first equation in (2.37) and we integrate over B_r with $r \in (0, \rho)$. Hence we get

$$\underbrace{4\pi r^2 w^2 \left(\frac{v_1}{w}\right)'}_{(A)} + \underbrace{a_\infty \int_{B_r} w^2 v_1}_{(B)} < 0.$$

If $(A) < 0$ then $\frac{v_1}{w}$ is a decreasing function. Hence for $r > 0$ we get

$$\frac{v_1(r)}{w(r)} < \frac{v_1(0)}{w(0)} = 0$$

and then $v_1(r) < 0$ for $0 < r < \rho$. This implies that also $(B) < 0$. If $(A) > 0$ then, necessarily, $(B) < 0$ and this means $v_1(r) < 0$ for $r \in (0, \rho)$. Hence

$$\Delta\psi_1 = -K_\infty w v_1 > 0$$

and this implies that ψ_1 has no local maximum point in $(0, \rho)$. Then $\psi_1(\rho) > \psi_1(0) > 0$ and this gives a contradiction since $\psi(\rho) = 0$.
Hence $v_1(0) > 0$. By standard ODE theory it follows that the dimension of the set of solutions is at most one.
On the other hand,

$$(v_1, \psi_1) = \left(2w + rw', \frac{2}{K_\infty}(K_\infty \tilde{\phi}_w - 1) + r\tilde{\phi}'_w\right)$$

is a solution of (2.37). Since the dimension of the solution set is at most one, we know that any solution satisfies

$$(v_1, \psi_1) = C\left(2w + rw', \frac{2}{K_\infty}(K_\infty \tilde{\phi}_w - 1) + r\tilde{\phi}'_w\right).$$

But since $\psi_1 \to -\frac{2}{K_\infty}$ as $r \to +\infty$, we conclude that $\psi_1 \notin W^{2,q}(\mathbb{R}^3)$. Therefore (2.37) has no solution in $H^1(\mathbb{R}^3) \times W^{2,q}(\mathbb{R}^3)$. □

4 Existence of bound states

To study the existence of bound states solutions for the problem (SN') we have to locate the intervals in which the compactness condition is preserved. In [20] we have proved the following result which gives a picture of the compactness situation for the problem (SN').

Lemma 4.1 *Let $(u_n)_n$ be a (PS) sequence of I constrained on $\mathcal{N}$, i.e. $u_n \in \mathcal{N}$ and*

$$\begin{aligned} &\text{a)}\ I(u_n) \textit{ is bounded;} \\ &\text{b)}\ \nabla I_{|\mathcal{N}}(u_n) \to 0 \textit{ strongly in } H^1(\mathbb{R}^3). \end{aligned} \tag{2.38}$$

Then replacing $(u_n)_n$, if necessary, with a subsequence, there exist a solution $\bar{u}$ of (SN'), a number $k \in \mathbb{N} \cup \{0\}$, k functions $u^1, ..., u^k$ of $H^1(\mathbb{R}^3)$ and k sequences of points (y_n^j), $y_n^j \in \mathbb{R}^3$, $0 \le j \le k$ such that

$$
\begin{aligned}
&\text{(i) } |y_n^j| \to +\infty, \ |y_n^j - y_n^i| \to +\infty \ \textit{if } i \neq j, \ \ n \to +\infty;\\
&\text{(ii) } u_n - \sum_{j=1}^{k} u^j(\cdot - y_n^j) \longrightarrow \bar{u}, \ \textit{in } H^1(\mathbb{R}^3);\\
&\text{(iii) } I(u_n) \to I(\bar{u}) + \sum_{j=1}^{k} \mathcal{J}(u^j);\\
&\text{(iv) } u^j \textit{ are non trivial weak solutions of } (\mathrm{SN}')_\infty.
\end{aligned}
\tag{2.39}
$$

Moreover we agree that in the case $k = 0$, the above holds without u^j.

If (H) holds it is quite clear that the energy of a solution of (SN') will be always above c_∞ and then the problem (SN') cannot be solved by using minimization arguments. More specifically the following result holds:

Proposition 4.2 *The relation $m = c_\infty$ holds and m is not attained.*

Proof As done in [7] one can show that $m \ge c_\infty$.
Let us now prove the opposite inequality. To do this, let us consider the sequence $u_n = t_n w_n$, where $w_n(\cdot) = w(\cdot - z_n)$, being w the unique positive solution centered at zero of $(\mathrm{SN}')_\infty$, $(z_n)_n$, $z_n \in \mathbb{R}^3$, is such that $|z_n| \to +\infty$ as $n \to +\infty$ and $t_n = t(w_n)$. We want to show that

$$\lim_{n \to +\infty} I(u_n) = c_\infty. \tag{2.40}$$

By using Lemma A.2 of [1] we have

$$\lim_{n \to +\infty} \int_{\mathbb{R}^3} a(x)|w_n|^3 dx = \lim_{n \to +\infty} \int_{\mathbb{R}^3} (a_\infty + \alpha(x))|w_n|^3 dx = \int_{\mathbb{R}^3} a_\infty |w|^3 dx. \tag{2.41}$$

We claim that

$$\lim_{n \to +\infty} \int_{\mathbb{R}^3} K(x)\phi_{w_n} w_n^2 dx = \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2 \, dx. \tag{2.42}$$

Indeed:

$$
\begin{aligned}
\left| \int_{\mathbb{R}^3} K(x)\phi_{w_n}(w_n)^2 \, dx - \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2 \, dx \right| \quad \le \quad & \underbrace{\int_{\mathbb{R}^3} |\eta(x)| \bar{\phi}_{w_n} w_n^2 \, dx}_{(I)} \\
& + 2 \underbrace{\int_{\mathbb{R}^3} |\eta(x)| \tilde{\phi}_{w_n} w_n^2 \, dx}_{(II)}
\end{aligned}
$$

Since w_n weakly converges to zero in $L^{p+1}(\mathbb{R}^3)$ then $w_n \to 0$ in $L^4_{loc}(\mathbb{R}^3)$. Hence for any choice of $\epsilon > 0$ and $\rho > 0$

$$|w_n|_{4, B_\rho} < \epsilon, \qquad n \text{ large}. \tag{2.43}$$

Furthermore $(w_n)_n$ is bounded then by 2) of Lemma 2.1 $\bar{\phi}_{w_n}$ and $\tilde{\phi}_{w_n}$ are bounded too. Now, since $\eta \in L^3(\mathbb{R}^3)$ for any $\epsilon > 0$ there exists $\bar{\rho} \equiv \bar{\rho}(\epsilon) > 0$ such that

$$|\eta|_{3,B_\rho^c} < \epsilon, \qquad \forall\ \rho \geq \bar{\rho}. \tag{2.44}$$

Hence

$$\begin{aligned}
(I) &\leq \bar{S}^{-1}\|\bar{\phi}_{w_n}\|_{D^{1,2}}\left(\int_{\mathbb{R}^3}|\eta(x)|^{6/5}(w_n)^{12/5}\,dx\right)^{5/6}\\
&\leq C\left(\int_{B_\rho}|\eta(x)|^{6/5}w_n^{12/5}\,dx+\int_{\mathbb{R}^3\setminus B_\rho}|\eta(x)|^{6/5}w_n^{12/5}\,dx\right)^{5/6}\\
&\leq C\left(|\eta|^{6/5}_{3,B_\rho^c}|w_n|^{12/5}_{4,B_\rho^c}+|\eta|^{6/5}_{3,B_\rho}|w_n|^{12/5}_{4,B_\rho(0)}\right)^{5/6}\\
&\leq \tilde{C}\left(\epsilon^{6/5}\|w\|^{12/5}+|\eta|_3^{6/5}\epsilon^{12/5}\right)^{5/6}
\end{aligned}$$

from which it follows $(I) = o(1)$. By similar arguments one can also show that $(II) = o(1)$ and the claim is proved.
Therefore, to obtain (2.40), we just need to show that $t_n \to 1$ as $n \to +\infty$.

1. $t_n \geq 1$:
   Indeed since (H) holds, $t_n w_n \in \mathcal{N}$ and $w \in \mathcal{M}$ then

$$\begin{aligned}
t_n^2\int_{\mathbb{R}^3}K_\infty\tilde{\phi}_w w^2\,dx+t_n\int_{\mathbb{R}^3}a_\infty|w|^3\,dx &= t_n^2\int_{\mathbb{R}^3}K_\infty\tilde{\phi}_{w_n}w_n^2\,dx+\\
&\quad +t_n\int_{\mathbb{R}^3}a_\infty|w_n|^3\,dx\\
&\leq t_n^2\int_{\mathbb{R}^3}K(x)\phi_{w_n}w_n^2\,dx+t_n\int_{\mathbb{R}^3}a(x)|w_n|^3\,dx\\
&= \|w_n\|^2=\|w\|^2\\
&= \int_{\mathbb{R}^3}K_\infty\tilde{\phi}_w w^2\,dx+\int_{\mathbb{R}^3}a_\infty|w|^3\,dx
\end{aligned}$$

   and hence

$$(t_n^2-1)\int_{\mathbb{R}^3}K_\infty\tilde{\phi}_w w^2\,dx+(t_n-1)\int_{\mathbb{R}^3}a_\infty|w|^3\,dx\geq 0$$

   from which it follows $t_n \geq 1$.

2. $(t_n)_n$ is bounded:
   Indeed, if, by contradiction, t_n is unbounded then, up to a subsequence, we can assume that $t_n \to +\infty$ as $n \to +\infty$. Thus, being $t_n w_n \in \mathcal{N}$, we obtain

$$\frac{1}{t_n^2}\|w_n\|^2=\frac{1}{t_n}\int_{\mathbb{R}^3}a(x)|w_n|^3dx+\int_{\mathbb{R}^3}K(x)\phi_{w_n}w_n^2dx. \tag{2.45}$$

   Hence, taking into account that $\|w_n\| = \|w\|$ for any n and using (2.42) we deduce that

$$\int_{\mathbb{R}^3}K_\infty\tilde{\phi}_w w^2\,dx=o(1).$$

   But we reach a contradiction since w is strictly positive and $K_\infty \neq 0$. Hence the sequence $(t_n)_n$ is bounded.

3. $t_n \to 1$ as $n \to +\infty$:
   In fact, if by contradiction $t_n \to \bar{t} \neq 1$ as $n \to +\infty$ then from (2.45) it follows that
$$\|w\|^2 = \bar{t} \int_{\mathbb{R}^3} a_\infty |w|^3\, dx + \bar{t}^2 \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\, dx$$
   and since $w \in \mathcal{M}$ we find
$$(1-\bar{t}) \int_{\mathbb{R}^3} a_\infty |w|^3\, dx + (1-\bar{t}^2) \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\, dx = 0$$
   from which it follows that $\bar{t} = 1$.

Finally, assume, by contradiction, that there exists $\bar{u} \in \mathcal{N}$ such that $I(\bar{u}) = m = c_\infty$. Let $\xi > 0$ be such that $\xi\bar{u} \in \mathcal{M}$. Then, using (H), (2.7) and (2.10), we have

$$\begin{aligned} c_\infty &\leq \mathcal{J}(\xi\bar{u}) = \frac{1}{2}\|\xi\bar{u}\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\xi\bar{u}} (\xi\bar{u})^2\, dx - \frac{1}{3}\int_{\mathbb{R}^3} a_\infty |\xi\bar{u}|^3\, dx \\ &\leq \frac{1}{2}\|\xi\bar{u}\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{\xi\bar{u}}(x)(\xi\bar{u})^2\, dx - \frac{1}{3}\int_{\mathbb{R}^3} a(x)|\xi\bar{u}|^3\, dx \\ &\leq \frac{1}{2}\|\bar{u}\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{\bar{u}}(x)\bar{u}^2\, dx - \frac{1}{3}\int_{\mathbb{R}^3} a(x)|\bar{u}|^3\, dx \\ &= I(\bar{u}) = m = c_\infty. \end{aligned}$$

So we infer $\xi = 1$ ans

$$\frac{1}{4}\left(\int_{\mathbb{R}^3} K(x)\phi_{\bar{u}}(x)\bar{u}^2\, dx - \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{u}}(x)\bar{u}^2\, dx\right) + \frac{1}{3}\int_{\mathbb{R}^3} (a(x) - a_\infty)|\bar{u}|^3\, dx = 0 \tag{2.46}$$

Thus $\bar{u} \in \mathcal{M}$ and $\mathcal{J}(\bar{u}) = c_\infty$, hence $\bar{u}$ is a ground state solution of $(\mathrm{SN}')_\infty$ on $\mathcal{M}$ and so $\bar{u}(\cdot) = w(\cdot - z) > 0$. Therefore by (H)

$$\int_{\mathbb{R}^3} K(x)\phi_{\bar{u}}(x)\bar{u}^2\, dx - \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{u}}(x)\bar{u}^2\, dx \leq 0$$

and
$$\int_{\mathbb{R}^3} (a(x) - a_\infty)|\bar{u}|^3\, dx < 0$$
contradicting (2.46). □

By the previous proposition, we can only hope to find critical points of I at levels higher than m.
Let $v = v^+ - v^-$ a changing sign solution of $(\mathrm{SN}')_\infty$. Hence v is a critical point of $\mathcal{J}$ constrained on $\mathcal{M}$. Then it is easy to see that the energy of v is greater or equal that $2c_\infty$.
Next Lemma provides a range of values greater than c_∞ such that the (PS) property holds.

Lemma 4.3 *The functional I satisfies the $(PS)_d$ condition for all $d \in (c_\infty, 2c_\infty)$*

Proof Let us consider a $(PS)_d$ sequence $(u_n)_n$ and apply to it Lemma 4.1. Then (2.39)-(iii) gives (up to a subsequence)

$$d = \lim_{n\to+\infty} I(u_n) = I(\bar{u}) + \sum_{j=0}^{k} \mathcal{J}(u^j), \tag{2.47}$$

where $\bar{u}$ is the weak limit of $(u_n)_n$ and $\mathcal{J}(u^j) \geq c_\infty$.
Thus, being $c_\infty < d < 2c_\infty$, (2.47) implies $k < 2$. If $k = 1$ there are two possibilities:

i) $\bar{u} \neq 0$, from which $I(\bar{u}) > c_\infty$ follows and

$$2c_\infty > d = \lim_{n\to+\infty} I(u_n) = I(\bar{u}) + \mathcal{J}(u^1) > 2c_\infty;$$

ii) $\bar{u} = 0$, then $I(\bar{u}) = 0$ and so

$$d = \lim_{n\to+\infty} I(u_n) = \mathcal{J}(u^1)$$

this is impossible because either $\mathcal{J}(u^1) = c_\infty$, or $\mathcal{J}(u^1) \geq 2c_\infty$ but $d \in (c_\infty, 2c_\infty)$.

Both cases bringing to a contradiction and so we conclude that $k = 0$. □

Now we need to build a suitable min-max scheme for our problem. To do that, we first remind the definition of the barycenter of a function $u \in H^1(\mathbb{R}^3)$, $u \neq 0$ given in [6]. Setting

$$\mu(u)(x) = \frac{1}{|B_1(0)|} \int_{B_1(x)} |u(y)| dy, \qquad \mu(u) \in L^\infty \text{ and is continuous}$$

$$\hat{u}(x) = \left[\mu(u)(x) - \frac{1}{2} \max \mu(x)\right]^+, \qquad \hat{u} \in C_0(\mathbb{R}^3);$$

we define the barycenter $\beta : H^1(\mathbb{R}^3) \setminus \{0\} \to \mathbb{R}^3$ by

$$\beta(u) = \frac{1}{|\hat{u}|_1} \int_{\mathbb{R}^3} x\hat{u}(x) dx \in \mathbb{R}^3.$$

Since $\hat{u}$ has compact support, β is well defined. Moreover the following properties hold:

1. β is continuous in $H^1(\mathbb{R}^3) \setminus \{0\}$;
2. If u is a radial function $\beta(u) = 0$;
3. For all $t \neq 0$ and for all $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, $\beta(tu) = \beta(u)$;
4. Given $z \in \mathbb{R}^3$ and setting $u_z(x) = u(x - z)$, $\beta(u_z) = \beta(u) + z$.

Let us define

$$b_0 := \inf\{I(u) \ : \ u \in \mathcal{N}, \ \beta(u) = 0\}. \tag{2.48}$$

As done in [7] one can show the following Lemmas.

Lemma 4.4 *$b_0 > m$.*

Now, let w the unique positive ground state of $(\mathrm{SN}')_\infty$ and we define the operator

$$\Gamma : \mathbb{R}^3 \to \mathcal{N}$$

as

$$\Gamma[z](x) = t_z w_z$$

where $w_z := w(x-z)$ and t_z is chosen such that $\Gamma[z] \in \mathcal{N}$. So arguing as done in Proposition 4.2 point 1., we find $t_z \geq 1$. By using the properties of the barycenter we find

$$\beta(\Gamma[z]) = \beta(t_z w(x-z)) = \beta(w(x-z)) = \beta(w(x)) + z = z. \tag{2.49}$$

Moreover, arguing as for proving (2.40), we obtain

Lemma 4.5 $\lim_{|z|\to+\infty} I(\Gamma(z)) = c_\infty$.

Lemma 4.6 *Assume that*

$$\frac{\max\{K_\infty^2, a_\infty\}}{\min\{\mathcal{K}^2, \mathcal{A}\}} < \sqrt[4]{2}. \tag{2.50}$$

Then

$$I(\Gamma[z]) < 2c_\infty. \tag{2.51}$$

Proof Since $\Gamma[z] \in \mathcal{N}$, using (2.7), (H), $w \in \mathcal{M}$ and $t_z \geq 1$ we infer

$$\begin{aligned} I(\Gamma[z]) &= \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{\Gamma[z]}(\Gamma[z])^2\,dx + \frac{1}{6}\int_{\mathbb{R}^3} a(x)|\Gamma[z]|^3 dx \\ &\leq t_z^4\left[\frac{1}{4}\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{w_z}(w_z)^2\,dx + \frac{1}{6}\int_{\mathbb{R}^3} a_\infty|w_z|^3 dx\right] \\ &= t_z^4 c_\infty. \end{aligned}$$

Therefore for proving (2.51) it is enough to show $t_z^4 < 2$ for all $z \in \mathbb{R}^3$. To do this, let us observe that t_z is such that

$$t_z^2\|w_z\|^2 = t_z^4\int_{\mathbb{R}^3} K(x)\phi_{w_z}(x)w_z^2 dx + t_z^3\int_{\mathbb{R}^3} a(x)|w_z|^3 dx$$

from which, using $t_z \geq 1$, we deduce

$$\begin{aligned} t_z &\leq \frac{\|w_z\|^2}{\displaystyle\int_{\mathbb{R}^3} K(x)\phi_{w_z}(x)w_z^2\,dx + \int_{\mathbb{R}^3} a(x)|w_z|^3\,dx} \\ &\leq \frac{\max\{K_\infty^2, a_\infty\}\left(\displaystyle\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}w_z^2(x)w_z^2(y)\,dx\,dy + \int_{\mathbb{R}^3}|w_z|^3\,dx\right)}{\min\{\mathcal{K}^2, \mathcal{A}\}\left(\displaystyle\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}w_z^2(x)w_z^2(y)\,dx\,dy + \int_{\mathbb{R}^3}|w_z|^3\,dx\right)} \\ &= \frac{\max\{K_\infty^2, a_\infty\}}{\min\{\mathcal{K}^2, \mathcal{A}\}} \end{aligned}$$

Then, by using (2.50) we obtain (2.51) as desidered. □

Proof[Theorem 1.1] The proof can be done in a similar way to that of Theorem 1.2 of [7]. We sketch it for the sake of completeness.
By Lemmas 4.4 and 4.5 there exists $\bar{\rho} > 0$ such that for all $\rho \geq \bar{\rho}$

$$c_\infty < \max_{|z|=\rho} I(\Gamma[z]) < b_0.$$

We take

$$Q := \Gamma(\bar{B}_{\bar{\rho}}(0)); \quad S := \{u \in \mathcal{N} \ : \ \beta(u) = 0\}.$$

As done in [7], we can prove that S and ∂Q link. Therefore we can define

$$d := \inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u))$$

where $\mathcal{H} := \left\{h \in C(Q, \mathcal{N}) \ : \ h_{|\partial Q} = id\right\}$. Then, by using also Lemma 4.6, $d \in (c_\infty, 2c_\infty)$ and hence (PS) condition is satisfied (Lemma 4.3). By the Linking Theorem d is a critical value of I. This shows the existence of a non trivial solution of (SN'). To get positive solution, let us observe that the energy of a changing sign solution $v = v^+ - v^-$ of (SN') is such that

$$I(v) \geq 2m = 2c_\infty.$$

Since d is less then $2c_\infty$, then we obtain the positivity of u. □

References

[1] A. Bahri, P.L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. Henri Poincaré **14**, (1997), no. 3, 365–413.

[2] V. Benci, G. Cerami, *Positive solutions of some nonlinear elliptic problems in exterior domains*, Arch. Ration. Mech. Anal. **99**, (1987) 283–300.

[3] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal. **82**, (1983), no. 4, 347–375.

[4] H. Brezis, E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. A.M.S. **88**, (1983), 486–490.

[5] G. Cerami, *Some nonlinear elliptic problems in unbounded domains*, Milan J. Math. **74**, (2006), 47–77.

[6] G. Cerami, D. Passaseo, *The effect of concentrating potentials in some singularly perturbed problems*, Calc. Var. PDE **17**, (2003), no. 3, 257–281.

[7] G. Cerami, G. Vaira, *Positive solutions for some non autonomous Schrödinger-Poisson Systems*, J. Diff. Equations **248**, (2010), no. 3, 521–543.

[8] W. X. Chen, C. M. Li, B. Ou, *Classification of solutions for an integral equations*, Comm. Pure. Appl. Math. **59**, (2006), 330–343.

[9] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47**, (1974), 324–353.

[10] C. M. Li, L. Ma, *Uniqueness of positive bound states to Schrödinger system with critical exponents*, SIAM J. Math. Anal. **40**, no. 3, (2008), 1049–1057.

[11] E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Stud. Appl. Math. **57**, (1977), 93–105.

[12] E. H. Lieb, M. Loss, *Analysis second ed.*, Graduate Studies in Mathematics, vol. **14**, American Math. Society, Providence, RI, 2001.

[13] L. Ma, D. Z. Chen, *Radial symmetry and uniqueness result for an integral system*, Math. Comput. Model, **49**, (2009), 379–385.

[14] L. Ma, D. Z. Chen, *Radial symmetry and monotonicity results for an integral system*, J. Math. Anal. Appl., **342**, (2008), 943–949.

[15] L. Ma, L. Zhao, *Classification of positive solitary solutions of the nonlinear Choquard equation*, Arch. Rational Mech. Anal., **195**, (2010), 455–467.

[16] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Analysis **237**, (2006), 655–674.

[17] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.

[18] M. Struwe, *Variational Methods*, Springer, 1996.

[19] M. Willem, *Minimax Theorems*, PNDEA Vol. 24, Birkhäuser, 1996.

[20] G. Vaira, *Ground states for Schrödinger-Poisson type systems*, preprint.