Stable determination of the surface impedance of an obstacle by far field measurements *

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Abstract

We deal with the inverse scattering problem of determining the surface impedance of a partially coated obstacle. We prove a stability estimate of logarithmic type for the impedance term by the far field measurements.

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1 Introduction

We consider the scattering of an acoustic incident time-harmonic plane wave, at a given wave number $k > 0$ and at a given incident direction $\omega \in \mathbb{S}^2$, by an obstacle $D \subset \mathbb{R}^3$ partially coated by a material with surface impedance λ. Such a problem is modeled by the following mixed boundary value problem for the Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ u = 0, & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \nu} + i\lambda(x)u = 0, & \text{on } \Gamma_I, \end{cases} \tag{1.1}$$

where $u = u^s + \exp(ikx \cdot \omega)$ is the total field, that is given as the sum of the scattered wave u^s and the incident plane waves $\exp(ikx \cdot \omega)$ and where Γ_I, Γ_D are two open and connected portions of the boundary ∂D such that $\partial D = \overline{\Gamma_I \cup \Gamma_D}$.
Moreover, the scattered field u^s is required to satisfy the so-called *Sommerfeld radiation condition*

$$\lim_{r\to\infty} r \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0, \qquad r = \|x\|. \tag{1.2}$$

It is well-known, that the scattered field u^s has the following asymptotic behavior

$$u^s(x) = \frac{\exp(ikr)}{r} \left\{ u_\infty(\hat{x}) + O\left(\frac{1}{r}\right) \right\} , \tag{1.3}$$

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as r tends to ∞, uniformly with respect to $\hat{x} = \frac{x}{\|x\|}$ and where u_∞ is the so-called far field pattern of the scattered wave (see for instance [11]).
The *inverse scattering problem* that we examine here consists in the determination of the surface impedance $\lambda(x)$ by the knowledge of the far field pattern, provided some suitable a priori assumptions on the impedance are made.
Such a problem, in two dimensions, has been recently studied by F. Cakoni and D. Colton in [7]. The authors have provided a variational method for the determination of the essential supremum of the surface impedance when the far field data are available.
In this paper, we shall deal with the stability issue, namely we will prove a stability estimate of logarithmic type for the surface impedance by the far field measurements.
Let us point out that a stability result for this type of problem has been proved in [15] by C. Labreuche under the assumption of an analytic boundary. The new feature of the present paper consists in a reduced assumption on the regularity of the boundary, namely we shall assume that Γ_I is a $C^{1,1}$ portion of ∂D. Thus it turns out that the argument of analytic extension used in [15] cannot be applied.
The stable recovering of the surface impedance needs some a priori mild assumptions on the impedance itself. The additional a priori information that we require on the unknown surface impedance λ, is an a priori bound on its Lipschitz continuity, that is we assume that for a given positive constant Λ, the following holds

$$\|\lambda\|_{C^{0,1}(\Gamma_I)} \leqslant \Lambda. \tag{1.4}$$

Moreover, we prescribe the following uniform lower bound

$$\lambda(x) \geqslant \lambda_0, \quad \text{for every } x \in \Gamma_I, \tag{1.5}$$

where λ_0 is a given positive constant.
In order to treat the inverse scattering problem we first need to analyze the *direct* one. In Section 3, indeed, following the arguments of potential theory developed in [8], we observe that the direct scattering problem is well posed (see Lemma 3.1). The proof relies on the fact that the mixed boundary value problem (1.1) can be reformulated as a system of boundary integral equations. Moreover, we prove, (see Theorem 3.2), that the solution and its first order derivatives are Hölder continuous in a neighborhood of the portion Γ_I, where the impedance takes place. The proof is based on the Moser's iteration technique. Finally in Corollary 3.3, we obtain a uniform lower bound for the total field u on sets away from the obstacle.
In Section 4, we deal with the inverse scattering problem. The underlying ideas and the main tools that lead to the stability result can be outlined as follows.

i) As first step we evaluate how much the error on the far field can affect the values of the field near the scatterer;

ii) in the second step we are concerned with a stability estimate of the field at the boundary in terms of the near field;

iii) finally, as last step, we obtain a stability result for the impedance λ by the estimate of the field at the boundary.

Let us start the analysis of Section 4 illustrating the arguments introduced in the step iii) of the list above.
By the impedance condition in (1.1) we can formally compute λ as

$$\lambda(x) = \frac{i}{u(x)} \frac{\partial u(x)}{\partial \nu(x)} \ . \tag{1.6}$$

Since u may vanish in some points of Γ_I, it follows that the quotient in (1.6) may be undetermined. In this respect, we found it necessary to evaluate the local vanishing rate of the solution on the boundary. To establish such a control we shall make use of quantitative estimates of unique continuation. We first obtain, in Lemma 4.5, a *volume doubling inequality* at the boundary, namely

$$\int_{\Gamma_{I,2\rho}(x_0)} |u|^2 \leqslant const. \int_{\Gamma_{I,\rho}(x_0)} |u|^2 \ , \tag{1.7}$$

where $\Gamma_{I,\rho}(x_0)$ and $\Gamma_{I,2\rho}(x_0)$ are the portions of the balls centered at the boundary point x_0 of radius ρ and 2ρ respectively, contained in $\mathbb{R}^3 \setminus \overline{D}$, (see (2.13) for a precise definition).
In order to obtain the formula in (1.7), we have adapted the arguments developed in [2] for the more general setting of complex valued solutions which is required by the boundary value problem (1.1).
A further difficulty in dealing with such arguments is due to the fact that the techniques used in [2] apply to an homogeneous Neumann boundary condition. We overcome such a difficulty by performing a suitable change of the independent variable, (see Proposition 4.3), that fits our problem under the assumptions required in [2]. Moreover, well-known stability estimates for the Cauchy problem [17], allow us to reformulate the *volume* doubling inequality at the boundary deriving in Theorem 4.6 a new one on the boundary, that is a *surface doubling inequality*

$$\int_{\Delta_{I,2\rho}(x_0)} |u|^2 \leqslant const. \int_{\Delta_{I,\rho}(x_0)} |u|^2 \ , \tag{1.8}$$

where $\Delta_{I,\rho}(x_0)$ and $\Delta_{I,2\rho}(x_0)$ are the portions of the boundary of $\Gamma_{I,\rho}(x_0)$ and $\Gamma_{I,2\rho}(x_0)$ respectively, which have non empty intersection with ∂D, (see (2.14) for a precise definition).
The surface doubling inequality allows us to apply the theory of *Muckenhoupt weights* [9] which, in particular, implies the existence of some exponent $p > 1$ such that $|u|^{-\frac{2}{p-1}}$ is integrable on an inner portion of Γ_I, see Corollary 4.7. This integrability property, as well as the Hölder continuity of the normal derivative, justifies the computation made in (1.6) in the $L^{\frac{2}{p-1}}$ sense.
Let us carry over our analysis by discussing the evaluation introduced in the step i). Such an evaluation, introduced by V. Isakov [13, 14], and then developed by I. Bushuyev [6], concerns a stability estimate for the *near field* in terms of the measurements of the *far field* (see Lemma 4.1). It means that if u_1 and u_2 are two acoustic fields corresponding to impedances λ_1 and λ_2 such that their scattering amplitudes, $u_{1,\infty}$ and $u_{2,\infty}$ respectively, are close

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leqslant \varepsilon, \tag{1.9}$$

then u_1 and u_2 satisfy

$$\|u_1 - u_2\|_{L^2(B_{R_1+1}(0)\setminus B_{R_1}(0))} \leqslant const.\varepsilon^{\alpha(\varepsilon)}, \tag{1.10}$$

where $R_1 > 0$ is a suitable radius such that $B_{R_1}(0) \supset \overline{D}$ and $\alpha(\varepsilon)$ is the function introduced in (2.18).
As last step of this treatment we provide the stability estimate introduced in ii). The proof is based on arguments of quantitative unique continuation, as the *three spheres inequality* and leads to the following estimate

$$\|u_1 - u_2\|_{C^1(\Gamma_I^\rho)} \leqslant const.|\log(\|u_1 - u_2\|^{-1}_{L^2(B_{R_1+1}(0)\setminus B_{R_1}(0))})|^{-\theta}, \tag{1.11}$$

where $\theta > 0$ and where Γ_I^ρ is a given inner portion of Γ_I (see (2.12) for a precise definition).
By combining the stability estimates listed in i) and ii), we obtain a stability result for the total field at the boundary in terms of the measurements of the far field, (see Theorem 4.2).
Finally, as a consequence of Theorem 4.2 and Corollary 4.7, let us formulate the main result of the present paper, that consists in a stability estimate of the surface impedance by the far field measurements, (see Theorem 2.1). Assuming that (1.9) holds, we have shown that the impedances λ_1, λ_2 agree up to an error

$$\left|\log\left(\varepsilon^{-\alpha(\varepsilon)}\right)\right|^{-\theta}. \tag{1.12}$$

Moreover, let us observe that the rate of stability in (1.12) is intermediate between a *log* and a *loglog* rate of stability.

2 Main assumptions and results

2.1 Main hypothesis and notations

Assumptions on the domain.
We shall assume through that D is a bounded domain in $\mathbb{R}^3$, such that diam$D \leqslant d$, with Lipschitz boundary ∂D with constants r_0, M. More precisely, for every $x_0 \in \partial D$, exists a rigid transformation of coordinates under which,

$$D \cap B_{r_0}(x_0) = \{(x', x_3) : x_3 > \gamma(x')\}\ , \tag{2.1}$$

where $x \in \mathbb{R}^3,\ x = (x', x_3)$, with $x' \in \mathbb{R}^2,\ x_3 \in \mathbb{R}$ and

$$\gamma : B'_{r_0}(x_0) \subset \mathbb{R}^2 \to \mathbb{R}$$

satisfying $\gamma(0) = 0$ and

$$\|\gamma\|_{C^{0,1}(B'_{r_0}(x_0))} \leqslant Mr_0,$$

where we denote by

$$\|\gamma\|_{C^{0,1}(B'_{r_0}(x_0))} = \|\gamma\|_{L^\infty(B'_{r_0}(x_0))} + r_0 \sup_{\substack{x,y \in B'_{r_0}(z_0) \\ x \neq y}} \frac{|\gamma(x) - \gamma(y)|}{|x - y|}$$

and $B'_{r_0}(x_0)$ denotes a ball in $\mathbb{R}^2$. Moreover, we assume that the portion of the boundary Γ_I is contained into a surface S_I, which is $C^{1,1}$ smooth with constants r_0, M.

More precisely, for any $x_0 \in S_I$, we have that up to a rigid change of coordinates,

$$S_I \cap B_{r_0}(x_0) = \{(x', x_3) : x_3 = \varphi_I(x')\}, \tag{2.2}$$

where

$$\varphi_I : B'_{r_0}(z_0) \subset \mathbb{R}^2 \to \mathbb{R} \tag{2.3}$$

is a $C^{1,1}$ function satisfying

$$\varphi_I(0) = |\nabla \varphi_I(0)| = 0 \tag{2.4}$$

and

$$\|\varphi_I\|_{C^{1,1}(B'_{r_0}(z_0))} \leqslant M r_0, \tag{2.5}$$

where we denote

$$\|\varphi_I\|_{C^{1,1}(B'_{r_0}(z_0))} = \|\varphi_I\|_{L^\infty(B'_{r_0}(z_0))} + r_0 \|\nabla \varphi_I\|_{L^\infty(B'_{r_0}(z_0))} + \tag{2.6}$$

$$+ {r_0}^2 \sup_{\substack{x,y \in B'_{r_0}(z_0) \\ x \neq y}} \frac{|\nabla \varphi_I(x) - \nabla \varphi_I(y)|}{|x-y|} \quad . \tag{2.7}$$

In particular it follows that, if

$$x_0 \in \Gamma_I \quad \text{and} \quad \text{dist}(x_0, \Gamma_D) > r_0 \ ,$$

then

$$D \cap B_{r_0}(x_0) = \{(x', x_3) \in B_{r_0}(x_0) : x_3 > \varphi_I(x')\} \ , \tag{2.8}$$

where φ_I is the Lipschitz function whose graph locally represents ∂D. Moreover, since $D \cap B_{r_0}(x_0) \cap \Gamma_D = \emptyset$, φ_I must also be the $C^{1,1}$ function whose graph locally represents S_I.

For a sake of simplicity we shall assume that $0 \in D$.
Fixed $R > d$, $\rho \in (0, r_0)$ and $x_0 \in \Gamma_I$, let us define the following sets

$$D^+ = \mathbb{R}^3 \setminus \overline{D}, \tag{2.9}$$

$$D^+_R = B_R(0) \cap D^+, \tag{2.10}$$

$$D^+_{R,\rho} = \{x \in \overline{D^+_R} \ : \text{dist}(x, \Gamma_D) > \rho\}, \tag{2.11}$$

$$\Gamma^\rho_I = \partial D^+_{R,\rho} \cap \Gamma_I, \tag{2.12}$$

$$\Gamma_{I,\rho}(x_0) = B_\rho(x_0) \setminus \overline{D}, \tag{2.13}$$

$$\Delta_{I,\rho}(x_0) = \overline{\Gamma_{I,\rho}(x_0)} \cap \partial D. \tag{2.14}$$

A priori information on the impedance term.
We assume that the impedance coefficient λ belongs to $C^{0,1}(\Gamma_I, \mathbb{R})$ and is such that

$$\lambda(x) \geqslant \lambda_0 > 0 \tag{2.15}$$

for every $x \in \Gamma_I$. Moreover we assume that, for a given constant $\Lambda > 0$, we have that

$$\|\lambda\|_{C^{0,1}(\Gamma_I)} \leqslant \Lambda. \tag{2.16}$$

From now on we shall refer to the *a priori data* as to the following set of quantities: $d, r_0, M, \lambda_0, \Lambda, k, \omega$.
In the sequel we shall denote with $\eta(t)$ a positive increasing function defined on $(0, +\infty)$, that satisfies

$$\eta(t) \leqslant C(\log(t^{-\alpha(t)}))^{-\vartheta}, \quad \text{for every} \ \ 0 < t < 1 \ , \tag{2.17}$$

where

$$\alpha(t) = \frac{1}{1 + \log(\log(t^{-1}) + e)} \ , \tag{2.18}$$

and $C > 0, \theta > 0$ are constants depending on the *a priori data* only.

2.2 The main result

Theorem 2.1 (Stability for λ). *Let $u_i, \ i = 1, 2$, be the weak solutions to the problem* (1.1) *with $\lambda = \lambda_i$ respectively and let $u_{i,\infty}$ be their respectively far field patterns. There exist $\delta > 0, \varepsilon_0 > 0$ constants only depending on the* a priori data*, such that, if for some $\varepsilon, \ 0 < \varepsilon < \varepsilon_0$, we have*

$$\|u_{1,\infty} - u_{2,\infty}\|_{L^2(\partial B_1(0))} \leqslant \varepsilon, \tag{2.19}$$

then

$$\|\lambda_1 - \lambda_2\|_{L^\infty(\Gamma_I^{r_0})} \leqslant \eta(\varepsilon), \tag{2.20}$$

where η is given by (2.17)*.*

3 The direct scattering problem

Let us introduce the following space

$$H^1_{\text{loc}}(D^+) = \{v \in D^*(D^+) : \ v|_{D_R^+} \in H^1(D_R^+), \text{ for every } R > 0 \text{ s.t. } \overline{D} \subset B_R(0)\}$$

where $D^*(D^+)$ is the space of distribution on D^+.
A weak solution to the problem (1.1) is a function $u = \exp(ik\omega \cdot x) + u^s$, where $u^s \in H^1_{\text{loc}}(D^+)$ is a weak solution to the problem

$$\begin{cases} \Delta u^s + k^2 u^s = 0, & \text{in } D^+, \\ u^s = -\exp(ik\omega \cdot x), & \text{on } \Gamma_D, \\ \dfrac{\partial u^s}{\partial \nu} + i\lambda(x) u^s = -\dfrac{\partial}{\partial \nu}\exp(ik\omega \cdot x) - i\lambda(x)\exp(ik\omega \cdot x), & \text{on } \Gamma_I, \\ \lim_{r\to\infty} r\left(\dfrac{\partial u^s}{\partial r}(r\hat{x}) - iku^s(r\hat{x})\right) = 0, & \text{uniformly in } \hat{x}. \end{cases} \tag{3.1}$$

Let us recall that a weak solution of (3.1) is a function $u^s \in H^1_{\mathrm{loc}}(D^+)$, with $u^s|_{\Gamma_D} = -\exp(ik\omega \cdot x)$ in the trace sense, such that, for all test functions $\eta \in H^1(D^+)$ with compact support in $\mathbb{R}^3$ and $\eta|_{\Gamma_D} = 0$, the following holds

$$\int_{D^+} \nabla u^s \cdot \nabla \overline{\eta} - k^2 \int_{D^+} u^s \overline{\eta} = \int_{\Gamma_I} \left(\frac{\partial}{\partial \nu} \exp(ik\omega \cdot x) + i\lambda(x) \exp(ik\omega \cdot x) \right) \overline{\eta} + + \int_{\Gamma_I} ik\lambda u^s \overline{\eta} \ . \tag{3.2}$$

Furthermore, u^s satisfies the asymptotic condition (1.2).

Lemma 3.1 (Well-posedness). *The problem* (3.1) *has one and only one weak solution u^s. Moreover, for every $R > d$, there exists a constant $C_R > 0$ depending on the* a priori data *and on R only, such that the following holds*

$$\|u^s\|_{H^1(D^+_R)} \leqslant C_R \ . \tag{3.3}$$

Proof For the proof we refer to [8, Theorem 2.5], in which the authors, among various results, show that the exterior mixed boundary value problem (3.1) can be reformulated as a 2×2 system of boundary integral equations. In [8], Theorem 2.5 has been proved in two dimensions for a constant λ, however it can be verified that the same techniques can be carried over in three dimensions and with $\lambda = \lambda(x) \in C^{0,1}(\Gamma_I)$. □

Theorem 3.2 ($C^{1,\alpha}$ regularity at the boundary). *Let u be the weak solution to* (1.1)*, then there exists a constant α, $0 < \alpha < 1$, such that for every $R > d$ and $\rho \in (0, r_0)$, $u \in C^{1,\alpha}(D^+_{R,\rho})$. Moreover, there exists a constant $C_{R,\rho} > 0$ depending on the* a priori data*, on R and on ρ only, such that*

$$\|u\|_{C^{1,\alpha}(D^+_{R,\rho})} \leqslant C_{R,\rho} \ . \tag{3.4}$$

Proof From the weak formulation (3.2), it follows that the total field u satisfies

$$\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u \cdot \nabla \overline{\eta} - k^2 \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} u \overline{\eta} = -i \int_{\Delta_{I,\frac{r_0}{2}}(x_0)} \lambda(x) u \overline{\eta} \ ,$$

where $x_0 \in \Gamma_I$ and η is any test function such that $\mathrm{supp}\eta \subset \overline{\Gamma}_{I,\frac{r_0}{2}}(x_0)$.
By (2.16) we have that

$$\left| \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u \cdot \nabla \overline{\eta} \right| \leqslant k^2 \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} |u \overline{\eta}| + \Lambda \int_{\Delta_{I,\frac{r_0}{2}}(x_0)} |u \overline{\eta}| \tag{3.5}$$

and by a trace inequality (see [1, p.114]) it follows that

$$\left| \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u \cdot \nabla \overline{\eta} \right| \leqslant k^2 \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} |u \overline{\eta}| + C\Lambda \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} |\nabla(u \overline{\eta})| \ , \tag{3.6}$$

where $C > 0$ is a constant depending on the *a priori data* only.

By the standard iteration techniques due to Moser (see for instance [12]), we obtain the following local bound for u

$$\|u\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))} \leqslant C\|u\|_{H^1\left(\Gamma_{I,\frac{r_0}{2}}(x_0)\right)} , \tag{3.7}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let us denote by u_1 and u_2 the real and the imaginary part of u respectively. Thus by the elliptic equations in weak form satisfied by u_1 and u_2, it follows that

$$\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u_1 \cdot \nabla \eta - k^2 \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} u_1 \eta = \int_{\Delta_{I,\frac{r_0}{2}}(x_0)} \lambda(x) u_2 \eta , \tag{3.8}$$

$$\int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} \nabla u_2 \cdot \nabla \eta - k^2 \int_{\Gamma_{I,\frac{r_0}{2}}(x_0)} u_2 \eta = -\int_{\Delta_{I,\frac{r_0}{2}}(x_0)} \lambda(x) u_1 \eta , \tag{3.9}$$

where η is any real valued test function such that supp$\eta \subset \overline{\Gamma}_{I,\frac{r_0}{2}}(x_0)$.
By applying again the Moser method to the weak formulations (3.8) and (3.9), we obtain the following bounds of the Hölder continuity of u_1 and u_2, namely

$$\|u_1\|_{C^{0,\alpha}(\Gamma_{I,\frac{r_0}{8}}(x_0))} \leqslant C(\|u_1\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))} + \|u_2\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))}) , \tag{3.10}$$

$$\|u_2\|_{C^{0,\alpha}(\Gamma_{I,\frac{r_0}{8}}(x_0))} \leqslant C(\|u_2\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))} + \|u_1\|_{L^\infty(\Gamma_{I,\frac{r_0}{4}}(x_0))}) , \tag{3.11}$$

where $\alpha, 0 < \alpha < 1, C > 0$ are constants depending on the *a priori data* only.
Combining the two last inequalities with (3.7), we obtain

$$\|u\|_{C^{0,\alpha}(\Gamma_I)} \leqslant C\|u\|_{H^1(D_R^+)} , \tag{3.12}$$

where $C > 0$ are constants depending on the *a priori data* only and $R = d + r_0$.
By (3.3) we have that

$$\|u^s\|_{H^1(D_R^+)} \leqslant C, \tag{3.13}$$

where C is a constant depending on the *a priori data* only. Moreover, since $u = \exp(ik\omega \cdot x) + u^s$, by (3.12) and (3.13), we have that

$$\|u\|_{C^{0,\alpha}(\Gamma_I)} \leqslant C, \tag{3.14}$$

where C is a constant depending on the *a priori data* only. By (3.14) and by (2.16), we have that

$$\frac{\partial u}{\partial \nu}(x) = -i\lambda(x)u(x) \ \in \ C^{0,\alpha}(\Gamma_I). \tag{3.15}$$

By well-known regularity bounds for the Neumann problem (see for instance [3, p.667]) it follows that, for every $R > d, \rho \in (0, r_0), u \in C^{1,\alpha}(D^+_{R,\rho})$ and the following estimate holds

$$\|u\|_{C^{1,\alpha}(D^+_{R,\rho})} \leqslant C_{R,\rho}\left(\|u\|_{C^{0,\alpha}(\Gamma_I^{\frac{\rho}{2}})} + \left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_I^{\frac{\rho}{2}})} + \|u\|_{H^1(D^+_{2R})}\right) , \tag{3.16}$$

where $C_{R,\rho} > 0$ is a constant depending on the *a priori data*, on R and on ρ only. We shall estimate the $C^{0,\alpha}$ norm of $\dfrac{\partial u}{\partial \nu}$ in terms of the *a priori data*, indeed

$$\begin{aligned}\left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_I^{\frac{\rho}{2}}))} &= \sup_{x\in\Gamma_I^{\frac{\rho}{2}}}\left|\frac{\partial u(x)}{\partial \nu}\right| + \left(\frac{\rho}{2}\right)^{\alpha}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{\left|\dfrac{\partial u(x)}{\partial \nu}-\dfrac{\partial u(y)}{\partial \nu}\right|}{|x-y|^{\alpha}} = \\ &\leqslant \sup_{x\in\Gamma_I^{\frac{\rho}{2}}}|\lambda(x)u(x)| + \left(\frac{\rho}{2}\right)^{\alpha}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{|\lambda(x)||u(x)-u(y)|}{|x-y|^{\alpha}} + \\ &\quad + \left(\frac{\rho}{2}\right)^{\alpha}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{|u(y)||\lambda(x)-\lambda(y)|}{|x-y|^{\alpha}} .\end{aligned}$$

Combining (2.16) and (3.14) we obtain

$$\begin{aligned}\left\|\frac{\partial u}{\partial \nu}\right\|_{C^{0,\alpha}(\Gamma_I^{\frac{\rho}{2}})} &\leqslant \Lambda\sup_{x\in\Gamma_I^{\frac{\rho}{2}}}|u(x)| + \Lambda\left(\frac{\rho}{2}\right)^{\alpha}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{|u(x)-u(y)|}{|x-y|^{\alpha}} + \\ &\quad + \left(\frac{\rho}{2}\right)^{\alpha}|\Gamma_I|^{1-\alpha}\|u\|_{C^{0,\alpha}(\Gamma_I)}\sup_{x,y\in\Gamma_I^{\frac{\rho}{2}}}\frac{|\lambda(x)-\lambda(y)|}{|x-y|} \leqslant \\ &\leqslant \bar{C}_{\rho}\end{aligned}$$

where $\bar{C}_{\rho} > 0$ is a constant depending on the *a priori data* and on ρ only. Moreover, since $u = \exp(ik\omega\cdot x) + u^s$, we have that (3.3) yields to

$$\|u\|_{H^1(D^+_{2R})} \leqslant C_R, \tag{3.17}$$

where $C_R > 0$ is a constant depending on the *a priori data* and on R only. Thus, inserting (3.14), (3.17) and (3.17) in (3.16), we obtain that

$$\|u\|_{C^{1,\alpha}(D^+_{R,\rho})} \leqslant C_{R,\rho}, \tag{3.18}$$

where $C_{R,\rho} > 0$ is a constant depending on the *a priori data*, on R and on ρ only. □

Corollary 3.3 (Lower bound). *Let u be the weak solution to* (1.1)*, then there exists a radius $R_0 > 0$ depending on the* a priori data *only, such that*

$$|u(x)| > \frac{1}{2} \ \textit{ for every } x,\ |x| > R_0 . \tag{3.19}$$

Proof Let us choose $R = 4d + 4r_0$. By Theorem 3.2 it follows that there exists a constant $C > 0$ depending on the *a priori data* only, such that

$$\|u\|_{C^{1,\alpha}\left(D^+_{2R,\frac{r_0}{2}}\right)} \leqslant C . \tag{3.20}$$

In particular, by (3.20), it follows that

$$|u^s| \leqslant C_1 ,\ \left|\frac{\partial u^s}{\partial \nu}\right| \leqslant C_1 \ \text{ on } \partial B_R(0), \tag{3.21}$$

where $C_1 > 0$ is a constant depending on the *a priori data* only.
By the Green's formula for the scattered wave u^s (see for instance [11, p.18]), we have that

$$u^s(x) = \int_{\partial B_R(0)} \left(u^s(y) \frac{\partial \phi(x,y)}{\partial \nu(y)} - \frac{\partial u^s(y)}{\partial \nu(y)} \phi(x,y) \right) \mathrm{d}s(y), \qquad |x| > R, \quad (3.22)$$

where

$$\phi(x,y) = \frac{1}{4\pi} \frac{\exp(ik|x-y|)}{|x-y|}, \quad x \neq y \ ,$$

is the fundamental solution to the Helmholtz equation in $\mathbb{R}^3$.
Thus, by (3.22) and by (3.21) it follows that

$$|u^s(x)| \leqslant C_1 \int_{\partial B_R(0)} \left| \frac{\partial \phi(x,y)}{\partial \nu(y)} \right| + |\phi(x,y)| \mathrm{d}s(y) \leqslant \quad (3.23)$$

$$\leqslant C_1 R^2 \left(\frac{kR}{||x|-R|^2} + \frac{R}{||x|-R|^3} + \frac{1}{||x|-R|} \right) . \quad (3.24)$$

Straightforward calculations show that

$$|u^s| < \frac{1}{2} \ , \text{ for every } x, |x| > R_0, \quad (3.25)$$

where $R_0 = (k+1)8R^3 C_1 + 2R$.
The thesis follows observing that $|u| \geqslant 1 - |u^s|$. □

4 The inverse scattering problem

Lemma 4.1 (From the far field to the near field). *Let* $u_i, u_{i,\infty},\ i = 1,2,$ *be as in Theorem 2.1. Suppose that, for some* $\varepsilon,\ 0 < \varepsilon < 1,$ (2.19) *holds, then there exist a radius* $R_1 > 0$ *and a constant* $C > 0$, *depending on the* a priori data *only, such that*

$$\|u_1 - u_2\|_{L^2(B_{R_1+1}(0) \setminus B_{R_1}(0))} \leqslant C \varepsilon^{\alpha(\varepsilon)}, \quad (4.1)$$

where $\alpha(\varepsilon)$ *is the function introduced in* (2.18).

Proof Let us choose $R = 4d + 4r_0$ and let us denote by $u_i^s,\ i = 1,2,$ the scattered wave of the problem (1.1) with $\lambda = \lambda_i$ respectively. By (3.21) it follows that

$$\|u_1^s - u_2^s\|_{L^2(\partial B_R(0))} \leqslant C \ , \quad (4.2)$$

where $C > 0$ is a constant depending on the *a priori data* only.
By the argument in [14] (see also [6]), it follows that there exists a constant $C > 0$ depending on the *a priori data* only, such that, for every $r \in (4R, 4R+1)$, the following holds

$$\|u_1^s - u_2^s\|_{L^2(\partial B_r(0))} \leqslant C \varepsilon^{\alpha(\varepsilon)}. \quad (4.3)$$

Integrating (4.3) with respect to r over $(4R, 4R+1)$, we obtain that

$$\|u_1^s - u_2^s\|_{L^2(B_{4R+1}(0)\setminus B_{4R}(0))} \leqslant C\varepsilon^{\alpha(\varepsilon)} , \tag{4.4}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Thus the thesis follows with $R_1 = 16d + 16r_0$ and by observing that $u_1^s - u_2^s = u_1 - u_2$.
Let us stress, that Hölder stability doesn't hold, indeed, in [6, Section 4], it has been proved that it is not possible to choose α independently on ε. □

Theorem 4.2 (Stability at the boundary). *Let $u_i, u_{i,\infty}$, $i = 1, 2$, be as in Theorem 2.1. We have that there exists $\varepsilon_0 > 0$ depending on the* a priori data *only, such that, if for some ε, $0 < \varepsilon < \varepsilon_0$,* (2.19) *holds, then for every $\rho \in (0, r_0)$ we have*

$$\|u_1 - u_2\|_{C^1(\Gamma_I^\rho)} \leqslant \eta(\varepsilon) , \tag{4.5}$$

where η is given by (2.17)*, with a constant $C > 0$ depending on the* a priori data *and on ρ only.*

Proof By the Lipschitz regularity of the boundary ∂D, it follows that the cone property holds. Namely, for every point $Q \in \partial D$, there exists a rigid transformation of coordinates under which we have $Q = 0$ and the finite cone

$$\mathcal{C} = \left\{ x : |x| < r_0,\ \frac{x \cdot \xi}{|x|} > \cos\theta \right\}$$

with axis in the direction ξ and width 2θ, where $\theta = \arctan\frac{1}{M}$, is such that $\mathcal{C} \subset D^+$.
Let Q be a point such that $Q \in \Gamma_I^{r_0}$ and let Q_0 be a point lying on the axis ξ of the cone with vertex in $Q = 0$ such that $d_0 = \mathrm{dist}(Q_0, 0) < \frac{r_0}{2}$.
Let us define $R_2 = 2R_1 + 2$, where R_1 is the radius introduced in the statement of Lemma 4.1. Dealing as in Lieberman [16], we consider a regularized distance $\tilde{d}$ from the boundary of ∂D such that, $\tilde{d} \in C^2(D^+_{R_2}) \cap C^{0,1}(\overline{D^+_{R_2}})$ and furthermore the following properties hold

- $\gamma_0 \leqslant \dfrac{\mathrm{dist}(x, \partial D)}{\tilde{d}(x)} \leqslant \gamma_1,$
- $|\nabla \tilde{d}(x)| \geqslant c_1,$ for every x such that $\mathrm{dist}(x, \partial D) \leqslant br_0,$
- $\|\tilde{d}\|_{C^{0,1}} \leqslant c_2 r_0,$

where $\gamma_0, \gamma_1, c_1, c_2, b$ are positive constants depending on M only, (see also [4, Lemma 5.2]).
Let us define for every $\rho > 0$

$$D^\rho = \{x \in D^+_{R_2} : \mathrm{dist}(x, \partial D) > \rho\} , \tag{4.6}$$

$$\tilde{D}^\rho = \{x \in D^+_{R_2} : \tilde{d}(x) > \rho\} . \tag{4.7}$$

It follows that there exists a, $0 < a \leqslant 1$, only depending on M such that for every ρ, $0 < \rho \leqslant ar_0$, $\tilde{D}^\rho$ is connected with boundary of class C^1 and

$$\tilde{c}_1\rho \leqslant \mathrm{dist}(x, \partial D) \leqslant \tilde{c}_2\rho \quad \text{for every } x \in \partial\tilde{D}^\rho, \tag{4.8}$$

where $\tilde{c}_1, \tilde{c}_2$, are positive constants depending on M only. By(4.8) we deduce that

$$D^{\tilde{c}_2\rho} \subset \tilde{D}^{\rho} \subset D^{\tilde{c}_1\rho} .$$

Let us now define $\rho_0 = \min\{\frac{1}{16}, \frac{r_0}{4}\sin\theta\}$ and let P be a point in the annulus $B_{R_1+1}(0)\setminus B_{R_1}(0))$, such that $B_{4\rho_0}(P) \subset B_{R_1+1}(0)\setminus B_{R_1}(0))$. Furthermore, let γ be a path in $\tilde{D}^{\frac{\rho_0}{\tilde{c}_1}}$ joining P to Q_0 and let us define $\{y_i\}$, $i = 0, \dots, s$ as follows $y_0 = Q_0$, $y_{i+1} = \gamma(t_i)$, where $t_i = \max\{t \text{ s.t. } |\gamma(t) - y_i| = 2\rho_0\}$ if $|P - y_i| > 2\rho_0$, otherwise let $i = s$ and stop the process.
Let us introduce the function $U \in H^1_{\text{loc}}(D^+)$ defined as follows

$$U(x) = u_1(x) - u_2(x). \tag{4.9}$$

We shall denote with U_1 and U_2 the real and the imaginary part of U respectively. Namely

$$U(x) = U_1(x) + iU_2(x).$$

It immediately follows that U_1, U_2, are both real valued solutions to the Helmholtz equation in D^+.
Thus, by the three spheres inequalities for elliptic system with Laplacian principal part, (see [5, Theorem 3.1]), we have that for every $\beta_1, \beta_2,\ 1 < \beta_1 < \beta_2$, there exist $\bar{r} > 0, \tau,\ 0 < \tau < 1$ and $C > 0$ depending on the *a priori data* and on β_1, β_2 only, such that for every $x \in D^{\beta_2\rho}$ the following holds

$$\int_{B_{\beta_1\rho}(x)} |U|^2 \leqslant C \left(\int_{B_{\rho}(x)} |U|^2 \right)^{\tau} \cdot \left(\int_{B_{\beta_2\rho}(x)} |U|^2 \right)^{1-\tau} \tag{4.10}$$

for every $\rho \in (0, \bar{r})$. By a possible replacement of ρ_0 with $\bar{r}$ if $\rho_0 > \bar{r}$ and choosing in (4.10) $\beta_1 = 3,\ \beta_2 = 4, \rho = \rho_0,\ x = y_0$, we infer that

$$\int_{B_{3\rho_0}(y_0)} |U|^2 \leqslant C \left(\int_{B_{\rho_0}(y_0)} |U|^2 \right)^{\tau} \cdot \left(\int_{B_{4\rho_0}(y_0)} |U|^2 \right)^{1-\tau} . \tag{4.11}$$

As a consequence of Lemma 3.1, we have that

$$\|U\|_{H^1(D^+_{R_2})} \leqslant C, \tag{4.12}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let us observe that $B_{4\rho_0}(y_0) \subset D^+_{R_2}$ and $B_{\rho_0}(y_0) \subset B_{3\rho_0}(y_1)$. Thus by (4.11) and (4.12) we deduce that

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leqslant C \left(\int_{B_{3\rho_0}(y_1)} |U|^2 \right)^{\tau} \cdot C^{1-\tau} .$$

An iterated application of the three spheres inequality leads to

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leqslant \left(\int_{B_{\rho_0}(y_s)} |U|^2 \right)^{\tau^s} \cdot C^{1-\tau^s} .$$

Finally, since $B_{\rho}(y_s) \subset B_{R_1+1}(0) \setminus B_{R_1}(0))$, by (4.1) we obtain that

$$\int_{B_{\rho_0}(y_0)} |U|^2 \leqslant C \big\{ \varepsilon^{\alpha(\varepsilon)} \big\}^{\tau^s} .$$

We shall construct a chain of balls $B_{\rho_k}(Q_k)$ centered on the axis of the cone, pairwise tangent to each other and all contained in the cone

$$\mathcal{C}' = \left\{ x : |x| < r_0,\ \frac{x\cdot\xi}{|x|} > \cos\theta' \right\} ,$$

where $\theta' = \arcsin\left(\frac{\rho_0}{d_0}\right)$. Let $B_{\rho_0}(Q_0)$ be the first of them, the following are defined by induction in such a way

$$\begin{aligned} &Q_{k+1} = Q_k - (1+\mu)\rho_k\xi \ , \\ &\rho_{k+1} = \mu\rho_k \ , \\ &d_{k+1} = \mu d_k \ , \end{aligned}$$

with

$$\mu = \frac{1-\sin\theta'}{1+\sin\theta'} \ .$$

Hence, with this choice, we have $\rho_k = \mu^k\rho_0$ and $B_{\rho_{k+1}}(Q_{k+1}) \subset B_{3\rho_k}(Q_k)$. Considering the following estimate obtained by a repeated application of the three spheres inequality, we have that

$$\begin{aligned} \|U\|_{L^2(B_{\rho_k}(Q_k))} &\leqslant \|U\|_{L^2(B_{3\rho_{k-1}}(Q_{k-1}))} \leqslant \\ &\leqslant \|U\|^{\tau}_{L^2(B_{\rho_{k-1}}(Q_{k-1}))}\|U\|^{1-\tau}_{L^2(B_{4\rho_{l-1}}(Q_{k-1}))} \\ &\leqslant C\|U\|^{\tau^k}_{L^2(B_{\rho_0}(Q_0))} \leqslant C\left\{\left[\varepsilon^{\alpha(\varepsilon)}\right]^{\tau^s}\right\}^{\tau^k} . \end{aligned} \tag{4.13}$$

For every r, $0 < r < d_0$, let $k(r)$ be the smallest positive integer such that $d_k \leqslant r$ then, since $d_k = \mu^k d_0$, it follows

$$\frac{|\log(\frac{r}{d_0})|}{\log\mu} \leqslant k(r) \leqslant \frac{|\log(\frac{r}{d_0})|}{\log\mu} + 1 \ , \tag{4.14}$$

and by (4.13) we deduce

$$\|U\|_{L^2(B_{\rho_{k(r)}}(Q_{k(r)}))} \leqslant C\left\{\left[\varepsilon^{\alpha(\varepsilon)}\right]^{\tau^s}\right\}^{\tau^{k(r)}} . \tag{4.15}$$

Let $\bar{x} \in \Gamma_I^{\frac{\rho}{2}}$ with $\rho \in (0, r_0)$ and let $x \in B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$. By Theorem 3.2, in particular, it follows that $U \in C^{1,\alpha}(D^+_{R_2,\frac{\rho}{4}})$ with

$$\|U\|_{C^{1,\alpha}(D^+_{R_2,\frac{\rho}{4}})} \leqslant C_\rho, \tag{4.16}$$

where $C_\rho > 0$ is a constant depending on the *a priori data* and on ρ only. Then (4.16) yields to

$$|U(\bar{x})| \leqslant |U(x)| + C_\rho|x-\bar{x}|^\alpha \leqslant |U(x)| + C_\rho\left(\frac{2}{\mu}r\right)^\alpha .$$

Integrating this inequality over $B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$, we have that

$$|U(\bar{x})|^2 \leqslant \frac{2}{\omega_3(\frac{\rho_{k-1}}{2})^3}\int_{B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})} |U(x)|^2 \mathrm{d}x + 2C_\rho^2\left(\frac{4r^2}{\mu^2}\right)^\alpha . \tag{4.17}$$

Being k the smallest integer such that $d_k \leqslant r$, then $d_{k-1} > r$ and thus (4.17) yields to

$$|U(\bar{x})|^2 \leqslant \frac{C}{\big(r\sin\theta'\big)^3}\int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})}|U(x)|^2 \mathrm{d}x + C_\rho r^{2\alpha} \ .$$

By (4.15) we deduce that

$$|U(\bar{x})|^2 \leqslant \frac{C}{r^3}\Big\{\big[\varepsilon^{\alpha(\varepsilon)}\big]^{\tau^s}\Big\}^{\tau^{k(r)-1}} + C_\rho r^{2\alpha} \ . \tag{4.18}$$

The estimate (4.16) also provides us that

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right| \leqslant \left|\frac{\partial U(x)}{\partial \nu}\right| + C_\rho\Big(\frac{2}{\mu}r\Big)^\alpha \ .$$

Integrating over $B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})$ we deduce that

$$\begin{aligned}\left|\frac{\partial U(\bar{x})}{\partial \nu}\right|^2 &\leqslant \frac{2}{\omega_3(\frac{\rho_{k-1}}{2})^3}\int_{B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})}\left|\frac{\partial U(x)}{\partial \nu}\right|^2 \mathrm{d}x + 2C_\rho^2\Big(\frac{4r^2}{\mu^2}\Big)^\alpha \leqslant \\ &\leqslant \frac{2}{\omega_3(\frac{\rho_{k-1}}{2})^3}\int_{B_{\frac{\rho_{k(r)-1}}{2}}(Q_{k(r)-1})}|\nabla U(x)|^2 \mathrm{d}x + 2C_\rho^2\Big(\frac{4r^2}{\mu^2}\Big)^\alpha \ .\end{aligned}$$

Applying the Caccioppoli inequality, we have

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right|^2 \leqslant \frac{C}{\big(\rho_{k-1}\big)^5}\int_{B_{\rho_{k(r)-1}}(Q_{k(r)-1})} U(x)^2 \mathrm{d}x + C_\rho r^{2\alpha} \ .$$

Dealing with the same arguments that lead to (4.18), we obtain that

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right|^2 \leqslant \frac{C}{r^5}\Big\{\big[\varepsilon^{\alpha(\varepsilon)}\big]^{\tau^s}\Big\}^{\tau^{k(r)-1}} + C_\rho r^{2\alpha} \ . \tag{4.19}$$

The choice in (4.14) guarantees that

$$\tau^{k(r)-1} \geqslant \Big(\frac{r}{d_0}\Big)^\nu,$$

where $\nu = -\log\big(\frac{1}{\mu}\big)\log\tau$. Thus, by (4.18) and by (4.19), it follows that

$$|U(\bar{x})| \leqslant C_\rho\bigg\{r^{-\frac{3}{2}}\Big[\big(\varepsilon^{\alpha(\varepsilon)}\big)^{\tau^s}\Big]^{\frac{r^\nu}{2}} + r^\alpha\bigg\} \ , \tag{4.20}$$

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right| \leqslant C_\rho\bigg\{r^{-\frac{5}{2}}\Big[\big(\varepsilon^{\alpha(\varepsilon)}\big)^{\tau^s}\Big]^{\frac{r^\nu}{2}} + r^\alpha\bigg\} \ . \tag{4.21}$$

Minimizing the right hand sides of the above inequalities with respect to r, with $r \in (0, \frac{r_0}{4})$, we deduce

$$|U(\bar{x})| \leqslant C_\rho\big(\log\big(\varepsilon^{-\alpha(\varepsilon)}\big)\big)^{-\frac{2\alpha}{\nu+2}} \ , \tag{4.22}$$

$$\left|\frac{\partial U(\bar{x})}{\partial \nu}\right| \leqslant C_\rho\big(\log\big(\varepsilon^{-\alpha(\varepsilon)}\big)\big)^{-\frac{2\alpha}{\nu+2}}, \tag{4.23}$$

where $C_\rho > 0$ is a constant depending on the *a priori data* and on ρ only. Thus, since $\bar{x}$ is an arbitrary point in $\Gamma_I^{\frac{\rho}{2}}$, by (4.22) and (4.23) we have that

$$\|U(\bar{x})\|_{L^\infty(\Gamma_I^{\frac{\rho}{2}})} \leqslant C_\rho\big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha}{\nu+2}}, \tag{4.24}$$

$$\left\|\frac{\partial U(\bar{x})}{\partial \nu}\right\|_{L^\infty(\Gamma_I^{\frac{\rho}{2}})} \leqslant C_\rho\big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha}{\nu+2}}. \tag{4.25}$$

By an interpolation inequality we have

$$\|\nabla_t(U)\|_{L^\infty(\Gamma_{1,\rho})} \leqslant c_\rho \|U\|^{\beta}_{L^\infty(\Gamma_{1,\frac{\rho}{2}})} \|U\|_{C^{1,\alpha}(\Gamma_{1,\rho})}{}^{1-\beta},$$

where $\beta = \frac{\alpha}{\alpha+1}$ and $c_\rho > 0$ depends on the *a priori data* and on ρ only. Thus, by (4.16), we obtain

$$\|\nabla_t(U)\|_{L^\infty(\Gamma_{1,\rho})} \leqslant c_\rho \|U\|^{\beta}_{L^\infty(\Gamma_{1,\frac{\rho}{2}})} C_\rho{}^{1-\beta}.$$

It follows that for every $\varepsilon < \varepsilon_0$, with ε_0 depending only on the *a priori data*,

$$\begin{aligned} \|\nabla(U)\|_{L^\infty(\Gamma_{1,\rho})} &\leqslant \left\|\frac{\partial U}{\partial \nu}\right\|_{L^\infty(\Gamma_{1,\rho})} + \|\nabla_t(U)\|_{L^\infty(\Gamma_{1,\rho})} \leqslant \\ &\leq C_\rho\big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha\beta}{\nu+2}}, \end{aligned} \tag{4.26}$$

where $C_\rho > 0$ depends on the *a priori data* and on ρ only. Hence, by a possible replacing of ε_0 with a smaller one depending on the *a priori data* only, we have that

$$\|u_1 - u_2\|_{C^1(\Gamma_{1,\rho})} \leq C_\rho\big(\log(\varepsilon^{-\alpha(\varepsilon)})\big)^{-\frac{2\alpha\beta}{\nu+2}} \text{ for every } \varepsilon, 0 < \varepsilon < \varepsilon_0. \tag{4.27}$$

Thus the thesis follows replacing in (2.17) C with C_ρ and θ with $\frac{2\alpha\beta}{\nu+2}$. □

Proposition 4.3. *There exists a radius $r_1 > 0$ depending on the* a priori data *only such that, for every $x_0 \in \Gamma_I^{r_0}$, the problem*

$$\begin{cases} \Delta\psi + k^2\psi = 0, & \text{in } \Gamma_{I,r_1}(x_0), \\ \dfrac{\partial \psi}{\partial \nu} + i\lambda(x)\psi = 0, & \text{on } \Delta_{I,r_1}(x_0), \end{cases} \tag{4.28}$$

admits a solution $\psi \in H^1(\Gamma_{I,r_1}(x_0))$ satisfying

$$|\psi(x)| \geqslant 1 \text{ for every } x \in \Gamma_{I,r_1}(x_0). \tag{4.29}$$

Moreover, there exists a constant $\bar{\psi} > 0$ depending on the a priori data *only, such that for every $x_0 \in \Gamma_I^{r_0}$*

$$\|\psi\|_{C^1(\Gamma_{I,r_1}(x_0))} \leqslant \bar{\psi}. \tag{4.30}$$

Proof Let us consider a point $x_0 \in \Gamma_I^{r_0}$. After a translation we may assume that $x_0 = 0$ and, fixing local coordinates, we can represent the boundary as a graph of a $C^{1,1}$ function. Namely, we have that

$$D^+ \cap B_{r_0}(0) = \{(x', x_3) \in B_{r_0}(0) : x_3 < \varphi_I(x')\}, \tag{4.31}$$

where φ_I is the $C^{1,1}$ function satisfying (2.3),(2.4),(2.5).
Let $\Phi \in C^{1,1}(B_{\frac{r_0}{4M}}, \mathbb{R}^3)$ be the map defined as follows

$$\Phi(y', y_3) = (y', y_3 + \varphi_I(y')) \ . \tag{4.32}$$

We have that there exist $\theta_1, \theta_2, \theta_1 > 1 > \theta_2 > 0$, constants depending on M and r_0 only, such that, for every $r \in (0, \frac{r_0}{4M})$, it follows that

$$\Gamma_{I,\theta_2 r}(0) \subset \Phi(B_r^-(0)) \subset \Gamma_{I,\theta_1 r}(0) \ , \tag{4.33}$$

where $B_r^-(0) = \{y \in \mathbb{R}^3 : |y| < r, \ y_3 < 0\}$ and furthermore we have

$$|\det \, D\Phi \ | = 1 \ . \tag{4.34}$$

The inverse map $\Phi^{-1} \in C^{1,1}(\Gamma_{I,r_0}(0), \mathbb{R}^3)$ and is defined by

$$\Phi^{-1}(x', x_3) = (x', x_3 - \varphi_I(x')) \ . \tag{4.35}$$

Denoting by

$$\sigma(y) = (\sigma_{i,j}(y))_{i,j=1}^3 = (D\Phi^{-1})(\Phi(y)) \cdot (D\Phi^{-1})^T(\Phi(y)) \ , \tag{4.36}$$

$$\lambda'(y) = \lambda(\Phi(y)) \ , \tag{4.37}$$

$$\lambda_0{}' = \lambda'(0) \ , \tag{4.38}$$

it follows that

$$\sigma(0) = \mathrm{I}, \tag{4.39}$$

$$\|\sigma_{i,j}\|_{C^{0,1}(\Gamma_{I,r_0})} \leqslant \Sigma, \quad \text{for} \ \ i,j = 1,2,3, \tag{4.40}$$

$$\frac{1}{2}|\xi|^2 \leqslant \sigma(y)\xi \cdot \xi \leqslant C_1|\xi|^2, \ \text{for every} \ y \in B^-_{(\frac{r_0}{4M})}(0) \ \text{and every} \ \xi \in \mathbb{R}^3, \tag{4.41}$$

$$\|\lambda'\|_{C^{0,1}(B'_{\frac{r_0}{4M}}(0))} \leqslant \Lambda' \ , \tag{4.42}$$

where $\Sigma > 0, C_1 > 0, \Lambda' > 0$ are constants depending on M, r_0, Λ only.

Claim 4.4. *There exists a radius r_2, $0 < r_2 < \frac{r_0}{4M}$ and a solution $\psi' \in H^1(B^-_{r_2}(0))$ to the problem*

$$\begin{cases} div(\sigma \nabla \psi') + k^2 \psi' = 0 \ , & in \quad B^-_{r_2}(0) \ , \\ \sigma \nabla \psi' \cdot \nu' + i\lambda' \psi' = 0 \ , & on \quad B'_{r_2}(0), \end{cases} \tag{4.43}$$

where $\nu' = (0,0,1)$ such that

$$|\psi'| \geqslant 1 \ in \ B^-_{r_2}(0).$$

Proof of Claim 4.4.
We look for a radius $r_2 > 0$ and for a solution of the form $\psi' = \psi_0 - s$ such that, $\psi_0 \in H^1(B^-_{r_2}(0))$ is a weak solution to the problem

$$\begin{cases} \Delta\psi_0 + k^2\psi_0 = 0 \ , & \text{in} \ \ B^-_{r_2}(0), \\ \dfrac{\partial\psi_0}{\partial\nu} + i{\lambda_0}'\psi_0 = 0 \ , & \text{on} \ \ B'_{r_2}(0), \end{cases} \tag{4.44}$$

satisfying $|\psi_0| \geqslant 2$ in $B^-_{r_2}(0)$.
Whereas $s \in H^1(B^-_{r_2}(0))$ is a weak solution to the problem

$$\begin{cases} \operatorname{div}(\sigma\nabla s) + k^2 s = \operatorname{div}((\sigma - I)\nabla\psi_0) \ , & \text{in} \ \ B^-_{r_2}(0), \\ \sigma\nabla s\cdot\nu + i\lambda' s = (\sigma - I)\nabla\psi_0\cdot\nu + i(\lambda' - {\lambda_0}')\psi_0 \ , & \text{on} \ \ B'_{r_2}(0), \\ s = 0 \ , & \text{on} \ \ |y| = r_2, \end{cases} \tag{4.45}$$

such that $s(y) = O(|y|^2)$ near the origin.
We can construct ψ_0 explicitly as follows

$$\psi_0(y_1,y_2,y_3) = 8\cosh\left(|{\lambda_0}'^2 - k^2|^{\frac{1}{2}}y_1\right)\left[\sin\left({\lambda_0}'y_3\right) + i\cos\left({\lambda_0}'y_3\right)\right], \ \text{if} \ k^2 < {\lambda_0}'^2,$$
$$\psi_0(y_1,y_2,y_3) = 8\cos\left(|k^2 - {\lambda_0}'^2|^{\frac{1}{2}}y_1\right)\left[\sin\left({\lambda_0}'y_3\right) + i\cos\left({\lambda_0}'y_3\right)\right], \ \text{if} \ k^2 > {\lambda_0}'^2,$$
$$\psi_0(y_1,y_2,y_3) = 8\sin\left({\lambda_0}'y_3\right) + i8\cos\left({\lambda_0}'y_3\right), \ \text{if} \ k^2 = {\lambda_0}'^2.$$

Denoting by

$$\tilde{r} = \frac{\pi}{4}\min\left\{\frac{1}{\sqrt{|k^2 - {\lambda_0}'^2|}}, \frac{1}{{\lambda_0}'}\right\} \ , \tag{4.46}$$

it follows, by straightforward calculations, that $\psi_0 \in H^1(B^-_{\tilde{r}}(0))$ is a weak solution of (4.44) with $r_2 = \tilde{r}$ and $|\psi_0| \geqslant 2$ in $B^-_{\tilde{r}}(0)$.
Let us now look for a solution s to the problem (4.45).
Fixed $r \in (0, \frac{r_0}{8M})$, let us define the space

$$H^1_{0^-}(B^-_r(0)) = \{\eta \in H^1(B^-_r(0)) \text{ such that } \eta(y) = 0 \text{ on } \ |y| = r\}, \tag{4.47}$$

endowed with the usual $\|\cdot\|_{H^1_0(B^-_r(0))}$ norm. Thus the weak formulation of the problem (4.45) reads in this way: find $s \in H^1_{0^-}(B^-_r(0))$ such that, for every $\eta \in H^1_{0^-}(B^-_r(0))$, the following holds

$$\begin{aligned} \int_{B^-_r(0)}\sigma\nabla s\cdot\nabla\bar{\eta} - \int_{B^-_r(0)}k^2 s\bar{\eta} - \int_{B'_r(0)}i\lambda' s\bar{\eta} \quad &= \int_{B^-_r(0)}(\sigma - I)\nabla\psi_0\cdot\nabla\bar{\eta} + \\ &+ i\int_{B'_r(0)}(\lambda' - {\lambda_0}')\psi_0\bar{\eta}. \end{aligned} \tag{4.48}$$

Let us introduce the following bilinear form

$$A \ : \ H^1_{0^-}(B^-_r(0)) \times H^1_{0^-}(B^-_r(0)) \to \mathbb{C} \tag{4.49}$$

such that

$$A(\eta_1,\eta_2) = \int_{B^-_r(0)}\sigma\nabla\eta_1\cdot\nabla\bar{\eta}_2 - \int_{B^-_r(0)}k^2\eta_1\bar{\eta}_2 - \int_{B'_r(0)}i\lambda'\eta_1\bar{\eta}_2 \tag{4.50}$$

and the following functional

$$F : H^1_{0^-}(B_r^-(0)) \to \mathbb{C} \tag{4.51}$$

such that

$$F(\eta) = \int_{B_r^-(0)} (\sigma - I)\nabla\psi_0 \cdot \nabla\bar{\eta} + i\int_{B_r'(0)} (\lambda' - {\lambda_0}')\psi_0\bar{\eta}\ . \tag{4.52}$$

It immediately follows that A and F are continuous on $H^1_{0^-}(B_r^-(0))$ as bilinear form and as a functional respectively.
Moreover, dealing as in [12, Lemma 8.4], we have that, by the Hölder inequality, it follows that for every $\eta \in H^1_{0^-}(B_r^-(0))$

$$\int_{B_r^-(0)} |\eta|^2 \leqslant \tilde{c_1} r^2 \Big(\int_{B_r^-(0)} |\eta|^6 \Big)^{\frac{1}{3}}, \tag{4.53}$$

where $\tilde{c_1} > 0$ is a constant depending on the *a priori data* only. Hence by the Sobolev Imbedding Theorem, (see [1, Chap.4]), and by (4.53), we have that

$$\int_{B_r^-(0)} |\eta|^2 \leqslant c_1 r^2 \int_{B_r^-(0)} |\nabla\eta|^2, \tag{4.54}$$

where $c_1 > 0$ is a constant depending on the *a priori data* only.
Analogously, by the Hölder inequality on the boundary, it follows that

$$\int_{B_r'(0)} |\eta|^2 \leqslant \tilde{c_2} r \Big(\int_{B_r'(0)} |\eta|^4 \Big)^{\frac{1}{2}}, \tag{4.55}$$

where $\tilde{c_2} > 0$ is a constant depending on the *a priori data* only. By a trace inequality (see for instance [1], Chap.5), it follows that

$$\int_{B_r'(0)} |\eta|^2 \leqslant c_2 r \int_{B_r^-(0)} |\nabla\eta|^2, \tag{4.56}$$

where $c_2 > 0$ is a constant depending on the *a priori data* only.
Thus, by (4.41),(4.54) and (4.56), we deduce that

$$|A(\eta,\eta)| \geqslant (\frac{1}{2} - c_1 r^2 k^2 - c_2 r \Lambda') \int_{B_r^-(0)} |\nabla\eta|^2.$$

Denoting by

$$r_3 = \min\left\{1, \frac{1}{8}(c_1 k^2 + c_2\Lambda), \frac{r_0}{8M}\right\}, \tag{4.57}$$

we have that for every $r \in (0, r_3)$

$$|A(\eta,\eta)| \geqslant \frac{1}{4}\int_{B_r^-(0)} |\nabla\eta|^2. \tag{4.58}$$

Thus it follows that, for every $r \in (0, r_3)$, the bilinear form A is coercive on $H^1_{0^-}(B_r^-(0))$. Hence by the Lax-Milgram theorem we can infer that, for every $r \in (0, r_3)$, there exists a unique solution $s \in H^1_{0^-}(B_r^-(0))$ to the problem (4.45).

Fixing $r \in (0, r_3)$ and choosing $\eta = s$ as test function in the weak formulation (4.48), we obtain

$$\int_{B_r^-(0)} \sigma \nabla s \cdot \nabla \bar{s} - \int_{B_r^-(0)} k^2 |s|^2 - \int_{B'_r(0)} i\lambda' |s|^2 = \int_{B_r^-(0)} (\sigma - I)\nabla \psi_0 \cdot \nabla \bar{s} + + i \int_{B'_r(0)} (\lambda' - {\lambda_0}')\psi_0 \bar{s}. \quad (4.59)$$

By (4.58), we have that

$$\frac{1}{4}\int_{B_r^-(0)} |\nabla s|^2 \leqslant \Big| \int_{B_r^-(0)} (\sigma - I)\nabla \psi_0 \cdot \nabla \bar{s} \Big| + \Big| \int_{B'_r(0)} (\lambda' - {\lambda_0}')\psi_0 \bar{s} \Big|. \quad (4.60)$$

By the Schwartz inequality, by (4.39) and by (4.40) we have that

$$\Big| \int_{B_r^-(0)} (\sigma - I)\nabla \psi_0 \cdot \nabla \bar{s} \Big| \leqslant 16\Sigma r^2 \int_{B_r^-(0)} |\nabla \psi_0|^2 + \frac{1}{16}\int_{B_r^-(0)} |\nabla s|^2 \ . \quad (4.61)$$

Analogously, we have that, by the Schwartz inequality, by (4.38) and by (4.42) it follows that

$$\Big| \int_{B'_r(0)} (\lambda' - {\lambda_0}')\psi_0 \bar{s} \Big| \leqslant 16 c_2 \Lambda' r^2 \int_{B'_r(0)} |\psi_0|^2 + \frac{1}{16 c_2}\int_{B'_r(0)} |s|^2 \ . \quad (4.62)$$

Moreover, by the inequality (4.56) and by (4.62) we deduce

$$\Big| \int_{B'_r(0)} (\lambda' - {\lambda_0}')\psi_0 \bar{s} \Big| \leqslant c_2^2 r^4 16\Lambda' \int_{B_r^-(0)} |\nabla \psi_0|^2 + \frac{1}{16} r \int_{B_r^-(0)} |\nabla s|^2 \ . \quad (4.63)$$

Hence inserting (4.61) and (4.63) in (4.60) we obtain that

$$\frac{1}{8}\int_{B_r^-(0)} |\nabla s|^2 \leqslant (16\Sigma + c_2^2 16\Lambda') r^2 \int_{B_r^-(0)} |\nabla \psi_0|^2 \ . \quad (4.64)$$

Denoting by

$$Q = \sup_{B^-_{\frac{r_0}{8M}}(0)} |\nabla \psi_0|^2,$$

we have that

$$\frac{1}{8}\int_{B_r^-(0)} |\nabla s|^2 \leqslant \frac{4}{3}\pi (16\Sigma + c_1^2 16\Lambda') r^5 Q \ . \quad (4.65)$$

By standard estimates for solutions of elliptic equations (see for instance [12], Chap.8) and observing that $Q > 0$ depends on the *a priori data* only, we can infer that for every $r \in (0, \frac{r_3}{2})$

$$\|s\|_{L^\infty(B_r^-(0))} \leqslant c_4 r^2,$$

where $c_4 > 0$ is a constant depending on the *a priori data* only.
Hence the Claim follows choosing $r_2 = \min\{\tilde{r}, \frac{r_3}{2}, \frac{1}{\sqrt{c_4}}\}$ and observing that

$$|\psi'| \geqslant |\psi_0| - |s| \geqslant 1 \quad \text{in } B^-_{r_2}(0) \ .$$

□

Let us notice that choosing $r_1 = \theta_2 r_2$ and $\psi(x', x_3) = \psi'(\Phi^{-1}(x', x_3))$, we have that $\psi \in H^1(\Gamma_{I,r_1}(0))$ is a weak solution to the problem (4.28) and is such that $|\psi| \geqslant 1$ in $\Gamma_{I,r_1}(0)$.
Finally, we conclude the proof of Proposition 4.3 observing that (4.30) follows dealing with the same argument used in the proof of Theorem 3.2. □

Lemma 4.5 (Volume doubling inequality). *Let u be the solution to the problem* (1.1)*, then there exists a radius $\bar{r} > 0$ such that for every $x_0 \in \Gamma_I^{r_0}$ the following holds*

$$\int_{\Gamma_{I,\beta r}} |u|^2 \leqslant C\beta^K \int_{\Gamma_{I,r}} |u|^2 \tag{4.66}$$

for every r, β such that $\beta > 1$ and $0 < \beta r < \bar{r}$, where $C > 0, K > 0$ are constants depending on the a priori data *only.*

Proof Let $x_0 \in \Gamma_I^{r_0}$ and let r_1 and ψ be, respectively, the radius and the function, introduced in Proposition 4.3. Denoting by

$$z = \frac{u}{\psi}, \tag{4.67}$$

it follows that $z \in H^1(\Gamma_{I,r_1}(x_0))$ is a weak solution to the problem

$$\begin{cases} \Delta z + 2\dfrac{\nabla\psi}{\psi}\cdot\nabla z = 0, & \text{in } \Gamma_{I,r_1}(x_0), \\ \dfrac{\partial z}{\partial\nu} = 0, & \text{on } \Delta_{I,r_1}(x_0). \end{cases} \tag{4.68}$$

Dealing as in Proposition 4.3, we may assume that, up to a rigid transformation of coordinates, $x_0 = 0$ and, by local coordinates, we can locally represent the boundary as a graph of a $C^{1,1}$ function as in (4.31).
Following [2, Theorem 0.8], (see also [4, Proposition 3.5]), we have that there exists a map $\Psi \in C^{1,1}(B_{\rho_2}(0), \mathbb{R}^3)$ such that

$$\Psi(B_{\rho_2}(0)) \subset B_{\rho_1}(0), \tag{4.69}$$

$$\Psi(y', 0) = (y', \varphi_I(y')), \qquad \text{for every } y' \in B'_{\rho_2}(0), \tag{4.70}$$

$$\Gamma_{I,\frac{\rho}{2}} \subset \Psi(B^-_\rho(0)) \subset \Gamma_{I,c_1\rho}, \quad \text{for every } \rho \in (0, \rho_2), \tag{4.71}$$

$$\frac{1}{8} \leqslant |\text{det} D\Psi| \leqslant c_2, \tag{4.72}$$

where $\rho_1, 0 < \rho_1 < r_0, \rho_2 > 0, c_1 > 0, c_2 > 0$ are constants depending on r_0, M, Λ only. Denoting by

$$A(y) = |\text{det} D\Psi(y)|(D\Psi^{-1})(\Psi(y))(D\Psi^{-1})^T(\Psi(y)), \tag{4.73}$$

$$B(y) = 2|\mathrm{det} D\Psi(y)|(D\Psi^{-1})(\Psi(y))\frac{\nabla\psi(\Psi(y))}{\psi(\Psi(y))}, \tag{4.74}$$

$$v(y) = z(\Psi(y)), \tag{4.75}$$

it follows that

$$A(0) = I \ , \tag{4.76}$$

$$A(y',0)(y',0)\cdot e_3 = 0, \text{ for every } \ y', |y'| \leqslant \rho_2, \tag{4.77}$$

$$c_3|\xi|^2 \leqslant A(y)\xi\cdot\xi \leqslant c_4|\xi|^2, \text{for every } \ y \in B^-_{\rho_2}(0) \text{ and for every } \xi \in \mathbb{R}^3, \tag{4.78}$$

$$|A(y_1) - A(y_2)| \leqslant c_5|y_1 - y_2|, \quad \text{for every } \ y_1, y_2 \in B^-_{\rho_2}(0), \tag{4.79}$$

$$|B(y)| \leqslant c_6, \text{ for every } \ y \in B^-_{\rho_2}(0), \tag{4.80}$$

where $c_4 > 0, c_5 > 0, c_6 > 0$ are constants depending on r_0, M, Λ only.
Let us observe that $v \in H^1(B^-_{\rho_2}(0))$ is a weak solution to the problem

$$\begin{cases} \mathrm{div}(A\nabla v) + B\nabla v = 0, & \text{in } B^-_{\rho_2}(0), \\ A(y',0)\nabla v\cdot\nu' = 0, & \text{on } B'_{\rho_2}(0). \end{cases} \tag{4.81}$$

Hence we are under the assumptions of Theorem 1.3 in [2] and thus we can infer that there exists a radius $\rho_3, 0 < \rho_3 < \rho_2$, depending on the *a priori data* only, such that

$$\int_{B^-_{\beta\rho}(0)} |v|^2 \leqslant c\beta^K \int_{B^-_{\rho}(0)} |v|^2 \ , \tag{4.82}$$

for every ρ, β such that $\beta > 1$ and $0 < \beta\rho \leqslant \rho_3$, where $c > 0$ is constant depending on the *a priori data* only, and $K > 0$ depends on the *a priori data* and increasingly on

$$N(\rho_3) = \rho_3 \frac{\int_{B^-_{\rho_3}(0)} A\nabla v\cdot\nabla\bar{v} + Re(\bar{v}\ \mathrm{div}(A\nabla v))}{\int_{\partial B^-_{\rho_3}(0)\setminus B'_{\rho_3(0)}} \mu|v|^2} \ , \tag{4.83}$$

where we denote

$$\mu(x) = \frac{A(x)x\cdot x}{|x|^2}, \quad \text{for every } x \in B^-_{\rho_2}(0). \tag{4.84}$$

By (4.78) it follows that

$$c_3 \leqslant \mu(x) \leqslant c_4, \quad \text{for every } x \in B^-_{\rho_2}(0). \tag{4.85}$$

Let us observe that the proof of Theorem 1.3 in [2] needs, in this context, a slight modification due to the fact that we deal with complex valued functions. We omit the details.

Denoting by

$$\tilde{N}(\rho_3) = \frac{\int_{B^-_{\rho_3}(0)} \rho_3^2 |\nabla v|^2 + |v|^2}{\int_{B^-_{\rho_3}(0)} |v|^2} \ , \tag{4.86}$$

we notice, following the arguments in [5, Lemma 3.3], that

$$N(\rho_3) \leqslant C\tilde{N}(\rho_3), \tag{4.87}$$

where $C > 0$ is a constant depending on the *a priori data* only.
By (4.71), it follows, that for every r and $\beta > 1$ such that $0 < r < \beta r < \frac{\rho_3}{2}$

$$\int_{\Gamma_{I,\beta r}(0)} |z|^2 \leqslant C \int_{B^-_{2\beta r}(0)} |v|^2 \ , \tag{4.88}$$

where $C > 0$ is a constant depending on r_0, M, Λ only. Moreover, by (4.82) and by (4.71) we have that

$$\int_{B^-_{2\beta r}(0)} |v|^2 \leqslant C(2\beta c_1)^K \int_{B^-_{\frac{r}{c_1}}(0)} |v|^2 \leqslant C(2\beta c_1)^K \int_{\Gamma_{I,r}(0)} |z|^2, \tag{4.89}$$

where $C > 0$ is a constant depending on r_0, M, Λ only.
Combining (4.88) and (4.89), we have that

$$\int_{\Gamma_{I,\beta r}} |z|^2 \leqslant C(2\beta c_1)^K \int_{\Gamma_{I,r}(0)} |z|^2 \ . \tag{4.90}$$

Finally the last inequality, (4.29),(4.30) imply that

$$\int_{\Gamma_{I,\beta r}} |u|^2 \leqslant C(\beta)^K \int_{\Gamma_{I,r}(0)} |u|^2 \ , \tag{4.91}$$

where $C > 0, K > 0$ are constants depending on *a priori data* and on $\tilde{N}(\rho_3)$ only. Thus the Lemma follows with

$$\bar{r} = \frac{\rho_3}{2}. \tag{4.92}$$

It only remains to majorize the quantity (4.86) by a constant depending on the *a priori data* only. Let us observe that by (4.71), by (4.29) and by (4.30), we have that

$$\int_{B^-_{\rho_3}(0)} |\nabla v|^2 + |v|^2 \leqslant C \int_{\Gamma_{I,\rho_3 c_1}(0)} |\nabla u|^2 + |u|^2, \tag{4.93}$$

where $C > 0$ is a constant depending on the *a priori data* only. Moreover, by the above inequality and by (3.4), we can conclude that

$$\int_{B^-_{\rho_3}(0)} |\nabla v|^2 + |v|^2 \leqslant C, \tag{4.94}$$

where $C > 0$ is a constant depending on *a priori data* only.

On the other hand, we have that choosing $P_0 = \frac{M}{8\sqrt{1+M^2}\rho_3}\nu$ and $\rho_4 = \frac{1}{32}\frac{M}{\sqrt{1+M^2}}\rho_3$, where ν is the outer unit normal to D at 0, it follows that $B_{\rho_4}(P_0) \subset \Gamma_{I,\frac{\rho_3}{2}}(0)$. Thus, by (4.71) and by (4.30) it follows that

$$\int_{B^-_{\rho_3}(0)} |v|^2 \geqslant C \int_{\Gamma_{I,\frac{\rho_3}{2}}(0)} |u|^2 \geqslant C \int_{B_{\rho_4}(P_0)} |u|^2 \ , \tag{4.95}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let us consider a point $Q \in \mathbb{R}^3 \setminus D^+_{2R_0}$ such that

$$B_{4\rho_4}(Q) \subset \mathbb{R}^3 \setminus \overline{D}^+_{2R_0}, \tag{4.96}$$

where R_0 is the radius introduced in Corollary 3.3. Dealing as in the proof of Theorem 4.2, we cover a path joining P_0 to Q by a chain of balls of radius ρ_4 pairwise tangent to each other. Hence, by an iterated use of the three spheres inequality, we have that the following holds

$$\|u\|_{L^2(B_{\frac{\rho_4}{4}}(Q))} \leqslant C \|u\|^{\tau^s}_{L^2(B_{\rho_4(P_0)})} \ , \tag{4.97}$$

where $C > 0, s > 0$ and $\tau, 0 < \tau < 1$ are constants depending on the *a priori data* only. By the last inequality, by (4.96) and by (3.19), we can infer that

$$\|u\|_{L^2(B_{\rho_4(P_0)})} \geqslant \left(\frac{\pi\rho_4^3}{C48}\right)^{\frac{1}{\tau^s}} . \tag{4.98}$$

Hence, by (4.98) and by (4.95), we have that

$$\int_{B^-_{\rho_3}(0)} |v|^2 \geqslant C, \tag{4.99}$$

where $C > 0$ is a constant depending on *a priori data* only. Hence, by (4.94) and by (4.99), we can majorize $\tilde{N}(\rho_3)$ by a constant depending on the *a priori data* only and thus the Lemma follows. □

Theorem 4.6 (Surface doubling inequality). *Let u be the solution to the problem* (1.1)*, then there exists a constant $C > 0$ depending on the* a priori data *only such that, for every $x_0 \in \Gamma_I^{r_0}$ and for every $r \in (0, \frac{\bar{r}}{4})$, the following holds*

$$\int_{\Delta_{I,2r}(x_0)} |u|^2 d\sigma \leqslant C \int_{\Delta_{I,r}(x_0)} |u|^2 d\sigma \ . \tag{4.100}$$

Proof Let $x_0 \in \Gamma_I^{r_0}$ and let $z \in H^1(\Gamma_{I,r_1}(x_0))$ and $\bar{r}$ be, respectively, the solution to the problem (4.68) defined by (4.67) and the radius introduced in (4.92). By a regularity estimate at the boundary, (see for instance [4, p.777]) we have that, for any $r \in (0, \frac{\bar{r}}{4})$, the following holds

$$\int_{\Delta_{I,r}(x_0)} |\nabla_t z|^2 \leqslant C \left(\frac{1}{r}\int_{\Gamma_{I,2r}(x_0)} |\nabla z|^2\right)^{1-\gamma} \left(\frac{1}{r^2}\int_{\Delta_{I,r}(x_0)} |z|^2\right)^{\gamma}, \tag{4.101}$$

where $C>0$ and $0<\gamma<1$ are constants depending on the *a priori data* only and where $\nabla_t z$ represents the tangential gradient.
Thus, by the Young inequality we have that for every $\varepsilon>0$ the following holds

$$\int_{\Delta_{I,r}(x_0)} |\nabla_t z|^2 \leqslant \frac{C\varepsilon^{\frac{1}{1-\gamma}}}{r}\int_{\Gamma_{I,2r}(x_0)} |\nabla z|^2 + \frac{C}{\varepsilon^{\frac{1}{\gamma}} r^2}\int_{\Delta_{I,r}(x_0)} |z|^2, \qquad (4.102)$$

where $C>0$ is a constant depending on the *a priori data* only.
Moreover, by a well-known estimate of stability for the Cauchy problem (see for instance [17]), we have that

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |z|^2 \leqslant Cr\left(\int_{\Delta_{I,r}(x_0)} |z|^2 + r^2\int_{\Delta_{I,r}(x_0)} |\nabla_t z|^2\right)^{1-\delta} \cdot \qquad (4.103)$$

$$\cdot\left(\int_{\Delta_{I,r}(x_0)} |z|^2 + r^2\int_{\Delta_{I,r}(x_0)} |\nabla_t z|^2 + r\int_{\Gamma_{I,r}(x_0)} |\nabla z|^2\right)^{\delta},$$

where $C>0$ and $0<\delta<1$ are constants depending on the *a priori data* only.
Hence, by (4.103) and by the Young inequality, we have that for every $\beta>0$ the following holds

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |z|^2 \leqslant \frac{C}{\varepsilon^{\frac{\beta}{1-\delta}}}\left(r\int_{\Delta_{I,r}(x_0)} |z|^2 + r^3\int_{\Delta_{I,r}(x_0)} |\nabla_t z|^2\right) + \qquad (4.104)$$

$$+C\varepsilon^{\frac{\beta}{\delta}}\left(r\int_{\Delta_{I,r}(x_0)} |z|^2 + r^3\int_{\Delta_{I,r}(x_0)} |\nabla_t z|^2 + r^2\int_{\Gamma_{I,r}(x_0)} |\nabla z|^2\right),$$

where $C>0$ is a constant depending on the *a priori data* only.
Choosing β in (4.104) such that $\beta=\frac{1-\delta}{1-\gamma}\gamma$ and inserting (4.102) in (4.104), we obtain

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |z|^2 \leqslant \frac{Cr}{\varepsilon^{\frac{\gamma^2+1-\gamma}{\gamma(1-\gamma)}}}\int_{\Delta_{I,r}(x_0)} |z|^2 + C\varepsilon r^2\int_{\Gamma_{I,2r}(x_0)} |\nabla z|^2,$$

where $C>0$ is a constant depending on the *a priori data* only. By the Caccioppoli inequality we have that

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |z|^2 \leqslant \frac{Cr}{\varepsilon^{\frac{\gamma^2+1-\gamma}{\gamma(1-\gamma)}}}\int_{\Delta_{I,r}(x_0)} |z|^2 + C\varepsilon\int_{\Gamma_{I,4r}(x_0)} |z|^2,$$

where $C>0$ is a constant depending on the *a priori data* only.
Thus by (4.29) and (4.30) we can infer that

$$\int_{\Gamma_{I,r}(x_0)} |u|^2 \leqslant \frac{Cr}{\varepsilon^{\frac{\gamma^2+1-\gamma}{\gamma(1-\gamma)}}}\int_{\Delta_{I,2r}(x_0)} |u|^2 + C\varepsilon\int_{\Gamma_{I,8r}(x_0)} |u|^2,$$

where $C>0$ is a constant depending on the *a priori data* only.
By (4.66) it follows that

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |u|^2 \leqslant \frac{Cr}{\varepsilon^{\frac{\gamma^2+1-\gamma}{\gamma(1-\gamma)}}}\int_{\Delta_{I,r}(x_0)} |u|^2 + C(8)^K\varepsilon\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |u|^2, \qquad (4.105)$$

where $C > 0$ is a constant depending on the *a priori data* only.
Hence, choosing ε in (4.105) such that $\varepsilon = \frac{1}{2C(8)^K}$, we obtain that

$$\int_{\Gamma_{I,\frac{r}{2}}(x_0)} |u|^2 \leqslant Cr \int_{\Delta_{I,r}(x_0)} |u|^2, \tag{4.106}$$

where $C > 0$ is a constant depending on the *a priori data* only.
By applying again (4.66) on the left hand side of (4.106), we obtain that

$$\int_{\Gamma_{I,2r}(x_0)} |u|^2 \leqslant Cr \int_{\Delta_{I,r}(x_0)} |u|^2, \tag{4.107}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Moreover, by a standard Dirichlet trace inequality, we have that

$$\int_{\Delta_{I,2r}(x_0)} |u|^2 \leqslant C \int_{\Delta_{I,r}(x_0)} |u|^2, \tag{4.108}$$

where $C > 0$ is a constant depending on the *a priori data* only.

□

Corollary 4.7 (A_p property on the boundary). *Let u be the solution to the problem* (1.1)*, then there exist $p > 1, A > 0$ constants depending on the* a priori data *only, such that, for every $x_0 \in \Gamma_I^{r_0}$ and every $r \in (0, \frac{\bar{r}}{4})$, the following holds*

$$\left(\frac{1}{|\Delta_{I,r}(x_0)|}\int_{\Delta_{I,r}(x_0)} |u|^2 d\sigma\right)\left(\frac{1}{|\Delta_{I,r}(x_0)|}\int_{\Delta_{I,r}(x_0)} |u|^{-\frac{2}{p-1}} d\sigma\right)^{p-1} \leqslant A. \tag{4.109}$$

Proof Let $x_0 \in \Gamma_I^{r_0}$ and let $r \in (0, \frac{\bar{r}}{4})$, then by a trace inequality, (see for instance [1], Chap. 5), it follows that

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leqslant C\|u\|_{H^1(\Gamma_{I,r}(x_0))}, \tag{4.110}$$

where $C > 0$ is a constant depending on the *a priori data* only. By the Caccioppoli inequality we deduce that

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leqslant \frac{C}{r}\|u\|_{L^2(\Gamma_{I,2r}(x_0))}. \tag{4.111}$$

Applying the Doubling inequality (4.66) on the right hand side of (4.111), we obtain that

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leqslant \frac{C}{r}\|u\|_{L^2(\Gamma_{I,r}(x_0))}, \tag{4.112}$$

where $C > 0$ is a constant depending on the *a priori data* only. Combining (4.106) and (4.112) we have that

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leqslant \frac{C}{\sqrt{r}}\|u\|_{L^2(\Delta_{I,2r}(x_0))}, \tag{4.113}$$

where $C > 0$ is a constant depending on the *a priori data* only. Thus by the doubling inequality (4.100) we have

$$\|u\|_{L^4(\Delta_{I,r}(x_0))} \leqslant \frac{C}{\sqrt{r}}\|u\|_{L^2(\Delta_{I,r}(x_0))}. \tag{4.114}$$

Hence, we infer that for every $r \in (0, \frac{\bar{r}}{4})$ and for every $x_0 \in \Gamma_I^{r_0}$, the following holds

$$\left(\frac{1}{r^2}\int_{\Delta_{I,r}}|u|^4\right)^{\frac{1}{4}} \leqslant \left(\frac{C}{r^2}\int_{\Delta_{I,r}}|u|^2\right)^{\frac{1}{2}},$$

obtaining a reverse Hölder inequality.
The result in [9] assures the existence of some $p > 1$ and $A > 0$ depending on the *a priori data* only such that (4.109) holds. □

Proof of Theorem 2.1. Let x_0 be a point in $\Gamma_I^{r_0}$. Let us pick $r = \frac{\bar{r}}{8}$, thus by (4.106) with $u = u_2$ it follows that

$$\int_{\Delta_{I,\frac{\bar{r}}{8}}(x_0)}|u_2|^2 \mathrm{d}\sigma \geqslant C\int_{\Gamma_{I,\frac{\bar{r}}{16}}(x_0)}|u_2|^2 \mathrm{d}x, \tag{4.115}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let P_0 and $\rho_4 > 0$ be, respectively a point and a radius, such that $B_{\rho_4}(P_0) \subset \Gamma_{I,\frac{\bar{r}}{16}(x_0)}$. By rephrasing the argument leading to (4.98) we deduce by (4.115) that

$$\int_{\Delta_{I,\frac{\bar{r}}{8}}(x_0)}|u_2|^2 \mathrm{d}\sigma \geqslant C, \tag{4.116}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Combining (4.109) and (4.116), we have that for every $x_0 \in \Gamma_I^{r_0}$ the following holds

$$\left(\int_{\Delta_{I,\frac{\bar{r}}{8}}(x_0)}|u_2|^{-\frac{2}{p-1}} \mathrm{d}\sigma\right)^{p-1} \leqslant C, \tag{4.117}$$

where $C > 0$ is a constant depending on the *a priori data* only.
Let us now consider $x \in \Delta_{I,\frac{\bar{r}}{8}}(x_0)$, then it follows that

$$\begin{aligned} |\lambda_1(x) - \lambda_2(x)| &= \left|-\lambda_1(x)\frac{u_1(x)-u_2(x)}{u_2(x)} + \frac{1}{iu_2(x)}\left(\frac{\partial u_2(x)}{\partial\nu} - \frac{\partial u_1(x)}{\partial\nu}\right)\right| \leqslant \\ &\leqslant |\lambda_1(x)|\frac{|u_1(x)-u_2(x)|}{|u_2(x)|} + \frac{1}{|u_2(x)|}\left|\frac{\partial u_2(x)}{\partial\nu} - \frac{\partial u_1(x)}{\partial\nu}\right|. \end{aligned}$$

Then by Theorem 4.2 and by (2.16) we have that, if $0 < \varepsilon < \varepsilon_0$, then

$$|\lambda_1(x) - \lambda_2(x)| \leqslant (\Lambda + 1)\eta(\varepsilon)\frac{1}{|u_2(x)|}. \tag{4.118}$$

Hence denoting by $\delta=\frac{2}{p-1}$, (4.118) yields to

$$\left(\int_{\Delta_{I,\frac{\bar{r}}{8}(x_0)}}|\lambda_1(x)-\lambda_2(x)|^{\delta}\right)^{\frac{1}{\delta}}\leqslant(\Lambda+1)\eta(\varepsilon)\left(\int_{\Delta_{I,\frac{\bar{r}}{8}(x_0)}}\frac{1}{|u_2(x)|^{\delta}}\right)^{\frac{1}{\delta}}. \tag{4.119}$$

By (4.117) and by a possible replacement of the constant C in (2.17), we have that

$$\left(\int_{\Delta_{I,\frac{\bar{r}}{8}(x_0)}}|\lambda_1(x)-\lambda_2(x)|^{\delta}\right)^{\frac{1}{\delta}}\leqslant\eta(\varepsilon). \tag{4.120}$$

By the a priori bound (2.16), we can infer that

$$|\lambda_1(x)-\lambda_2(x)|\leqslant|\lambda_1(x)-\lambda_2(x)|^{\frac{\delta}{2}}(2\Lambda)^{1-\frac{\delta}{2}}\ . \tag{4.121}$$

Integrating the above inequality with respect to x over $\Delta_{I,\frac{\bar{r}}{8}(x_0)}$ we have

$$\|\lambda_1(x)-\lambda_2(x)\|_{L^2(\Delta_{I,\frac{\bar{r}}{8}(x_0)})}\leqslant(2\Lambda)^{1-\frac{\delta}{2}}\left(\int_{\Delta_{I,\frac{\bar{r}}{8}(x_0)}}|\lambda_1(x)-\lambda_2(x)|^{\delta}\right)^{\frac{1}{2}}. \tag{4.122}$$

Hence, by a possible further replacement of the constants C,θ in (2.17), we can infer that the last inequality and (4.120) yield to

$$\|\lambda_1(x)-\lambda_2(x)\|_{L^2(\Delta_{I,\frac{\bar{r}}{8}(x_0)})}\leqslant\eta(\varepsilon)\ . \tag{4.123}$$

By an interpolation inequality, see for instance [4, p.777], we have that

$$\|\lambda_1-\lambda_2\|_{L^\infty(\Delta_{I,\frac{\bar{r}}{8}(x_0)})}\leqslant C\|\lambda_1-\lambda_2\|^{\frac{1}{2}}_{L^2(\Delta_{I,\frac{\bar{r}}{8}(x_0)})}\|\lambda_1-\lambda_2\|^{\frac{1}{2}}_{C^{0,1}(\Delta_{I,\frac{\bar{r}}{8}(x_0)})}, \tag{4.124}$$

where $C>0$ is a constant depending on the *a priori data* only. Hence by (2.16), it follows that

$$\|\lambda_1-\lambda_2\|_{L^\infty(\Delta_{I,\frac{\bar{r}}{8}(x_0)})}\leqslant C(2\Lambda)^{\frac{1}{2}}\|\lambda_1-\lambda_2\|^{\frac{1}{2}}_{L^2(\Delta_{I,\frac{\bar{r}}{8}(x_0)})}. \tag{4.125}$$

Combining (4.123) with (4.125) we obtain, by a possible further replacement of the constants C,θ in (2.17), that

$$\|\lambda_1-\lambda_2\|_{L^\infty(\Delta_{I,\frac{\bar{r}}{8}(x_0)})}\leqslant\eta(\varepsilon). \tag{4.126}$$

Let us cover $\Gamma_I^{r_0}$ with the sets $\Delta_{I,\frac{\bar{r}}{8}}(x_j),\ j=1,\ldots,J$, with $x_j\in\Gamma_I^{r_0}$.
Let i be an index such that

$$\|\lambda_1-\lambda_2\|_{L^\infty(\Delta_{I,\frac{\bar{r}}{8}(x_i)})}=\|\lambda_1-\lambda_2\|_{L^\infty(\Gamma_I^{r_0})}. \tag{4.127}$$

Thus, by a further possible replacement of the constant C,θ in (2.17), we deduce (2.20) by combining (4.127) and (4.126) with $x_0=x_i$. □

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