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Standing Waves of Some Coupled Nonlinear Schrödinger Equations

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Abstract

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1 Introduction

This work is motivated by the recent paper [10], dealing with systems of nonlinear Schrödinger (NLS) equation like

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 &= \mu_1 u_1^3 + \beta u_2^2 u_1, \quad u_1 \in W^{1,2}(\mathbb{R}^n), \\ -\Delta u_2 + \lambda_2 u_2 &= \mu_2 u_2^3 + \beta u_1^2 u_2, \quad u_2 \in W^{1,2}(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $n = 2, 3$, $\lambda_j, \mu_j > 0$, $j = 1, 2$, and $\beta \in \mathbb{R}$. Coupled NLS equations arise in nonlinear Optics. For example, if $\mathbf{E}(x, z)$ denotes the complex envelope of an Electric field, planar stationary light beams propagating in the z direction in a

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non-linear medium are described, up to rescaling, by a nonlinear Schrödinger (NLS) equation like

$$\mathrm{i}\,\mathbf{E}_z + \mathbf{E}_{xx} + \kappa|\mathbf{E}|^2\mathbf{E} = 0,$$

where i denotes the imaginary unit and subscripts denote derivatives. In the sequel the constant κ is assumed to be *positive*, corresponding to the fact that the medium is *self-focusing.* Without loss of generality we will put $\kappa = 1$. If $\mathbf{E}$ is the sum of two right- and left-hand polarized waves a_1E_1 and a_2E_2, $a_j \in \mathbb{R}$, the preceding equation gives rise to the following system of NLS equations for E_j, $j = 1,2$ (see e.g. [1, 12, 13])

$$\begin{cases} \mathrm{i}\,(E_1)_z + (E_1)_{xx} + (a_1^2|E_1|^2 + a_2^2|E_2|^2)E_1 &= 0, \\ \mathrm{i}\,(E_2)_z + (E_2)_{xx} + (a_1^2|E_1|^2 + a_2^2|E_2|^2)E_2 &= 0. \end{cases} \tag{1.2}$$

We will look for *standing waves*, namely for solutions to (1.2) of the form $E_j(z,x) = e^{\mathrm{i}\lambda_j z}u_j(x)$, where $\lambda_j > 0$ and $u_j(x)$ are real valued functions which solve the system

$$\begin{cases} -(u_1)_{xx} + \lambda_1 u_1 &= (a_1^2u_1^2 + a_2^2u_2^2)u_1, \\ -(u_2)_{xx} + \lambda_2 u_2 &= (a_1^2u_1^2 + a_2^2u_2^2)u_2. \end{cases} \tag{1.3}$$

If we take the coupling factor $a_2^2 := \beta$ as a parameter and let the coefficients of u_j^3 be different, say $\mu_j > 0$, (1.3) becomes

$$\begin{cases} -u_1'' + \lambda_1 u_1 &= \mu_1 u_1^3 + \beta u_2^2 u_1, \\ -u_2'' + \lambda_2 u_2 &= \mu_2 u_2^3 + \beta u_1^2 u_2, \end{cases} \tag{1.4}$$

which is the ODE counterpart of (1.1).

Roughly, we will show that there exist $\Lambda' \geq \Lambda > 0$, depending upon λ_j, μ_j, such that (1.4) has a radially symmetric solution $(u_1,u_2) \in W^{1,2}(\mathbb{R}^n)\times W^{1,2}(\mathbb{R}^n)$, with $u_1, u_2 > 0$, provided $\beta \in (0,\Lambda) \cup (\Lambda', +\infty)$. Moreover, for $\beta > \Lambda'$, these solutions are ground states, in the sense that they have minimal energy and their Morse index is 1. It is worth pointing out that for any β (1.1) has a pair of *semi-trivial* solutions having one component equal to zero. These solutions have the form $(U_1, 0)$, $(0, U_2)$ where U_j is the positive radial solution of $-\Delta u + \lambda_j u = \mu_j u^3$, $u \in W^{1,2}(\mathbb{R}^n)$. Of course, we look for solutions different from the preceding ones. On the other hand, the presence of $(U_1, 0)$ and $(0, U_2)$ can be usefully exploited to prove our existence results. Actually, the main idea is to show that the Morse index of $(U_1, 0)$ and $(0, U_2)$ changes with β: for $\beta < \Lambda$ small their index is 1, while for $\beta > \Lambda'$ their index is greater or equal than 2. This fact, jointly an appropriate use of the method of natural constraint, allows us to prove the existence of bound and ground states as outlined before.

Most of the papers on NLS systems deal with the existence of specific explicit solutions, see e.g. [7], or with results based on numerical arguments. Recently, some more general rigorous achievements have been obtained, see [5, 10, 14]. In particular, our results on the existence of bound states improve and precise the ones in [10]. The reader should pay attention that these bound states have Morse index greater or equal than 2. In spite of this fact, they are called ground states in [10]. See Remarks 2.2 and 5.8 later on. The existence results for $\beta > \Lambda'$ is new.

A preliminary announcements of the present work appeared in [2]. Let us also mention that, when [2] was completed, we became aware of [11] which contains results similar to some of ours.

The paper contains 5 more sections. In Section 2 we introduce notation and give the definition of bound and ground state. Sections 3 and 4 contain, respectively, some preliminary material on the method of the *natural constraint* and the key lemmas for getting the main existence results, which are stated and proved in Section 5. Extensions to systems with more than two equations are discussed in Section 6.

2 Notation and Preliminary Definitions

Let us introduce the following notation

- $E = W^{1,2}(\mathbb{R}^n)$, the standard Sobolev space, endowed with scalar product and norm
$$(u \mid v)_j = \int_{\mathbb{R}^n} [\nabla u \cdot \nabla v + \lambda_j uv] dx, \qquad \|u\|_j^2 = (u \mid u)_j, \quad j = 1, 2;$$
- $\mathbb{E} = E \times E$; the elements in $\mathbb{E}$ will be denoted by $\mathbf{u} = (u_1, u_2)$; as a norm in $\mathbb{E}$ we will take $\|\mathbf{u}\|^2 = \|u_1\|_1^2 + \|u_2\|_2^2$;
- we set $\mathbf{0} = (0, 0)$;
- for $\mathbf{u} \in \mathbb{E}$, the notation $\mathbf{u} \geq \mathbf{0}$, resp. $\mathbf{u} > \mathbf{0}$, means that $u_j \geq 0$, resp. $u_j > 0$, for all $j = 1, 2$;
- H denotes the space of *radially symmetric* functions in E;
- $\mathbb{H} = H \times H$.

For $u \in E$, resp. $\mathbf{u} \in \mathbb{E}$, we set

$$\begin{aligned}
I_j(u) &= \tfrac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + \lambda_j u^2) dx - \tfrac{1}{4} \mu_j \int_{\mathbb{R}^n} u^4 dx, \\
F(\mathbf{u}) &= \tfrac{1}{4} \int_{\mathbb{R}^n} (\mu_1 u_1^4 + \mu_2 u_2^4)\, dx. \\
G(\mathbf{u}) &= G(u_1, u_2) = \tfrac{1}{2} \int_{\mathbb{R}^n} u_1^2 u_2^2 dx, \\
\Phi(\mathbf{u}) &= \Phi(u_1, u_2) = I_1(u_1) + I_2(u_2) - \beta\, G(u_1, u_2) \\
&= \tfrac{1}{2}\|\mathbf{u}\|^2 - F(\mathbf{u}) - \beta\, G(\mathbf{u}).
\end{aligned}$$

Let us remark that F and G make sense because $E \hookrightarrow L^4(\mathbb{R}^n)$ for $n = 2, 3$ Any critical point $\mathbf{u} \in \mathbb{E}$ of Φ gives rise to a solution of (1.1). If $\mathbf{u} \neq \mathbf{0}$ we say that such a critical point (solution) is non-trivial. We say that a solution $\mathbf{u}$ of (1.1) is *positive* if $\mathbf{u} > \mathbf{0}$.

Among non-trivial solutions of (1.1), we shall distinguish the *bound states* from the *ground states.*

Definition 2.1 *We say that* $\mathbf{u} \in \mathbb{E}$ *is a non-trivial bound state of (1.1) if* $\mathbf{u}$ *is a non-trivial critical point of* Φ. *A positive bound state* $\mathbf{u} > \mathbf{0}$ *such that its energy is minimal among all the non-trivial bound states, namely*

$$\Phi(\mathbf{u}) = \min\{\Phi(\mathbf{v}) : \mathbf{v} \in \mathbb{E} \setminus \{\mathbf{0}\},\ \Phi'(\mathbf{v}) = 0\}, \tag{2.1}$$

is called a ground state of (1.1).

About the definition of ground states, some remark is in order.

Remark 2.2 In the case of a single NLS equation,

$$-\Delta u + \lambda u = \mu u^3, \qquad u \in E, \tag{2.2}$$

a ground state is a solution $u^* > 0$ of (2.2) such that $I(u^*) = \min\{I(u) : u \in E \setminus \{0\},\ u \geq 0,\ I'(u) = 0\}$, where

$$I(u) = \tfrac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + \lambda u^2)dx - \tfrac{1}{4}\mu \int_{\mathbb{R}^n} u^4 dx.$$

A main feature of a ground state is that it arises as a Mountain-Pass critical point of I and hence has Morse index equal to 1. Moreover, it is known that a necessary condition for the standing wave $E(z,x) = e^{i\lambda z}u^*(x)$ to be orbitally stable is that the Morse index of u^* is at most 1, see [6]. It is natural to expect that similar properties are shared by a ground state of coupled NLS systems. In particular, a ground state should have Morse index 1. We will show, see Remark 5.7, that the ground states we will find have this property. We also anticipate that in [10] a different definition of ground state is given, according to which it might have Morse index 2. See Remark 5.8 later on. ■

3 The Natural Constraint

In order to find critical points of Φ, let us set

$$\Psi(\mathbf{u}) = (\Phi'(\mathbf{u}) \mid \mathbf{u}) = \|\mathbf{u}\|^2 - 4F(\mathbf{u}) - 4\beta G(\mathbf{u}),$$

and introduce the so called Nehari manifold:

$$\mathcal{M} = \{\mathbf{u} \in \mathbb{H} \setminus \{\mathbf{0}\} : \Psi(\mathbf{u}) = 0\}.$$

Plainly, $\mathcal{M}$ contains all the non-trivial critical points of Φ on $\mathbb{H}$. Let us recall, for the reader convenience, some well known facts. First of all, for any $\mathbf{v} \in \mathbb{H} \setminus \{\mathbf{0}\}$ one has that

$$t\mathbf{v} \in \mathcal{M} \iff t^2\|\mathbf{v}\|^2 = t^4\left[4F(\mathbf{v}) + 4\beta G(\mathbf{v})\right].$$

As a consequemce, for all $\mathbf{v} \in \mathbb{H} \setminus \{\mathbf{0}\}$, there exists a unique $t > 0$ such that $t\mathbf{v} \in \mathcal{M}$. Moreover, since F, G are homogeneous with degree 4, that $\exists\, \rho > 0$ such that

$$\|\mathbf{u}\|^2 \geq \rho, \qquad \forall\, \mathbf{u} \in \mathcal{M}. \tag{3.1}$$

Furthermore, from (3.1) it follows that

$$(\Psi'(\mathbf{u}) \mid \mathbf{u}) = -2\|\mathbf{u}\|^2 < 0, \qquad \forall\, \mathbf{u} \in \mathcal{M}. \tag{3.2}$$

From (3.1) and (3.2) we infer that $\mathcal{M}$ is a smooth complete manifold of codimension 1 in $\mathbb{E}$. Moreover, if $\mathbf{u} \in \mathcal{M}$ is a critical point of Φ constrained on $\mathcal{M}$, then there exists $\omega \in \mathbb{R}$ such that

$$\Phi'(\mathbf{u}) = \omega \Psi'(\mathbf{u}).$$

Then one finds $\Psi(\mathbf{u}) = (\Phi'(\mathbf{u}) \mid \mathbf{u}) = \omega(\Psi'(\mathbf{u}) \mid \mathbf{u})$. Since $\Psi(\mathbf{u}) = 0$ while, by (3.2), $(\Psi'(\mathbf{u}) \mid \mathbf{u}) < 0$, we infer that $\omega = 0$ and thus $\Phi'(\mathbf{u}) = 0$.

In conclusion, we can state the following Proposition.

Proposition 3.1 *$\mathbf{u} \in \mathbb{H}$ is a non-trivial critical point of Φ if and only if $\mathbf{u} \in \mathcal{M}$ and is a constrained critical point of Φ on $\mathcal{M}$.*

Because of this, $\mathcal{M}$ is called a *natural constraint* for Φ. A remarkable advantage of working on the Nehari manifold is that Φ is bounded from below on $\mathcal{M}$. Actually, from $\Psi(\mathbf{u}) = 0$ and the definition of $\mathcal{M}$, it follows that

$$\|\mathbf{u}\|^2 = 4F(\mathbf{u}) + 4\beta G(\mathbf{u}). \tag{3.3}$$

Substituting into Φ we get

$$\Phi(\mathbf{u}) = \tfrac{1}{4}\|\mathbf{u}\|^2, \qquad \forall\, \mathbf{u} \in \mathcal{M}. \tag{3.4}$$

This jointly with (3.1) implies there exists $C > 0$ such that

$$\Phi(\mathbf{u}) \geq C > 0, \qquad \forall\, \mathbf{u} \in \mathcal{M}. \tag{3.5}$$

Concerning the Palais-Smale (PS) condition, the following Lemma holds.

Lemma 3.2 *Φ satisfies the (PS) condition on $\mathcal{M}$.*

Proof. Let $\mathbf{u}_n \in \mathcal{M}$ be a sequence such that $\Phi(\mathbf{u}_n) \to c > 0$. From (3.4) it follows that $\mathbf{u}_n$ is bounded and, without relabeling, we can assume that $\mathbf{u}_n \rightharpoonup \mathbf{u}_0$. Since H is compactly embedded into $L^4(\mathbb{R}^n)$, $n = 2, 3$ (see [15]), we infer that $F(\mathbf{u}_n) + \beta G(\mathbf{u}_n) \to F(\mathbf{u}_0) + \beta G(\mathbf{u}_0)$. Moreover using (3.3) jointly with (3.1), one has that $\exists\, c > 0$ such that $F(\mathbf{u}_n) + \beta G(\mathbf{u}_n) \geq c$ and then $\mathbf{u}_0 \neq 0$. Letting $\nabla_{\mathcal{M}}\Phi(\mathbf{u}) = \Phi'(\mathbf{u}) - \omega\Psi'(\mathbf{u})$, $\omega \in \mathbb{R}$, denote the constrained gradient of Φ on $\mathcal{M}$, suppose that $\nabla_{\mathcal{M}}\Phi(\mathbf{u}_n) \to 0$. Taking the scalar product with $\mathbf{u}_n$ and recalling that $(\Phi'(\mathbf{u}_n) \mid \mathbf{u}_n) = \Psi(\mathbf{u}_n) = 0$, we find that $\omega_n(\Psi'(\mathbf{u}_n) \mid \mathbf{u}_n) \to 0$ and this, jointly with (3.2), implies that $\omega_n \to 0$. Since, in addition, $\|\Psi(\mathbf{u}_n)\| \leq c_1 < +\infty$, we deduce that $\Phi'(\mathbf{u}_n) \to 0$. Finally, from $\lim(\Phi'(\mathbf{u}_n) \mid \mathbf{u}_0) = 0$ it readily follows that $\mathbf{u}_n \to \mathbf{u}_0$ strongly. ■

Remark 3.3 From the preceding arguments it follows immediately that $\min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{M}\}$ is achieved giving rise to a non-negative solution of (1.1). However, such an existence result is useless without any further specification. Actually, for every $\beta \in \mathbb{R}$, (1.1) already possesses two explicit solutions given by

$$\mathbf{u}_1 = (U_1, 0), \quad \mathbf{u}_2 = (0, U_2),$$

where U_j is radial positive and satisfies $-\Delta u + \lambda_j u = \mu_j u^3$. In other words, to find a non-trivial existence result, one has to find solutions having *both the components* not identically zero. ■

4 Evaluating the Morse index of $\mathbf{u}_j$

The aim of the following arguments is to show that there exist non-negative solutions of (1.1) different from $\mathbf{u}_j$, $j = 1, 2$. First, let us remark that if we let U denote the unique positive radial solution of $-\Delta u + u = u^3$, there holds

$$U_j(x) = \sqrt{\frac{\lambda_j}{\mu_j}}\, U(\sqrt{\lambda_j}\, x), \quad j = 1, 2.$$

Next, we set

$$\gamma_1^2 = \inf_{\varphi \in H\setminus\{0\}} \frac{\|\varphi\|_2^2}{\int U_1^2 \varphi_+^2}, \qquad \gamma_2^2 = \inf_{\varphi \in H\setminus\{0\}} \frac{\|\varphi\|_1^2}{\int U_2^2 \varphi_+^2},$$

and

$$\Lambda = \min\{\gamma_1^2, \gamma_2^2\}, \qquad \Lambda' = \max\{\gamma_1^2, \gamma_2^2\}.$$

The next Proposition shows that the nature of $\mathbf{u}_j$ changes in dependence of β, Λ, Λ'.

Proposition 4.1 *(i)* $\forall \beta < \Lambda$, $\mathbf{u}_j$, $j = 1, 2$, *are strict local minima of* Φ *on* $\mathcal{M}$.
(ii) If $\beta > \Lambda'$ *then* $\mathbf{u}_j$ *are saddle points of* Φ *on* $\mathcal{M}$. *In particular,* $\inf_{\mathcal{M}} \Phi < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.

To prove this Proposition, we will evaluate the Morse index of $\mathbf{u}_j$, as critical points of Φ constrained on $\mathcal{M}$. Let $D^2\Phi_{\mathcal{M}}(\mathbf{u}_j)$ denote the second derivative of Φ constrained on $\mathcal{M}$. Since $\Phi'(\mathbf{u}_j) = 0$, then one has that

$$D^2\Phi_{\mathcal{M}}(\mathbf{u}_j)[\mathbf{h}]^2 = \Phi''(\mathbf{u}_j)[\mathbf{h}]^2, \qquad \forall\, \mathbf{h} \in T_{\mathbf{u}_j}\mathcal{M}. \tag{4.1}$$

Similarly, if $\mathcal{N}_j$ denotes the Nehari manifolds relative to I_j, $j = 1, 2$,

$$\mathcal{N}_j = \{u \in H \setminus \{0\} : (I_j'(u)|u) = 0\} = \{u \in H \setminus \{0\} : \|u\|_j^2 = \mu_j \int u^4\},$$

then, from the fact that $I_j'(U_j) = 0$ it follows

$$D^2(I_j)_{\mathcal{N}_j}(U_j)[h]^2 = I_j''(U_j)[h]^2, \qquad \forall\, h \in T_{U_j}\mathcal{N}_j. \tag{4.2}$$

Notice that U_j is the minimum of I_j on $\mathcal{N}_j$ and thus, using also (4.2), one has that $\exists\, c_j > 0$ such that

$$I_j''(U_j)[h_j]^2 \geq c_j \|h_j\|_j^2, \qquad j = 1, 2. \tag{4.3}$$

The next lemma shows the relationship between $T_{\mathbf{u}_j}\mathcal{M}$ and $T_{U_j}\mathcal{N}_j$.

Lemma 4.2 *There holds:* $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_j}\mathcal{M} \Leftrightarrow h_j \in T_{U_j}\mathcal{N}_j, \quad j = 1, 2.$

Proof. One has that $h_j \in T_{U_j}\mathcal{N}_j$ iff $(U_j \mid \phi)_j = 2\mu_j \int U_j^3 \phi$, while $\mathbf{h} \in T_{\mathbf{u}}\mathcal{M}$ iff

$$(u_1 \mid h_1)_1 + (u_2 \mid h_2)_2 = 2 \int_{\mathbb{R}^n} (\mu_1 u_1^3 h_1 + \mu_2 u_2^3 h_2) + \beta \int_{\mathbb{R}^n} (u_1 h_1 u_2^2 + u_1^2 u_2 h_2).$$

Thus $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_j}\mathcal{M}$, iff $(U_j \mid h_j)_j = 2\mu_j \int U_j^3 h_j$. ∎

Proof of Proposition 4.1-(i). If $\mathbf{u} = (u_1, u_2) \in \mathbb{H}$ and $\mathbf{h} = (h_1, h_2) \in \mathbb{H}$ one has

$$\Phi''(\mathbf{u})[\mathbf{h}]^2 = I_1''(u_1)[h_1]^2 + I_2''(u_2)[h_2]^2 - \beta \int_{\mathbb{R}^n} \left(u_1^2 h_2^2 + u_2^2 h_1^2 + 4 u_1 u_2 h_1 h_2\right).$$

In particular, if $\mathbf{u} = \mathbf{u}_1$ we get

$$\Phi''(\mathbf{u}_1)[\mathbf{h}]^2 = I_1''(U_1)[h_1]^2 + \|h_2\|_2^2 - \beta \int U_1^2 h_2^2. \tag{4.4}$$

Now, let us take $\mathbf{h} = (h_1, h_2) \in T_{\mathbf{u}_1}\mathcal{M}$. Then, using Lemma 4.2, we have that $h_1 \in T_{U_1}\mathcal{N}_1$ and hence (4.3) yields

$$I_1''(U_1)[h_1]^2 \geq c_1 \|h_1\|_1^2.$$

Substituting into (4.4) we infer.

$$\Phi''(\mathbf{u}_1)[\mathbf{h}]^2 \geq c_1 \|h_1\|_1^2 + \|h_2\|_2^2 - \beta \int U_1^2 h_2^2, \qquad \forall\, \mathbf{h} \in T_{\mathbf{u}_1}\mathcal{M},$$

and this, using the definition of γ_1, yields

$$\Phi''(\mathbf{u}_1)[\mathbf{h}]^2 \geq c_1 \|h_1\|_1^2 + \|h_2\|_2^2 - \frac{\beta}{\gamma_1^2} \|h_2\|_2^2, \qquad \forall\, \mathbf{h} \in T_{\mathbf{u}_1}\mathcal{M}.$$

Therefore, if $\beta < \gamma_1^2$ there exists $c_2 > 0$ such that

$$\Phi''(\mathbf{u}_1)[\mathbf{h}]^2 \geq c_1 \|h_1\|_1^2 + c_2 \|h_2\|_2^2, \qquad \forall\, \mathbf{h} \in T_{\mathbf{u}_1}\mathcal{M}. \tag{4.5}$$

Taking into account (4.1), we infer that (4.5) implies that $\mathbf{u}_1$ is a local strict minimum of Φ on $\mathcal{M}$.

Similarly, if $\beta < \gamma_2^2$, $\exists\, c_i' > 0$ such that

$$\Phi''(\mathbf{u}_2)[\mathbf{h}]^2 \geq c_1' \|h_1\|_1^2 + c_2' \|h_2\|_2^2, \qquad \forall\, \mathbf{h} \in T_{\mathbf{u}_2}\mathcal{M}.$$

proving that $\mathbf{u}_2$ is a local strict minimum of Φ on $\mathcal{M}$. ∎

Proof of Proposition 4.1-(ii). We will evaluate $\Phi''(\mathbf{u}_1)$ on tangent vectors of the form $(0, h_2)$. According to (4.4) one has

$$\Phi''(\mathbf{u}_1)[(0,h_2)]^2 = \|h_2\|_2^2 - \beta \int_{\mathbf{R}^n} U_1^2 h_2^2.$$

Moreover, Lemma 4.2 implies that $(0, h_2) \in T_{\mathbf{u}_1}\mathcal{M}$, for all $h_2 \in H$. If $\beta > \gamma_1^2$, there exists $\widetilde{h_2} \in H \setminus \{0\}$ such that

$$\gamma_1^2 < \frac{\|\widetilde{h_2}\|_2^2}{\int U_1^2 \widetilde{h_2}^2} < \beta,$$

and hence

$$\Phi''(\mathbf{u}_1)[(0,\widetilde{h_2})]^2 = \|\widetilde{h_2}\|_2^2 - \beta \int_{\mathbf{R}^n} U_1^2 \widetilde{h_2}^2 < 0.$$

Similarly, if $\beta > \gamma_2^2$, there exist $\widehat{h_1} \in H \setminus \{0\}$ such that $\Phi''(\mathbf{u}_2)[(\widehat{h_1}, 0)]^2 < 0$. ∎

Remark 4.3 What we have really proved is that $\mathbf{u}_j$ is a minimum, resp. a saddle point, provided $\beta < \gamma_j^2$, resp. $\beta > \gamma_j^2$, $j = 1, 2$. ∎

5 Existence Results

According to Proposition 3.1, in order to find a non-trivial solution of (1.1) it suffices to find a critical point of Φ constrained on $\mathcal{M}$. The following lemma is a direct consequence of Proposition 4.1 and Lemma 3.2.

Lemma 5.1 *(i) If $\beta < \Lambda$, then Φ has a Mountain-Pass critical point $\mathbf{u}^*$ on $\mathcal{M}$, and there holds $\Phi(\mathbf{u}^*) > \max\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.*

(ii) If $\beta > \Lambda'$ then Φ has a positive global minimum $\widetilde{\mathbf{u}}$ on $\mathcal{M}$, and there holds $\Phi(\widetilde{\mathbf{u}}) < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.

Proof. (*i*) Proposition 4.1-(*i*) and Lemma 3.2 allow us to apply the Mountain Pass theorem to Φ on $\mathcal{M}$, yielding a critical point $\mathbf{u}^*$ of Φ. By the Mountain Pass Theorem, it also follows that $\Phi(\mathbf{u}^*) > \max\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.

(*ii*) By Lemma 3.2 the $\inf_{\mathcal{M}} \Phi$ is achieved at some $\widetilde{\mathbf{u}} > 0$. Moreover, if $\beta > \Lambda'$, Proposition 4.1-(*ii*) implies $\Phi(\mathbf{u}^*) < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$. ∎

Remark 5.2 In order to prove the preceding Lemma, it would be enough that only one among $\mathbf{u}_j$ is a minimum or a saddle. For example, if $\Phi(\mathbf{u}_1) < \Phi(\mathbf{u}_2)$ to prove (*i*) it suffices that the $\mathbf{u}_2$ is a minimum. According to Remark 4.3, this is the case provided $\beta < \gamma_2^2$. Unfortunately, a straight calculation shows that if $\Phi(\mathbf{u}_1) < \Phi(\mathbf{u}_2)$ then $\gamma_2^2 < \gamma_1^2$. Hence $\mathbf{u}_1$ is a minimum as well. Same remark holds for the case (*ii*). ∎

We are now in position to state our general existence results.

5.1 Existence of ground states

Concerning ground states, our main result is the following

Theorem 5.3 *If $\beta > \Lambda'$ then* (1.1) *has a (positive) radial ground state* $\widetilde{\mathbf{u}}$.

Proof. Lemma 5.1-(*ii*) yields a critical point $\widetilde{\mathbf{u}} \in \mathcal{M}$ which is a non-trivial solution of (1.1). To complete the proof we have to show that $\widetilde{\mathbf{u}} > \mathbf{0}$ and is a ground state in the sense of Definition 2.1. To prove this fact, we argue as follows. Since $|\widetilde{\mathbf{u}}| = (|\widetilde{u}_1|, |\widetilde{u}_2|)$ also belongs to $\mathcal{M}$ and $\Phi(|\widetilde{\mathbf{u}}|) = \Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{M}\}$, we can assume that $\widetilde{\mathbf{u}} \geq \mathbf{0}$. By the maximum principle, $\widetilde{\mathbf{u}} > \mathbf{0}$. It remains to prove that

$$\Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{v}) : \mathbf{v} \in \mathbb{E} \setminus \{\mathbf{0}\},\ \Phi'(\mathbf{v}) = 0\}. \tag{5.1}$$

By contradiction, let $\widetilde{\mathbf{v}} \in \mathbb{E}$ be a non-trivial critical point of Φ such that

$$\Phi(\widetilde{\mathbf{v}}) < \Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{M}\}. \tag{5.2}$$

Setting $\mathbf{w} = |\widetilde{\mathbf{v}}|$ there holds

$$\Phi(\mathbf{w}) = \Phi(\widetilde{\mathbf{v}}), \qquad \Psi(\mathbf{w}) = \Psi(\widetilde{\mathbf{v}}). \tag{5.3}$$

Let $\mathbf{w}^\star \in \mathbb{H} \setminus \{\mathbf{0}\}$ denote the Schwartz symmetric function associated to $\mathbf{w}$. Using the property of Schwartz symmetrization, there holds $\|\mathbf{w}^\star\|^2 \leq \|\mathbf{w}\|^2$ and $F(\mathbf{w}^\star) + \beta G(\mathbf{w}^\star) \geq F(\mathbf{w}) + \beta G(\mathbf{w})$. Thus $\Psi(\mathbf{w}^\star) \leq \Psi(\mathbf{w})$. Using the second of (5.3) and the fact that $\widetilde{\mathbf{v}}$ is a critical point of Φ, we get $\Psi(\mathbf{w}) = \Psi(\widetilde{\mathbf{v}}) = 0$ and there exists a unique $t \in (0, 1]$ such that $t\,\mathbf{w}^\star \in \mathcal{M}$. Moreover,

$$\Phi(t\,\mathbf{w}^\star) = \tfrac{1}{4} t^2 \|\mathbf{w}^\star\|^2 \leq \tfrac{1}{4} \|\mathbf{w}\|^2 = \Phi(\mathbf{w}).$$

This, the first of (5.3) and (5.2) yield

$$\Phi(t\,\mathbf{w}^\star) \leq \Phi(\mathbf{w}) = \Phi(\widetilde{\mathbf{v}}) < \Phi(\widetilde{\mathbf{u}}) = \min\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathcal{M}\},$$

which is a contradiction, since $t\,\mathbf{w}^\star \in \mathcal{M}$. This shows that (5.1) holds and completes the proof of Theorem 5.3. ■

5.2 Existence of bound states

Concerning the existence of positive bound states, the following result holds.

Theorem 5.4 *If $\beta < \Lambda$, then* (1.1) *has a radial bound state $\mathbf{u}^*$ such that $\mathbf{u}^* \neq \mathbf{u}_j$, $j = 1, 2$. Furthermore, if $\beta \in (0, \Lambda)$, then $\mathbf{u}^* > 0$.*

Proof. If $\beta < \Lambda$, a straight application of Lemma 5.1-(*i*) yields a non-trivial solution $\mathbf{u}^* \in \mathcal{M}$ of (1.1), which corresponds to a mountain-Pass critical point of Φ on $\mathcal{M}$. Moreover, $\Phi(\mathbf{u}^*) > \max\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$ implies that $\mathbf{u}^* \neq \mathbf{u}_j$, $j = 1, 2$.

To show that $\mathbf{u}^* > \mathbf{0}$ provided $\beta \in (0, \Lambda)$, let us introduce the functional

$$\Phi^+(\mathbf{u}) = \tfrac{1}{2}\|\mathbf{u}\|^2 - F(\mathbf{u}^+) - \beta\, G(\mathbf{u}^+),$$

where $\mathbf{u}^+ = (u_1^+, u_2^+)$ and $u^+ = \max\{u, 0\}$. Consider the corresponding Nehari manifold

$$\mathcal{M}^+ = \{\mathbf{u} \in \mathbb{H} \setminus \{0\} : (\nabla\Phi^+(\mathbf{u}) \mid \mathbf{u}) = 0\}.$$

Repeating with minor changes the arguments carried out in Section 3, one readily shows that what is proved in such a section, still holds with Φ and $\mathcal{M}$ substituted by Φ^+ and $\mathcal{M}^+$. In particular, Proposition 3.1 and Lemma 3.2 hold true for Φ^+ and $\mathcal{M}^+$. On the other hand, Proposition 4.1-(i) cannot be proved as before, because Φ^+ is not C^2. To circumvent this difficulty, we argue as follows.

Consider an ε-neighborhood $V_\varepsilon \subset \mathcal{M}$ of $\mathbf{u}_1$. For each $\mathbf{u} \in V_\varepsilon$ there exists $T(\mathbf{u}) > 0$ such that $T(\mathbf{u})\mathbf{u} \in \mathcal{M}^+$. Actually $T(\mathbf{u})$ satisfies

$$\|\mathbf{u}\|^2 = 4\,T^2(\mathbf{u})\left[F(\mathbf{u}^+) + \beta\, G(\mathbf{u}^+)\right],$$

and since $\|\mathbf{u}\|^2 = 4\,[F(\mathbf{u}) + \beta\, G(\mathbf{u})]$, we get

$$[F(\mathbf{u}) + \beta\, G(\mathbf{u})] = T^2(\mathbf{u})\left[F(\mathbf{u}^+) + \beta\, G(\mathbf{u}^+)\right]. \tag{5.4}$$

Let us point out that $F(\mathbf{u}^+) + \beta\, G(\mathbf{u}^+) \le F(\mathbf{u}) + \beta\, G(\mathbf{u})$ and this implies that $T(\mathbf{u}) \ge 1$. Moreover, since $\lim_{\mathbf{u}\to\mathbf{u}_1} F(\mathbf{u}^+) + \beta\, G(\mathbf{u}^+) = F(\mathbf{u}_1) > 0$ it follows that there exist $\varepsilon > 0$ and $c > 0$ such that

$$F(\mathbf{u}^+) + \beta\, G(\mathbf{u}^+) \ge c, \qquad \forall\, \mathbf{u} \in V_\varepsilon.$$

This and (5.4) imply that the map $\mathbf{u} \to T(\mathbf{u})\mathbf{u}$ is a homeomorphism, locally near $\mathbf{u}_1$. In particular, there are ε-neighborhoods $V_\varepsilon \subset \mathcal{M}$, $W_\varepsilon \subset \mathcal{M}^+$ of $\mathbf{u}_1$ such that for all $\mathbf{v} \in W_\varepsilon$, there exists $\mathbf{u} \in V_\varepsilon$ such that $\mathbf{v} = T(\mathbf{u})\mathbf{u}$. Finally, from $\Phi^+(\mathbf{v}) = \frac{1}{4}\|\mathbf{v}\|^2$, see (3.4), and the fact that $T(\mathbf{u}) \ge 1$, we infer

$$\Phi^+(\mathbf{v}) = \tfrac{1}{4}\|\mathbf{v}\|^2 = \tfrac{1}{4}T^2(\mathbf{u})\|\mathbf{u}\|^2 \ge \tfrac{1}{4}\|\mathbf{u}\|^2 = \Phi(\mathbf{u}).$$

Since, according to Proposition 3.1, $\mathbf{u}_1$ is a local minimum of Φ on $\mathcal{M}$, and thus

$$\Phi^+(\mathbf{v}) \ge \Phi(\mathbf{u}) \ge \Phi(\mathbf{u}_1) = \Phi^+(\mathbf{u}_1), \quad \forall\, \mathbf{v} \in W_\varepsilon,$$

proving that $\mathbf{u}_1$ is a local strict minimum for Φ^+ on $\mathcal{M}^+$. A similar proof can be carried out for $\mathbf{u}_2$.

From the preceding arguments, it follows that Φ^+ has a Mountain Pass critical point $\mathbf{u}^* \in \mathcal{M}^+$, which gives rise to a solution of

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 &= \mu_1 (u_1^+)^3 + \beta (u_2^+)^2 u_1^+, \quad u_1 \in W^{1,2}(\mathbb{R}^n), \\ -\Delta u_2 + \lambda_2 u_2 &= \mu_2 (u_2^+)^3 + \beta (u_1^+)^2 u_2^+, \quad u_2 \in W^{1,2}(\mathbb{R}^n). \end{cases} \tag{5.5}$$

In particular, one finds that $u_j \ge 0$. In addition, since $\mathbf{u}^*$ is a Mountain-Pass critical point, one has that $\Phi^+(\mathbf{u}^*) > \max\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$. Let us also remark that $\mathbf{u}^* \in \mathcal{M}^+$ implies that $\mathbf{u}^* \neq \mathbf{0}$ and hence $u_2^* \equiv 0$ implies that $u_1^* \not\equiv 0$. From $\Phi'(u_1^*, 0) = 0$ it follows that u_1^* is a non-trivial solution of

$$-\Delta u + \lambda_1 u = \mu_1 u_+^3, \qquad u \in H.$$

Since $u_1^* \geq 0$ and $u_1^* \not\equiv 0$, then $u_1^* = U_1$, namely $\mathbf{u}^* = (U_1, 0) = \mathbf{u}_1$. This is in contradiction to $\Phi^+(\mathbf{u}^*) > \Phi(\mathbf{u}_1)$, proving that $u_2^+ \not\equiv 0$. A similar argument proves that $u_1^* \not\equiv 0$. Since both u_1^* and u_2^* are $\not\equiv 0$, using the maximum principle we get $u_1^* > 0$ and $u_2^* > 0$. ■

5.3 Further existence results

It can be useful to state an existence result based upon a more explicit (though less precise) estimates on β. For this, it is convenient to set

$$\alpha_j = \mu_j \left(\frac{\lambda_k}{\lambda_j}\right)^{1-\frac{n}{4}}, \quad k, j = 1, 2,\ k \neq j \tag{5.6}$$

$$\zeta_j = \max\left\{\mu_j \left(\frac{\lambda_k}{\lambda_j}\right)^{1-\frac{n}{2}}, \mu_j \frac{\lambda_k}{\lambda_j}\right\}, \quad k, j = 1, 2,\ k \neq j. \tag{5.7}$$

Lemma 5.5 *There holds* $\alpha_j \leq \gamma_j^2 \leq \zeta_j, \qquad j = 1, 2.$

Proof. Let

$$\sigma_j^2 \equiv \inf_{\varphi \in E \setminus \{0\}} \frac{\|\varphi\|_j^2}{\|\varphi\|_{L^4}^2} = \inf_{\varphi \in E,\, \|\varphi\|_{L^4} = 1} \|\varphi\|_j^2,$$

denote the best Sobolev constant in the embedding of $(W^{1,2}(\mathbb{R}^n), \|\cdot\|_j)$ into $L^4(\mathbb{R}^n)$. It is easy to see that σ_j is achieved at

$$\widehat{v}_j(x) = \sigma_j^{-1} \sqrt{\lambda_j} U(\sqrt{\lambda_j} x),$$

and one has $\sigma_j^4 = \lambda_j^2 \int U^4(\sqrt{\lambda_j} x) dx = \lambda_j^{2-\frac{n}{2}} \int U^4(x) dx$. In particular, there holds

$$\frac{\sigma_k^2}{\sigma_j^2} = \left(\frac{\lambda_k}{\lambda_j}\right)^{1-\frac{n}{4}}, \quad (k \neq j).$$

Using this equation, the Hölder inequality and the fact that $\sigma_j^4 = \mu_j^2 \int U_j^4$, we get for $j \neq k$,

$$\begin{aligned}\gamma_j^2 &= \inf_{\varphi \in H \setminus \{0\}} \frac{\|\varphi\|_k^2}{\int_{\mathbb{R}^n} U_j^2 \varphi^2} \geq \inf_{\varphi \in H \setminus \{0\}} \left[\frac{\|\varphi\|_k^2}{\left(\int_{\mathbb{R}^n} U_j^4\right)^{1/2} \left(\int_{\mathbb{R}^n} \varphi^4\right)^{1/2}}\right] \\ &= \frac{\sigma_k^2}{\left(\int_{\mathbb{R}^n} U_j^4\right)^{1/2}} = \frac{\mu_j \sigma_k^2}{\sigma_j^2} = \mu_j \left(\frac{\lambda_k}{\lambda_j}\right)^{1-\frac{n}{4}} = \alpha_j.\end{aligned}$$

As for the upper bound, we take e.g. $\varphi = U_2$. Then

$$\gamma_1^2 \le \frac{\|U_2\|_2^2}{\int_{\mathbf{R}^n} U_1^2 U_2^2} \le \frac{\mu_1\mu_2\|U_2\|_2^2}{\lambda_1\lambda_2 \int_{\mathbf{R}^n} U^2(\sqrt{\lambda_1}x)U^2(\sqrt{\lambda_2}x)}.$$

If $\lambda_1 > \lambda_2$ (which implies that $U(\sqrt{\lambda_1}x) \le U(\sqrt{\lambda_2}x)$), we get

$$\gamma_1^2 \le \frac{\mu_1\mu_2\|U_2\|_2^2}{\lambda_1\lambda_2 \int_{\mathbf{R}^n} U^4(\sqrt{\lambda_1}x)} = \frac{\lambda_1\mu_1\mu_2^2 \int_{\mathbf{R}^n} U_2^4}{\lambda_2\sigma_1^4} = \frac{\lambda_1\mu_1\sigma_2^4}{\lambda_2\sigma_1^4} = \mu_1 \left(\frac{\lambda_2}{\lambda_1}\right)^{1-\frac{n}{2}}.$$

Similarly, if $\lambda_1 \le \lambda_2$ then one has $\gamma_1^2 \le \mu_1 \frac{\lambda_2}{\lambda_1}$. Same arguments lead to prove that $\gamma_2^2 \le \max\{\mu_2 \frac{\lambda_1}{\lambda_2}, \mu_2 (\frac{\lambda_1}{\lambda_2})^{1-\frac{n}{2}}\}$. ■

The preceding lemma, jointly with Theorems 5.3 and 5.4, immediately implies

Theorem 5.6 *(i) If* $0 < \beta < \min\{\alpha_1, \alpha_2\}$, *then* (1.1) *has a positive radial bound state* $\mathbf{u}^*$.

(ii) If $\beta > \max\{\zeta_1, \zeta_2\}$. *Then* (1.1) *has a positive radial ground state* $\tilde{\mathbf{u}}$.

5.4 Some Remarks

We end this section with some remarks. The first one deals with the Morse index of the solutions found above.

Remark 5.7 We can evaluate the Morse index of the solutions found above. Here, by definition, the Morse index of a critical point $\mathbf{v}$ of Φ is the maximal dimension of the subspace where $\Phi''(\mathbf{v})$ is negative defined. Now, from (3.2) it readily follows that for every $\mathbf{u} \in \mathcal{M}$, $\Phi''(\mathbf{u})[\mathbf{u}]^2 < 0$. This implies that the Morse index of any critical point $\mathbf{v}$ of Φ equals its Morse index as constrained critical point of Φ on $\mathcal{M}$, augmented by 1. As a consequence, the bound states $\mathbf{u}^*$ found for $\beta < \Lambda$ have Morse index ≥ 2, while the Morse index of the ground states $\tilde{\mathbf{u}}$, found for $\beta > \Lambda'$ is one. Let us point out explicitly that this agrees with the discussion carried out in Remark 2.2-(*i*) about the features of ground states. ■

We now make a comparison with [10] and [11].

Remark 5.8 In [10] a *ground state* is defined as a positive solution of (1.1) whose energy is minimal among all the *positive solutions* of (1.1). A ground state in this weaker sense is found as a constrained minimum on the manifold

$$\mathbf{N} = \{\mathbf{u} = (u_1, u_2) \in \mathbb{E} : u_j \ge 0,\ u_j \not\equiv 0,\ (I_j(u_j) \mid u_j) = 2\beta G(\mathbf{u}),\ j = 1, 2\},$$

under the assumption that the matrix A with entries $a_{jj} = \mu_j$, $a_{jk} = \beta$ $(j \ne k)$ is positive definite, which holds provided $0 < \beta < \sqrt{\mu_1\mu_2}$, [10, Thm. 2]. Due to the fact that $\mathbf{N}$ has codimension 2, these solutions have Morse index 2. Indeed, we suspect that they are nothing but the bound states we have found in Theorem

5.4. Moreover, in [10] (we restrict our comments to a system of 2 equations, but similar remarks holds for systems of more equations) nothing is proved in the case that the matrix A is not positive definite, while we do not make any assumption of this kind. In addition, no result dealing with the existence for β large is given in [10]. Furthermore, another new feature of our results is that the bounds Λ, Λ' depend not only on μ_j, but also on the ratio λ_1/λ_2. On the other hand, let us point out that the case in which $\lambda_1 \neq \lambda_2$ is the most interesting one. Actually, if $\lambda_1 = \lambda_2$ one has that $\Lambda = \min\{\mu_1, \mu_2\}$, $\Lambda' = \max\{\mu_1, \mu_2\}$ and for all $\beta \in (0, \Lambda) \cup (\Lambda', +\infty)$, there are explicit solutions of (1.1). For example, if say $\lambda_1 = \lambda_2 = 1$, they are given by $(a_{1\beta}U_1, a_{2\beta}U_2)$ where

$$a_{k\beta}^2 = \frac{\mu_k(\mu_j - \beta)}{\mu_k\mu_j - \beta^2}, \quad k \neq j, \ k = 1, 2. \tag{5.8}$$

Unfortunately, these solutions might coincide with those found in [10, Thm. 2] or in our Theorems 5.3, 5.4.

In [11, Theorem 2.3] an existence result like our Theorem 5.3 is proved. As for bound states, [11] only contains a result dealing with the $\min_{\mathcal{M}} \Phi$, like the one indicated above in Remark 3.3. No existence of *positive* bound states such as Theorem 5.4 is proved. ■

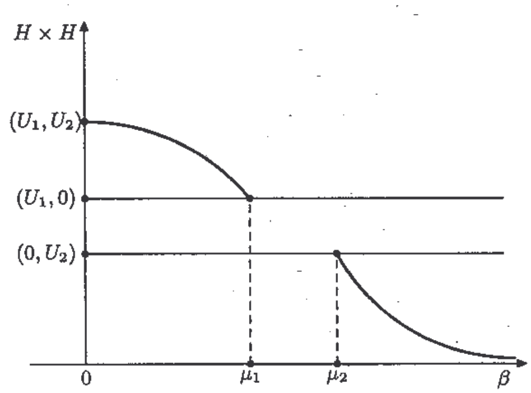

Figure 1: Bifurcation diagram of the family $(a_{1\beta}U_1, a_{2\beta}U_2)$, in the case $\lambda_1 = \lambda_2 = 1$ and $\mu_1 < \mu_2$ (the diagram is suggestive, only).

Our last remark is concerned with some open questions.

Remark 5.9 If $\lambda_1 = \lambda_2 = 1$ and $\mu_1 < \mu_2$, the family $(a_{1\beta}U_1, a_{2\beta}U_2)$, $\beta \in (0, \mu_1) \cup (\mu_2, +\infty)$, described in (5.8) is the union of two branches as indicated in the figure 1 above: a first branch joins (U_1, U_2) (for $\beta = 0$) and $(U_1, 0)$ (for $\beta = \mu_1$); the second one starts from $(0, U_2)$ at $\beta = \mu_2$ and tends to $\mathbf{0}$ as $\beta \to +\infty$. It would be interesting to see if the set of solutions of (1.1) has a global feature of this sort in the general case that $\lambda_1 \neq \lambda_2$. Let us remark that we can use perturbation arguments, see Theorem 6.4 in the next section, to prove that from $\beta = 0$ and $\mathbf{u} = \mathbf{u}_{12}$) emanates a branch Σ of positive solutions $(\beta, \mathbf{u}_\beta) \in \mathbb{R} \times \mathbb{H}$ of $\Phi'_\beta(\mathbf{u}) = \mathbf{0}$, provided $\beta = \varepsilon$ is sufficiently small. Here we write Φ_β instead of Φ to emphasize the dependence on β.

Another natural question is to study the case in which $\beta \in [\Lambda, \Lambda']$, which is not studied in the present work. Moreover, if $\Lambda = \Lambda'$, it would be interesting to see whether the set of bound and ground states found before is a continuous curve. If this happens, Remark 5.7 implies that the Morse index of the solutions along the branch changes across $\beta = \Lambda = \Lambda'$, which is therefore a bifurcation point, according to the results of [8].

Finally, for all $\beta \in \mathbb{R}$ the equation $\Phi'_\beta(\mathbf{u}) = \mathbf{0}$ has the "semi-trivial" solutions given by $(\beta, \mathbf{u}_j)$, $j = 1, 2$. From Proposition 4.1, see also Remark 5.7 above, we deduce that the Morse index of these solutions is 1 if $\beta < \gamma_j^2$, while it is ≥ 2 as $\beta > \gamma_j^2$. Then $\beta = \gamma_j^2$ is a bifurcation point from the family $\{(\beta, \mathbf{u}_j)\}$. We neither know if the new solutions bifurcating from γ_j^2 have both the components positive, nor their relationship with the set of ground and bound states found before. ■

6 Systems with more than 2 equations

In this final section we shortly discuss the case of systems with more than 2 equations. To simplify notation, we focus on a system of 3 equations like

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 &= \mu_1 u_1^3 + \beta_{12} u_1 u_2^2 + \beta_{13} u_1 u_3^2, \quad u_1 \in W^{1,2}(\mathbb{R}^n), \\ -\Delta u_2 + \lambda_2 u_2 &= \mu_2 u_2^3 + \beta_{12} u_2 u_1^2 + \beta_{23} u_2 u_3^2, \quad u_2 \in W^{1,2}(\mathbb{R}^n), \\ -\Delta u_3 + \lambda_3 u_3 &= \mu_3 u_3^3 + \beta_{13} u_3 u_1^2 + \beta_{23} u_3 u_2^2, \quad u_3 \in W^{1,2}(\mathbb{R}^n). \end{cases} \tag{6.1}$$

It would be easy to extend these results to systems with any number of equations.

To be short, we shall indicate only the main changes needed in the preceding arguments. We do not change notation, which have a natural meaning. For example, $\mathbb{E} = E \times E \times E$, $\mathbf{u} = (u_1, u_2, u_3)$, $F(\mathbf{u}) = \sum_{j=1,2,3} \mu_j \int_{\mathbb{R}^n} u_j^4 dx$, $G(\mathbf{u}) = \sum_{j \neq k} \beta_{jk} \int_{\mathbb{R}^n} u_j^2 u_k^2 dx$,

$$\mathcal{M} = \{\mathbf{u} \in \mathbb{E} \setminus \{\mathbf{0}\} : \|\mathbf{u}\|^2 = 4F(\mathbf{u}) + 4G(\mathbf{u})\},$$

etc.

Remark 6.1 There are now 3 explicit solutions of (6.1) given by $\mathbf{u}_1 = (U_1, 0, 0)$, $\mathbf{u}_2 = (0, U_2, 0)$ and $\mathbf{u}_3 = (0, 0, U_3)$. Furthermore, there could be solutions $\mathbf{u} = (u_1, u_2, u_3)$ of (6.1) having one component equal to zero. Precisely, if the component u_k is identically zero , then the remaining pair (u_i, u_j). $i, j \neq k$ solves the system

$$\begin{cases} -\Delta u_i + \lambda_i u_i &= \mu_i u_i^3 + \beta_{ij} u_i u_j^2, \quad u_i \in W^{1,2}(\mathbb{R}^n), \\ -\Delta u_j + \lambda_j u_j &= \mu_j u_j^3 + \beta_{ij} u_j u_i^2, \quad u_j \in W^{1,2}(\mathbb{R}^n), \end{cases}$$

which nothing but (1.1) with $\beta = \beta_{ij}$. In other words, for any pair (u_i, u_j) solving the preceding system, the function $\mathbf{u}$ having the remaining component $= 0$ is a solution of (6.1). We will denote by $\mathbf{u}_{ij}$ these specific solutions. ■

To repeat the arguments carried out in Sections 4 and 5, we need to introduce

$$\gamma_{jk}^2 = \inf_{\varphi \in H\setminus\{0\}} \frac{\|\varphi\|_k^2}{\int U_j^2 \varphi^2}, \qquad j,k = 1,2,3, \quad k \neq j.$$

Let us point out that, with this notation, the γ_j, $j = 1,2$, introduced in Section 4 would become

$$\gamma_1 = \gamma_{12}, \qquad \gamma_2 = \gamma_{21}. \tag{6.2}$$

As before, one has that $\mathbf{h} = (h_1, h_2, h_3)$ belongs to the tangent space $T_{\mathbf{u}_j}\mathcal{M}$ iff $h_j \in T_{U_j}\mathcal{N}_j$. Then, repeating the proof of Proposition 3.1, we find that $\mathbf{u}_j$ is a strict local minimum on $\mathcal{M}$ provided

$$\beta_{jk} < \gamma_{jk}^2, \qquad \forall\, k \neq j,$$

Similarly, $\mathbf{u}_j$ is a saddle point on $\mathcal{M}$ provided

$$\exists\, k \neq j \quad \text{such that} \quad \beta_{jk} > \gamma_{jk}^2.$$

From this it follows that all $\mathbf{u}_j$ are strict local minima of Φ on $\mathcal{M}$ provided

$$\beta_{jk} < \gamma_{jk}^2 \qquad \forall\, j,k = 1,2,3,\ k \neq j, \tag{6.3}$$

while all the $\mathbf{u}_j$ are saddle points provided

$$\forall\, j = 1,2,3, \quad \exists k \neq j \quad \text{such that} \quad \beta_{jk} > \gamma_{jk}^2. \tag{6.4}$$

As in Lemma 5.1, we infer that Φ has a Mountain-Pass critical point $\mathbf{u}^*$, resp. a minimum $\tilde{\mathbf{u}}$, on $\mathcal{M}$ provided (6.3), resp. (6.4), holds. Moreover, one has that

$$\Phi(\mathbf{u}^*) > \max_{j=1,2,3} \Phi(\mathbf{u}_j), \qquad \Phi(\tilde{\mathbf{u}}) < \min_{j=1,2,3} \Phi(\mathbf{u}_j). \tag{6.5}$$

Repeating the arguments carried out in Section 5 one shows that $\mathbf{u}^* \geq 0$ and $\tilde{\mathbf{u}} \geq 0$. Finally, using the second of (6.5) one proves that $\tilde{\mathbf{u}} > 0$. On the other hand, the first of (6.5) implies that $\mathbf{u}^* \neq \mathbf{u}_j$, $j = 1,2,3$, but does not exclude that $\mathbf{u}^*$ coincides with one of the solutions $\mathbf{u}_{ij}$, discussed in Remark 6.1.

In conclusion, we can state

Theorem 6.2 *(i) If* (6.3) *holds, then* (6.1) *has a radial bound state* $\mathbf{u}^*$ *such that* $\mathbf{u}^* \neq \mathbf{u}_j$, $j = 1,2,3$. *Moreover, if* $\beta_{jk} > 0$ *(and* (6.3) *holds), then* $\mathbf{u}^* \geq 0$.

(ii) If (6.4) *holds, then* (6.1) *has a positive radial ground state* $\tilde{\mathbf{u}}$.

Remark 6.3 If $j,k = 1,2$ the preceding assumptions on β_{jk} are the the counterpart of the assumptions made in Theorems 5.3 and 5.4. Actually, since $\beta_{jk} = \beta_{kj}$, (6.4), resp. (6.4), is equivalent to require that $\beta = \beta_{12} = \beta_{21} < \min\{\gamma_{12}^2, \gamma_{21}^2\}$, resp. $\beta = \beta_{12} = \beta_{21} > \max\{\gamma_{12}^2, \gamma_{21}^2\}$. Since $\gamma_1 = \gamma_{12}$ and $\gamma_2 = \gamma_{21}$, see (6.2), these are nothing but the assumptions made in Theorems 5.3 and 5.4. ■

As anticipated before, statement $(i)$of Theorem 6.2 is weaker that the corresponding statement in Theorem 5.4, since $\mathbf{u}^*$ might coincide with one of $\mathbf{u}_{ij}$. Motivated by this, let us prove that a *positive* bound state exists provided the β_{jk} are sufficiently small. Let $\mathbf{z} = (U_1, U_2, U_3)$.

Theorem 6.4 *If $\beta_{jk} = \varepsilon b_{jk} \geq 0$ for all $j, k = 1, 2, 3$, $j \neq k$, then for ε sufficiently small* (6.1) *has a radial bound state $\mathbf{u}_\varepsilon > 0$ such that $\mathbf{u}_\varepsilon \to \mathbf{z}$ as $\varepsilon \to 0$.*

Proof. If $\beta_{jk} = \varepsilon b_{jk}$, then Φ takes the form

$$\Phi(\mathbf{u}) = \Phi_\varepsilon(\mathbf{u}) = \Phi_0(\mathbf{u}) - \varepsilon \widetilde{G}(\mathbf{u}),$$

where

$$\Phi_0(\mathbf{u}) = \sum_{j=1,2,3} I_j(u_j), \qquad \widetilde{G}(\mathbf{u}) = \int_{\mathbf{R}^n} \sum_{j \neq k} b_{jk} u_j^2 u_k^2 dx.$$

Let us consider the critical point of the unperturbed functional Φ_0 given by $\mathbf{z} = (U_1, U_2, U_3)$. It is well known that each U_j is a non-degenerate critical point of I_j on H, see [9]. This immediately implies that $\mathbf{z}$ is a non-degenerate critical point of Φ_0 on $\mathbb{H}$ and a straight application of the Local Inversion Theorem yields the existence of critical points $\mathbf{u}_\varepsilon$ of Φ_ε on $\mathbb{H}$, provided ε is sufficiently small. Moreover, $\mathbf{u}_\varepsilon \to \mathbf{z}$ as $\varepsilon \to 0$. To complete the proof it remains to show that $\mathbf{u}_\varepsilon > 0$. We will follow, with suitable modifications, an arguments of [4], see also the proof of Theorem 5.10 in [3]. Let us set $\mathbf{u}_\varepsilon^+ = (u_{1\varepsilon}^+, u_{2\varepsilon}^+, u_{3\varepsilon}^+)$ and $\mathbf{u}_\varepsilon^- = (u_{1\varepsilon}^-, u_{2\varepsilon}^-, u_{3\varepsilon}^-)$. Since

$$\|U_j\|_j = \inf_{u \in H \setminus \{0\}} \frac{\|u\|_j^2}{\left(\mu_j \int_{\mathbf{R}^n} u^4 dx\right)^{1/2}},$$

it follows that

$$\|u_{j\varepsilon}^\pm\|_j^2 \geq \|U_j\|_j \left(\mu_j \int_{\mathbf{R}^n} (u_{j\varepsilon}^\pm)^4 dx\right)^{1/2}. \tag{6.6}$$

From (6.1) one infers

$$\begin{aligned} \|u_{j\varepsilon}^\pm\|_j^2 &= \mu_j \int_{\mathbf{R}^n} (u_{j\varepsilon}^\pm)^4 dx + \varepsilon \int_{\mathbf{R}^n} \left[(u_{j\varepsilon}^\pm)^2 \sum_{k \neq j} b_{jk} u_{k\varepsilon}^2 \right] dx \\ &\leq \mu_j \int_{\mathbf{R}^n} (u_{j\varepsilon}^\pm)^4 dx + \varepsilon \left(\int_{\mathbf{R}^n} (u_{j\varepsilon}^\pm)^4 dx \right)^{1/2} \sum_{k \neq j} b_{jk} \left(\int_{\mathbf{R}^n} u_{k\varepsilon}^4 \right)^{1/2}. \end{aligned}$$

This jointly with (6.6) yields

$$\|u_{j\varepsilon}^\pm\|_j^2 \leq \frac{\|u_{j\varepsilon}^\pm\|_j^4}{\|U_j\|_j^2} + \varepsilon\, \vartheta_\varepsilon \frac{\|u_{j\varepsilon}^\pm\|_j^2}{\|U_j\|_j},$$

where

$$\vartheta_\varepsilon = \mu_j^{-1/2} \sum_{k \neq j} b_{jk} \left(\int_{\mathbf{R}^n} u_{k\varepsilon}^4 \right)^{1/2}.$$

Notice that $\varepsilon\vartheta_\varepsilon \to 0$ as $\varepsilon \to 0$. Hence, for each j such that $\|u_{j\varepsilon}^\pm\|_j > 0$, one has

$$\|u_{j\varepsilon}^\pm\|_j^2 \geq \|U_j\|_j^2 + o(1), \qquad (\|u_{j\varepsilon}^\pm\|_j > 0). \tag{6.7}$$

Since $\mathbf{u}_\varepsilon \to \mathbf{z} > 0$, then $\|u_{j\varepsilon}^+\|_j > 0$ for all $j = 1,2,3$ and (6.7) yields

$$\|u_{j\varepsilon}^+\|_j^2 \geq \|U_j\|_j^2 + o(1), \qquad \forall\, j = 1,2,3. \tag{6.8}$$

Now, suppose by contradiction that there exists $k \in \{1,2,3\}$ such that $\|u_{k\varepsilon}^-\|_k > 0$. Then (6.7) implies $\|u_{k\varepsilon}^-\|_k^2 \geq \|U_k\|_k^2 + o(1)$ and hence

$$\|\mathbf{u}_\varepsilon^-\|^2 = \sum_{j=1,2,3} \|u_{j\varepsilon}^-\|_j^2 \geq \|U_k\|_k^2 + o(1). \tag{6.9}$$

Next, we evaluate

$$\Phi(\mathbf{u}_\varepsilon) = \tfrac{1}{4}\|\mathbf{u}_\varepsilon\|^2 = \tfrac{1}{4}\left[\|\mathbf{u}_\varepsilon^+\|^2 + \|\mathbf{u}_\varepsilon^-\|^2\right].$$

Using (6.8) and (6.9), we infer

$$\Phi(\mathbf{u}_\varepsilon) \geq \tfrac{1}{4} \sum_{j=1,2,3} \|U_j\|_j^2 + \tfrac{1}{4}\|U_k\|_k^2 + o(1). \tag{6.10}$$

On the other hand, since $\mathbf{u}_\varepsilon \to \mathbf{z}$ we also find

$$\Phi(\mathbf{u}_\varepsilon) = \tfrac{1}{4}\|\mathbf{u}_\varepsilon\|^2 \to \tfrac{1}{4}\|\mathbf{z}\|^2 = \tfrac{1}{4} \sum_{j=1,2,3} \|U_j\|_j^2.$$

This is in contradiction with (6.10), proving that $\mathbf{u}_\varepsilon \geq 0$. Finally, using once more that $\mathbf{u}_\varepsilon \to \mathbf{z}$ and applying the maximum principle it follows that $\mathbf{u}_\varepsilon > 0$. ∎

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