On the Maz'ya inequalities: existence and multiplicity results for an elliptic problem involving cylindrical weights *

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Abstract. In this paper we discuss existence and multiplicity of solutions to the following class of superlinear problems:

$$(1) \qquad \begin{cases} -\mathrm{div}(|x|^a \nabla u) = |x|^{-b_a} \ u^{p-1} & \text{in } \mathbb{R}^N \ , \ x \neq 0 \\ u > 0 \ . \end{cases}$$

Here $u = u(x,y) : \mathbb{R}^k \times \mathbb{R}^{N-k} \to \mathbb{R}$, $1 \le k < N$, $b_a = N - p\frac{N-2+a}{2}$ and $a > (2-N)\frac{k}{N}$. The main tools are some Sobolev-type inequalities with cylindrical weights that were proved by Maz'ya in 1980.

We also get new existence results for the singular superlinear problem

$$(2) \qquad \begin{cases} -\Delta v = \lambda |x|^{-2} \ v + |x|^{-b_0} \ v^{p-1} & \text{in } \mathbb{R}^N \ , \ x \neq 0 \\ v > 0 \ , \end{cases}$$

by compounding a simple functional change with our results for (1). Here $\lambda \le \left(\frac{k-2}{2}\right)^2$ and $b_0 = N - p\frac{N-2}{2}$. In particular, we give an alternative proof of a result by Tertikas and Tintarev [16] in case $\lambda = \left(\frac{k-2}{2}\right)^2$. For $\lambda < \left(\frac{k-2}{2}\right)^2$ we prove some existence results of singular solutions $v \in L^p(\mathbb{R}^N; |x|^{-b_0} d\xi)$.

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Introduction

Let k, N be positive integers with $1 \le k < N$. We put $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, and we denote points ξ in $\mathbb{R}^N$ as pairs $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. In this paper we study existence and multiplicity for problem

$$\begin{cases} -\mathrm{div}(|x|^a \nabla u) = |x|^{-b_a} u^{p-1} & \text{in } \mathbb{R}^N \ , \ x \neq 0 \\ u > 0 \ , \end{cases} \tag{0.1}$$

where the exponent b_a is given by

$$b_a = b_a(p) := N - p\frac{N-2+a}{2} \ , \tag{0.2}$$

and the real parameters a, p satisfy

$$(2-N)\frac{k}{N} < a \le 2-k \ , \quad \frac{2(N-k)}{N-2+a} < p \le 2^* \ . \tag{0.3}$$

Here, as usual, $2^* = +\infty$ if $N = 2$, and $2^* = \frac{2N}{N-2}$ if $N \ge 3$.

If $k = N$ problem (0.1) is related to the Caffarelli-Kohn-Nirenberg inequality [3]. There is a large number bibliographical references for (0.1) in this case: we quote for example [4], [6], [8], [17] and references there-in. Concerning the cylindrical case $k < N$ we cite [1], [5], [11], [12], where $2 \le k < N$ and $a = 0$ are assumed. In [16], Tertikas and Tintarev studied problem (0.1) for $a = 2 - k$, $N \ge 3$ and $p = 2^*$. In [15] some existence results of solutions $u \in D^1_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{a-2} d\xi)$ are given under the assumption $a \neq 2 - k$. Finally, the paper [10] is concerned with symmetric solutions.

Our interest in problem (0.1) is motivated by an inequality that is peculiar to the cylindrical case $k < N$, and that was proved by Maz'ya in [14] (see also [5] for the case $k = 1$,

$N=2$): if $a, p, b_a \in \mathbb{R}$ satisfy

$$a > (2-N)\frac{k}{N} \ , \quad \max\left\{2, \frac{2(N-k)}{N-2+a}\right\} < p \le 2^* \ , \quad b_a = N - p\frac{N-2+a}{2} \ , \qquad (0.4)$$

then there exists $c_{a,p} > 0$ such that

$$c_{a,p}\left(\int_{\mathbb{R}^N} |x|^{-b_a}|u|^p \ d\xi\right)^{2/p} \le \int_{\mathbb{R}^N} |x|^a|\nabla u|^2 \ d\xi \ , \quad \forall u \in C_c^\infty(\mathbb{R}^N). \qquad (0.5)$$

Thanks to inequality (0.5), we can define the Hilbert space $D_0^1(\mathbb{R}^N; |x|^a d\xi)$ by completing $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|^2 = \int_{\mathbb{R}^N} |x|^a|\nabla u|^2$. Then, every minimizer for

$$S_{a,p}^D := \inf_{D_0^1(\mathbb{R}^N;|x|^a d\xi)} \frac{\int_{\mathbb{R}^N} |x|^a|\nabla u|^2 \ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_a}|u|^p \ d\xi\right)^{2/p}} \qquad (0.6)$$

is, up to a Lagrange multiplier, a solution to (0.1). In case $a > 2-k$, the existence of solutions to (0.6) follows from Theorems B1 and B2 in [15]. In the present paper we focus our attention on the case $a \le 2-k$.

While studying (0.6) one has to take into account the action of the groups of dilations in $\mathbb{R}^N$ and of translations in $\mathbb{R}^{N-k}$. Indeed, it is quite easy to exhibit non compact minimizing sequences. To overcome this difficulty we use a suitable Rellich type theorem and a rescaling argument. Our existence result is stated in Theorem 2.1. If $N \ge 3$ and $p = 2^*$, the group of translations in the x-variable produces some extra lack of compactness phenomena. For this reason, to construct a compact minimizing sequence we need the additional assumption $S_{a,2^*}^D < S$, where S is the Sobolev constant. Some sufficient conditions for $S_{a,2^*}^D < S$ can be found in Section 3.2. Our main existence result in the critical case is stated in Theorem 3.9.

In Section 3 we take $a < 2-k$ and we compare the solution u^D to the minimization problem (0.6) with the solution $u^X \in D_0^1(\mathbb{R}^N; |x|^a d\xi) \cap L^2(\mathbb{R}^N; |x|^{a-2}d\xi)$ to (0.1), whose existence was proved in [15]. Under suitable assumptions on the exponents a and p, we are able to prove that u^X do not solve (0.6). Following this idea we prove several multiplicity results for (0.1). This will be done in Section 3.1 for $p < 2^*$ and in Section 3.2 in the limiting case $N \ge 3$, $p = 2^*$.

In the last Section we address our attention on the existence of classical solutions to

$$\begin{cases} -\Delta v = \lambda|x|^{-2} \ v + |x|^{-b_0} \ v^{p-1} & \text{in } \mathbb{R}^N \ , \ x \ne 0 \\ v > 0 \ , \end{cases} \qquad (0.7)$$

where $\lambda, p, b_0 \in \mathbb{R}$ satisfy

$$\lambda \le \left(\frac{k-2}{2}\right)^2 \ , \quad p \in (2, 2^*] \ , \quad b_0 = N - p\frac{N-2}{2} \ .$$

First of all we use a functional change and Theorem 2.1 to prove the existence of a cylindrically symmetric solution for $\lambda = \left(\frac{k-2}{2}\right)^2$ (see Theorem 4.1). Existence was already proved in [16] in case $p = 2^*$.

In Subsection 4.2 we state some existence results of singular solutions to (0.7) under the assumption $\lambda < \left(\frac{k-2}{2}\right)^2$. In this Introduction we just mention an unexpected consequence of our arguments in the limiting case $p = 2^*$. It is well known that problem

$$\begin{cases} -\Delta v = v^{2^*-1} & \text{in } \mathbb{R}^N \\ v > 0 \end{cases} \tag{0.8}$$

has a solution in $D_0^1(\mathbb{R}^N)$ which is unique up to translations and up to changes of scale in $\mathbb{R}^N$. Moreover, its L^{2^*}- norm is $S^{N/2}$, where S is the Sobolev constant. In Corollary 4.4 we prove that, for $N \ge 7$, problem (0.8) has a smooth cylindrically symmetric solution $v_\infty : (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^{N-3} \to \mathbb{R}$, such that

$$\int_{\mathbb{R}^N} |\nabla v_\infty|^2 \, d\xi = +\infty \ , \quad \int_{\mathbb{R}^N} |v_\infty|^{2^*} \, d\xi < S^{N/2} \ .$$

Notation For any integer $j \ge 1$, we denote by $B_R^j(z)$ the j-dimensional ball of radius R centered at $z \in \mathbb{R}^j$. We denote by ω_j the surface measure of the unit sphere $\mathbb{R}^j$.

It will be often convenient to set

$$\mathbb{R}_0^N := (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k} \ .$$

If $p \in [1, +\infty)$, $\alpha \in \mathbb{R}$, and if Ω is a domain in $\mathbb{R}^N$, then $L^p(\Omega; |x|^\alpha d\xi)$ is the space of measurable maps u on Ω with $\int_\Omega |x|^\alpha |u|^p \, d\xi < +\infty$. Therefore $L^p(\Omega; |x|^0 d\xi) \equiv L^p(\Omega)$ is the standard Lebesgue space. The Hilbert space $D_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the L^2-norm of ∇u.

For $N \ge 3$ the critical Sobolev exponent is $2^* = \frac{2N}{N-2}$. The Sobolev constant

$$S := \inf_{D_0^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |u|^{2^*} \, d\xi\right)^{2/2^*}} \ ,$$

is achieved by the map

$$v_T(x, y) := \left(1 + |x|^2 + |y|^2\right)^{-\frac{N-2}{2}} \ . \tag{0.9}$$

Accordingly with [12], a smooth map v on $\mathbb{R}_0^N = (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$ is said to be *cylindrically symmetric* if $u(\cdot, y)$ is symmetric decreasing in $\mathbb{R}^k$ for any choice of $y \in \mathbb{R}^{N-k}$, and if there exists $y_0 \in \mathbb{R}^{N-k}$ such that $u(x, \cdot)$ is symmetric decreasing about $y_0 \in \mathbb{R}^{N-k}$, for any choice of $x \in \mathbb{R}^k \setminus \{0\}$.

1 Preliminaries

One of the main features in problem (0.1) is its invariance with respect to transforms

$$u(x,y) \longrightarrow (T(\tau,\eta)u)(x,y) := \tau^{\frac{N-2+a}{2}} u(\tau x, \tau y + \eta) \ , \tag{1.1}$$

where $\tau \in (0,+\infty)$, $\eta \in \mathbb{R}^{N-k}$. In addition, it holds also that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^a |\nabla (T(\tau,\eta)u)|^2 \ d\xi &= \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \ d\xi \ , \\ \int_{\mathbb{R}^N} |x|^{-b_a} |T(\tau,\eta)u|^p \ d\xi &= \int_{\mathbb{R}^N} |x|^{-b_a} |u|^p \ d\xi \end{aligned} \tag{1.2}$$

for any $u \in C^\infty_c(\mathbb{R}^N)$, whenever the weights $|x|^a$ and $|x|^{-b_a}$ are locally integrable on $\mathbb{R}^N$. Indeed, these invariances are responsible for the integral inequalities that are the starting point of the present paper and that were proved by Maz'ya [14] in 1980.

1.1 Maz'ya inequality. The space $D^1_0(\mathbb{R}^N; |x|^a d\xi)$

Assume that a, p, b_a in $\mathbb{R}$ satisfy (0.4). Then the weights $|x|^a$ and $|x|^{-b}$ are locally integrable on $\mathbb{R}^N$. The *Maz'ya inequality* states that there exists $c_{a,p} > 0$ such that

$$c_{a,p} \left(\int_{\mathbb{R}^N} |x|^{-b_a} |u|^p \ d\xi \right)^{2/p} \le \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \ d\xi \ , \quad \forall u \in C^\infty_c(\mathbb{R}^N). \tag{1.3}$$

Inequality (1.3) was proved by Maz'ya in [14], Section 2.1.6, Corollary 2, under the assumptions $N \ge 3$ and $p \le 2^*$. As observed in [5], Section 4, if $N = 2$ and $k = 1$, then the same inequality holds true for any $p \in (2, +\infty)$.

Thanks to (1.3), for any $a > (2-N)\frac{k}{N}$ we can define the Hilbert space $D^1_0(\mathbb{R}^N; |x|^a d\xi)$ by completing $C^\infty_c(\mathbb{R}^N)$ with respect to the scalar product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} |x|^a \ \nabla u \cdot \nabla v \ d\xi \ .$$

Notice that for $N \ge 3$, $D^1_0(\mathbb{R}^N; |x|^0 d\xi) \equiv D^1_0(\mathbb{R}^N)$. From (1.3) it follows that the infimum

$$S^D_{a,p} := \inf_{u \in D^1_0(\mathbb{R}^N, |x|^a d\xi)} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_a} |u|^p \ d\xi \right)^{2/p}}$$

is positive, provided that a, p, b_a are as in (0.4). It is clear that if $S^D_{a,p}$ is achieved by $u \in D^1_0(\mathbb{R}^N; |x|^a d\xi)$ then u is a classical solution to (0.1), up to multiplicative constant. By standard elliptic regularity theory and by the maximum principle, u is smooth and positive on $\{x \neq 0\}$. In addition, u is a *weak solution* to (0.1) on $\mathbb{R}^N$, that is,

$$\int_{\mathbb{R}^N} |x|^a \nabla u \cdot \nabla \Phi \ d\xi = \int_{\mathbb{R}^N} |x|^{-b_a} u^{p-1} \Phi \ d\xi \ , \qquad \forall \Phi \in C^\infty_c(\mathbb{R}^N) \ , \tag{1.4}$$

and it is *entire*, namely,

$$\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi = \int_{\mathbb{R}^N} |x|^{-b_a} u^p \, d\xi < +\infty \ .$$

The problem of the existence of minimizers for $S^D_{a,p}$ will be discussed in Section 2.

1.2 Hardy-Sobolev-Maz'ya inequality. The space $X^1_0(\mathbb{R}^N; |x|^a d\xi)$

We first recall the *Hardy inequality* with cylindrical weights. It states that

$$\left(\frac{k-2+a}{2}\right)^2 \int_{\mathbb{R}^N} |x|^{a-2}|u|^2 \, d\xi \le \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi \tag{1.5}$$

for any $u \in C^\infty_c(\mathbb{R}^N)$ if $a > 2-k$, and for any $u \in C^\infty_c(\mathbb{R}^N_0)$ if $a \le 2-k$ (see for example [14], Section 2.1.6, or [7], Theorem 3.5). The constant

$$\lambda_1(a) := \left(\frac{k-2+a}{2}\right)^2 \tag{1.6}$$

is sharp, and it is not achieved. Notice that Hardy's inequality implies that $D^1_0(\mathbb{R}^N; |x|^a d\xi)$ is continuously embedded into $L^2(\mathbb{R}^N; |x|^{a-2} d\xi)$, provided that $a > 2-k$.

The *Hardy-Sobolev-Maz'ya inequality* was proved in [14] (Section 2.1.6, Corollary 3) in case $N \ge 3$, and in Section 4 of [5] in case $N = 2$, $k = 1$. It states that for any $a \in \mathbb{R}$ and for every real exponent $p \in (2, 2^*]$ there exists a constant $c_{a,p} > 0$ such that

$$c_{a,p} \left(\int_{\mathbb{R}^N} |x|^{-b_a} |u|^p \, d\xi\right)^{2/p} \le \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi - \lambda_1(a) \int_{\mathbb{R}^N} |x|^{a-2} |u|^2 \, d\xi \tag{1.7}$$

for any $u \in C^\infty_c(\mathbb{R}^N)$ if $a > 2-k$, and for any $u \in C^\infty_c(\mathbb{R}^N_0)$ if $a \le 2-k$. We remark that (1.7) fails if $k = N$. We refer to [9] for a discussion on this subject.

Inequality (1.7) provides the starting point for applying variational methods to the singular problem

$$\begin{cases} -\mathrm{div}(|x|^a \nabla u) = \lambda |x|^{a-2} u + |x|^{-b_a} u^{p-1} & \text{in } \mathbb{R}^N \ , \ x \neq 0 \\ u > 0 \ , \end{cases} \tag{1.8}$$

where λ is a real parameter. For future convenience we recall here the approach used in [15] to study (1.8) in case $\lambda < \lambda_1(a)$. For any $a \in \mathbb{R}$ define the Hilbert space

$$X^1_0(\mathbb{R}^N; |x|^a d\xi) := D^1_0(\mathbb{R}^N; |x|^a d\xi) \cap L^2(\mathbb{R}^N; |x|^{a-2} d\xi) \ .$$

Notice that $X^1_0(\mathbb{R}^N; |x|^a d\xi) = D^1_0(\mathbb{R}^N; |x|^a d\xi)$ if $a > 2-k$, by (1.5). It turns out that $C^\infty_c((\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k})$ is dense in $X^1_0(\mathbb{R}^N; |x|^a d\xi)$ ([15], Appendix B) and that (1.7) holds for any $u \in X^1_0(\mathbb{R}^N; |x|^a d\xi)$. The paper [15] deals with the existence of extremals for

$$\inf_{u \in X^1_0(\mathbb{R}^N; |x|^a d\xi)} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi - \lambda \int_{\mathbb{R}^N} |x|^{a-2} |u|^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_a} |u|^p \, d\xi\right)^{2/p}}$$

under the assumption $\lambda < \lambda_1(a)$. In particular, for $\lambda = 0$ and $a \neq 2-k$, every solution $u^X \in X_0^1(\mathbb{R}^N; |x|^a d\xi)$ to the minimization problem

$$S_{a,p}^X := \inf_{u \in X_0^1(\mathbb{R}^N;|x|^a d\xi)} \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_a} |u|^p \, d\xi\right)^{2/p}} \tag{1.9}$$

is, up to a Lagrange multiplier, an entire classical solution to (0.1). Notice also that u^X is a weak entire solution to (0.1) on $(\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$. If $a \leq 2-k$, in general, one can not conclude that u^X is indeed a weak solution on the whole of $\mathbb{R}^N$. On the other hand, for $a > 2-k$ one can take advantage of the Hardy inequality (1.5) to show that $S_{a,p}^D = S_{a,p}^X$, and that u^X satisfies (1.4) (see [15], Theorem A.2).

Finally we notice that in general $S_{a,p}^D \leq S_{a,p}^X$ for any $a > (2-N)\frac{k}{N}$, since $X_0^1(\mathbb{R}^N; |x|^a d\xi)$ is contained in $D_0^1(\mathbb{R}^N; |x|^a d\xi)$. In Section 3 we illustrate some examples in which the strict inequality $S_{a,p}^D < S_{a,p}^X$ holds true.

Concerning the existence of minimizers for $S_{a,p}^X$ we can state the following result.

Theorem 1.1 *Assume $a \neq 2-k$, and let $p > 2$. The infimum $S_{a,p}^X$ is achieved by a map $u^X \in X_0^1(\mathbb{R}^N; |x|^a d\xi)$ provided that one of the following conditions holds:*

a) $N = 2$ or $p \in (2, 2^*)$;

b) $N \geq 3$, $p = 2^*$ and $S_{a,2^*}^X < S$.

Moreover, if $k = 1$ then the support of u^X is a half-plane.

Theorem 1.1 is a direct consequence of Theorems B.1 and B.2 in [15] in case $N \geq 3$; the same proof can be carried out in case $N = 2$, $k = 1$.

We conclude this subsection with a few remarks on the infimum $S_{a,p}^X$. First we state a useful Lemma that was proved in [15], Appendix B.

Lemma 1.2 *For any $a \in \mathbb{R}$ the linear operator $L_a(u) := |x|^{\frac{a}{2}} u$ is a bi-continuous isomorphism between $X_0^1(\mathbb{R}^N; |x|^a d\xi)$ and $X_0^1(\mathbb{R}^N; d\xi) = D_0^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2} d\xi)$. Moreover,*

$$\int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \; - \lambda_1(a) \int_{\mathbb{R}^N} |x|^{a-2} |u|^2 \; = \int_{\mathbb{R}^N} |\nabla (L_a u)|^2 \; - \lambda_1(0) \int_{\mathbb{R}^N} |x|^{-2} |L_a u|^2 \tag{1.10}$$

for any $u \in X_0^1(\mathbb{R}^N; |x|^a d\xi)$.

In the next lemma we point out some remarks on the behavior of the map $a \to S_{a,p}^X$.

Lemma 1.3 *Assume $2 < p < 2^*$. Then the map $a \to S_{a,p}^X$ is strictly increasing for $a \geq 2-k$, and it is strictly decreasing for $a \leq 2-k$.*

Proof. For $a \in \mathbb{R}$ let $\lambda_1(a)$ be as in (1.6). Set $\bar{a} := 2(2-k)-a$ and notice that $\lambda_1(a) = \lambda_1(\bar{a})$. Then, by (1.10), it turns out that

$$
\begin{aligned}
S^X_{a,p} &= \inf_{v \in X^1_0(\mathbb{R}^N; d\xi)} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, d\xi - (\lambda_1(0) - \lambda_1(a)) \int_{\mathbb{R}^N} |x|^{-2} v^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_a} |v|^p \, d\xi\right)^{2/p}} \\
&= \inf_{v \in X^1_0(\mathbb{R}^N; d\xi)} \frac{\int_{\mathbb{R}^N} |\nabla v|^2 \, d\xi - (\lambda_1(0) - \lambda_1(\bar{a})) \int_{\mathbb{R}^N} |x|^{-2} v^2 \, d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_a} |v|^p \, d\xi\right)^{2/p}} = S^X_{\bar{a},p} \; .
\end{aligned}
$$

Thus $S^X_{a,p} = S^X_{\bar{a},p}$. The Lemma is readily proved, since the map $a \to \lambda_1(a)$ is increasing for $a \ge 2-k$, and since $S^X_{a,p}$ is achieved for any $a \ne 2-k$. □

The case $N \ge 3$, $p = 2^*$ is more difficult. We recall that $S^X_{a,2^*} \le S$ for any $a \in \mathbb{R}$ (see [15], Theorem B.5), and that the map v_T defined in (0.9) achieves the best Sobolev constant S on $D^1_0(\mathbb{R}^N)$. Moreover $v_T \in L^2(\mathbb{R}^N; |x|^{-2} d\xi)$, that is, $v_T \in X^1_0(\mathbb{R}^N; d\xi)$, if and only if $k \ge 3$. In this case, a direct computation shows that the map

$$
u_T(x,y) := |x|^{k-2} v_T(x,y)
$$

belongs to $X^1_0(\mathbb{R}^N; |x|^{2(2-k)} d\xi)$ and achieves $S^X_{2(2-k),2^*} = S$. On the contrary, if $k = 1, 2$ then $v_T \notin L^2(\mathbb{R}^N; |x|^{-2} d\xi)$, and $S^X_{0,2^*} = S^X_{2(2-k),2^*} = S$ are not achieved.

In the next Lemma we collect some remarks on the behavior of the map $a \to S^X_{a,2^*}$.

Lemma 1.4 *Assume $N \ge 3$ and let $a \in \mathbb{R}$. Then the map $a \to S^X_{a,2^*}$ is increasing for $a \ge 2-k$, and it is decreasing for $a \le 2-k$. Moreover:*

1. *$S^X_{a,2^*} \le S$ for any $a \in \mathbb{R}$.*
2. *$S^X_{a,2^*} = S$ and $S^X_{a,2^*}$ is not achieved in the following cases:*
   $k = 1$ and $N = 3$, or $k = 1$, $N \ge 4$ and $a \notin (0,2)$;
   $k = 2$
   $k \ge 3$, $a \notin [2(2-k), 0]$.
3. *$S^X_{a,2^*}$ is achieved in the following cases:*
   $k = 1$, $N \ge 4$, $a \in (0,2)$, $a \ne 1$;
   $k \ge 3$, $a \in [2(2-k), 0]$, $a \ne 2-k$.

Proof. The monotonicity properties of the map $a \to S^X_{a,2^*}$ can be checked as in Lemma 1.3. For the proof of 1. we refer to [15], Theorem B.5. By contradiction, assume that $k = 1$, $N = 3$ and that $S^X_{a,6}$ is achieved by a map $u \in X^1_0(\mathbb{R}^3; |x|^a d\xi)$, for some $a \in \mathbb{R}$. Then $a \in (0,2)$ by [15], Theorem B.5. Set $v(x,y) = |x|^{a/2} u(x,y)$. Then, by Proposition B.3 and by Lemma 1.2 of [15], we can assume that the support of v is contained in the half space $(0,+\infty)\times\mathbb{R}^2$, and that v solves

$$\begin{cases} -\Delta v = \lambda |x|^{-2} v + v^5 & \text{in } (0,+\infty)\times\mathbb{R}^2 \\ v > 0 & \\ v \in D_0^1((0,+\infty)\times\mathbb{R}^2)\ , & \end{cases}$$

where

$$\lambda = \frac{1}{4} - \left(\frac{a-1}{2}\right)^2 \quad , \quad 0 < \lambda < \frac{1}{4}\ .$$

This clearly contradicts the nonexistence result in [13], Section 6. The proofs of the remaining statements can be found in [15], Appendix B. □

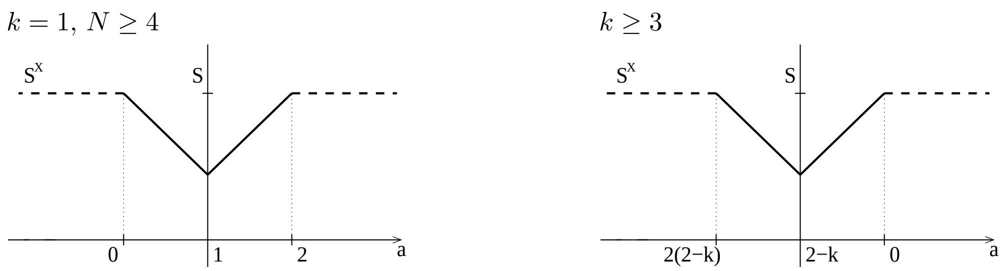

Fig.1 Graphics of $a \to S^X_{a,2^*}$. - - - $= S^X_{a,2^*}$ not achieved; —— $= S^X_{a,2^*}$ achieved.

1.3 The space $\hat{X}_0^1(\mathbb{R}^k \times \mathbb{R}^{N-k})$

For $a = 0$ inequality (1.7) becomes

$$c_p \left(\int_{\mathbb{R}^N} |x|^{-b_0} |v|^p \, d\xi \right)^{2/p} \le \int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0) |x|^{-2} |v|^2 \ \right] \, d\xi \tag{1.11}$$

for any $v \in C_c^\infty(\mathbb{R}_0^N)$, where, as before, $\mathbb{R}_0^N = (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$ and $b_0 = N - p\frac{N-2}{2}$.

Define as in [16] the Hilbert space $\hat{X}_0^1(\mathbb{R}^k \times \mathbb{R}^{N-k})$ by completing $C_c^\infty(\mathbb{R}_0^N)$ with respect to the scalar product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \left[\nabla u \cdot \nabla v - \lambda_1(0) |x|^{-2} uv \right] \, d\xi \ .$$

Notice that $X_0^1(\mathbb{R}^N; d\xi) = D_0^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2} d\xi) \subset \hat{X}_0^1(\mathbb{R}^k \times \mathbb{R}^{N-k})$. In particular, $D_0^1(\mathbb{R}^N)$ is contained in $\hat{X}_0^1(\mathbb{R}^k \times \mathbb{R}^{N-k})$ if and only if $k \ge 2$, and $D_0^1(\mathbb{R}^N) = \hat{X}_0^1(\mathbb{R}^2 \times \mathbb{R}^{N-2})$ if $k = 2$. In Section 4.1 we will study the existence of extremals for the inequality (1.11). For future convenience we point out a Lemma.

Lemma 1.5 *The linear operator* $L_{(k-2)}v := |x|^{\frac{k-2}{2}}v$ *is a bi-continuous isomorphism between* $\hat{X}^1_0(\mathbb{R}^k \times \mathbb{R}^{N-k})$ *and* $D^1_0(\mathbb{R}^N; |x|^{2-k}d\xi)$*. Moreover,*

$$\int_{\mathbb{R}^N} |x|^{2-k}|\nabla(L_{(k-2)}v)|^2 \, d\xi = \int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0)|x|^{-2}|v|^2\right] \, d\xi \tag{1.12}$$

for any $v \in \hat{X}^1_0(\mathbb{R}^k \times \mathbb{R}^{N-k})$.

Proof. An application of the divergence theorem shows that (1.12) holds for any $v \in C^\infty_c(\mathbb{R}^N_0)$. We have to prove that $C^\infty_c(\mathbb{R}^N_0)$ is dense in $D^1_0(\mathbb{R}^N; |x|^{2-k}d\xi)$. Fix any map $v \in C^\infty_c(\mathbb{R}^N)$. For $\varepsilon > 0$ set

$$\varphi_\varepsilon(|x|) = \begin{cases} 0 & \text{if } |x| \le \varepsilon^2 \\ \frac{\log |x|/\varepsilon^2}{|\log \varepsilon|} & \text{if } \varepsilon^2 < |x| < \varepsilon \\ 1 & \text{if } |x| \ge \varepsilon \ . \end{cases}$$

It is clear that $\varphi_\varepsilon v \in D^1_0(\mathbb{R}^N; |x|^{2-k}d\xi)$ and that $\nabla(v - \varphi_\varepsilon v) = (1 - \varphi_\varepsilon)\nabla v - v\nabla\varphi_\varepsilon \to 0$ a.e. on $\mathbb{R}^N$ as $\varepsilon \to 0$. To prove that $\varphi_\varepsilon v \to v$ in $D^1_0(\mathbb{R}^N; |x|^{2-k}d\xi)$ it suffices to remark that

$$\int_{\mathbb{R}^N} |x|^{2-k}|v\nabla\varphi_\varepsilon|^2 \, d\xi \le C_v \int_{\mathbb{R}^k} |x|^{2-k}|\varphi'_\varepsilon|^2 \, dx = O(|\log \varepsilon|^{-1}) \ .$$

The conclusion follows via Lebesgue's theorem, since $|(1 - \varphi_\varepsilon)\nabla v| \le |\nabla v|$ on $\mathbb{R}^N$, and since $|\nabla v| \in L^2(\mathbb{R}^N; |x|^{2-k}d\xi)$. □

2 Existence results for (0.1)

The main result in this Section concerns the existence of extremal functions for the Maz'ya inequality.

Theorem 2.1 *Assume that (0.3) is satisfied. Then* $S^D_{a,p}$ *is achieved on* $D^1_0(\mathbb{R}^N; |x|^a d\xi)$ *by a weak entire solution* $u^D \in D^1_0(\mathbb{R}^N; |x|^a d\xi)$ *to (0.1) provided that one of the following conditions are satisfied:*

a) $N = 2$ *or* $p < 2^*$*;*

b) $N \ge 3$*,* $p = 2^*$ *and* $S^D_{a,2^*} < S$*.*

Remark 2.2 Assume in addition that $k = 1$ and $a = 0$, hence $p > 2(N-1)/(N-2)$. Then the solution u^D in Theorem 2.1 is cylindrically symmetric by [10], Theorem 0.2. If $k \ge 3$ and $2(2-k) \le a \le 2-k$, then u^D turns out to be radially symmetric in the x-variable, by Corollary 5.2 in [10].

Remark 2.3 Assume $N \ge 3$, $p = 2^*$ and $a \in ((2-N)k/N), 2-k]$, $a \ne 0$. If $k = 1$ assume in addition that $N \ge 4$ or $a < 0$. In Section 3.2 we will show that under these assumptions condition $S^D_{a,2^*} < S$ is always satisfied. The case $k = 1$, $N = 3$ and $a \in (0, 1]$ is still open.

The first step in the proof is a Rellich-type theorem.

Lemma 2.4 *Assume* $\frac{2-N}{N}k < a \le 2-k$*, and let* Ω *be a bounded domain. Then*

$$D^1_0(\mathbb{R}^N; |x|^a d\xi) \hookrightarrow L^2(\Omega, |x|^a d\xi)$$

with compact inclusion.

Proof. Fix a map $u \in C^\infty_c(\mathbb{R}^N)$ and choose any exponent $p > 2$, satisfying also $p \le 2^*$ if $N \ge 3$. Let $b_a = N - p\frac{N-2+a}{2}$ as in (0.2). Notice that

$$b_a + \frac{ap}{2} \ge 0 \ .$$

Thus, Hölder and Maz'ya inequalities give

$$\int_\Omega |x|^a |u|^2 \ d\xi \le |\Omega|^{\frac{p}{p-2}} \left(\int_\Omega |x|^{\frac{ap}{2}} |u|^p \ d\xi \right)^{\frac{2}{p}} \le c|\Omega|^{\frac{p}{p-2}} \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \ d\xi \ , \qquad (2.1)$$

where $|\Omega|$ is the Lebesgue measure of the domain Ω and the constant c does not depend on u. This proves the continuity of the embedding. Now, let u_h be a sequence in $D^1_0(\mathbb{R}^N; |x|^a d\xi)$, with $u_h \to 0$ weakly in $D^1_0(\mathbb{R}^N; |x|^a d\xi)$. For $\varepsilon > 0$ small take a smooth function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^k)$ such that $\varphi_\varepsilon(x) = 0$ if $|x| \le \varepsilon^2$, and $\varphi_\varepsilon(x) = 1$ if $|x| \ge \varepsilon$. Notice that for ε fixed it results that

$$\int_\Omega |x|^a |\varphi_\varepsilon u_h|^2 \ d\xi = o(1)$$

as $h \to +\infty$ by the standard Rellich Theorem, since $|x|$ is bounded away from 0 on the support of φ_ε. Set $\Omega_\varepsilon := \{(x,y) \in \Omega \mid |x| < \varepsilon \}$, and notice that

$$\begin{aligned} \int_\Omega |x|^a |u_h|^2 \ d\xi &= \int_\Omega |x|^a |\varphi_\varepsilon u_h + (1-\varphi_\varepsilon) u_h|^2 \ d\xi \\ &\le 2\int_\Omega |x|^a |\varphi_\varepsilon u_h|^2 \ d\xi + 2\int_{\Omega_\varepsilon} |x|^a |1-\varphi_\varepsilon|^2 |u_h|^2 \ d\xi \\ &\le o(1) + c\int_{\Omega_\varepsilon} |x|^a |u_h|^2 \ d\xi \ . \end{aligned}$$

Next, fix $p > 2$ as before, and argue as for (2.1) to get

$$\int_\Omega |x|^a |u_h|^2 \ d\xi \le o(1) + c|\Omega_\varepsilon|^{\frac{p}{p-2}} \int_{\mathbb{R}^N} |x|^a |\nabla u_h|^2 \ d\xi \ .$$

Notice that the sequence u_h is bounded in $D^1_0(\mathbb{R}^N; |x|^a d\xi)$ and that $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$. Therefore, passing to the limit first as h go to 0, and then as $\varepsilon \to 0$, we get that

$$\int_\Omega |x|^a |u_h|^2 \ d\xi \to 0,$$

that proves the compactness of the embedding. □

The next Lemma will allow us to use suitable test functions.

Lemma 2.5 *If $\Psi \in C_c^\infty(\mathbb{R}^N)$ then $\Psi u \in D_0^1(\mathbb{R}^N; |x|^a d\xi)$ for any $u \in D_0^1(\mathbb{R}^N; |x|^a d\xi)$.*

Proof. We can approximate any fixed $u \in D_0^1(\mathbb{R}^N; |x|^a d\xi)$ with a sequence $u_h \in C_c^\infty(\mathbb{R}^N)$. Using Lemma 2.4 it is easy to prove that $\Psi u_h \to \Psi u$ in $D_0^1(\mathbb{R}^N; |x|^a d\xi)$, hence $\Psi u \in D_0^1(\mathbb{R}^N; |x|^a d\xi)$. □

The main step in the proof of Theorem 2.1 is the following Lemma. It is based on a rescaling argument that is quite close to that used in [15].

Lemma 2.6 *Let u_h be a bounded sequence in $D_0^1(\mathbb{R}^N; |x|^a d\xi)$, and let $f_h \to 0$ be a sequence in the dual of $D_0^1(\mathbb{R}^N; |x|^a d\xi)$. Assume that for a, p, b_a as in (0.3), (0.2) it holds that*

$$-\mathrm{div}(|x|^a \nabla u_h) = |x|^{-b_a} |u_h|^{p-2} u_h + f_h \ .$$

Then either (up to a subsequence) $u_h \to 0$ strongly in $L^p(\mathbb{R}^N; |x|^{-b_a} d\xi)$, or there exist sequences $(t_h)_h \subset (0, +\infty)$ and $(\eta_h)_h \subset \mathbb{R}^{N-k}$, such that

$$\lim_{h\to+\infty} \int_K |x|^{-b_a} |\tilde{u}_h|^p \ d\xi > 0 \ ,$$

where $\tilde{u}_h(x, y) = t_h^{\frac{N-2+a}{2}} u_h(t_h x, t_h y + \eta_h)$ and

$$K = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \mid \frac{1}{2} < |x| < 1 \ , \quad |y| < 1 \ \} \ .$$

Proof. We can assume that there exists $u \in D_0^1(\mathbb{R}^N; |x|^a d\xi)$ such that $u_h \to u$ weakly in $D_0^1(\mathbb{R}^N; |x|^a d\xi)$ and in $L^p(\mathbb{R}^N; |x|^{-b_a} d\xi)$. Notice that u solves (0.1). If $u \neq 0$ then we are done, since by elliptic regularity theory and by the standard maximum principle u can not vanish on K. Therefore, we only have to investigate the case $u = 0$, $\lim_{h\to+\infty} \int_{\mathbb{R}^N} |x|^{-b_a} |u_h|^p \ d\xi > 0$. We argue as follows. Firstly we fix some $\varepsilon_0 > 0$ in such a way that

$$\varepsilon_0^{\frac{p}{p-2}} < \lim_{h\to+\infty} \int_{\mathbb{R}^N} |x|^{-b_a} |u_h|^p \ d\xi \ , \qquad 2\varepsilon_0 < S_{a,p}^D \ .$$

Using in a standard way the concentration function

$$Q_h(t) := \sup_{\eta \in \mathbb{R}^{N-k}} \int_{B_t^k(0) \times B_t^{N-k}(\eta)} |x|^{-b_a} \ |u_h|^p \ d\xi \ ,$$

it is possible to select sequences $t_h > 0$ and $\eta_h \in \mathbb{R}^{N-k}$ such that, for a subsequence, the rescaled sequence

$$\tilde{u}_h(x, y) := t_h^{\frac{N-2+a}{2}} u_h(t_h x, t_h y + \eta_h)$$

is bounded in $D_0^1(\mathbb{R}^N;|x|^a d\xi)$ (with the same $D_0^1(\mathbb{R}^N;|x|^a d\xi)$ and $L^p(\mathbb{R}^N;|x|^{-b_a}d\xi)$-norms as u_h), and satisfies

$$\int_{B_1^k(0)\times B_1^{N-k}(y)} |x|^{-b_a}|\tilde{u}_h|^p \ d\xi \le (2\varepsilon_0)^{\frac{p}{p-2}} \quad \forall y\in\mathbb{R}^{N-k} \ , \tag{2.2}$$

$$\int_{B_1^k(0)\times B_1^{N-k}(0)} |x|^{-b_a}|\tilde{u}_h|^p \ d\xi \ge (\varepsilon_0)^{\frac{p}{p-2}} > 0 \ , \tag{2.3}$$

$$-\mathrm{div}(|x|^a\nabla\tilde{u}_h) = |x|^{-b_a}|\tilde{u}_h|^{p-2}\tilde{u}_h + \tilde{f}_h \ , \tag{2.4}$$

with $\tilde{f}_h\to 0$ in the dual of $D_0^1(\mathbb{R}^N;|x|^a d\xi)$. As before, if (up to a subsequence) it happens that $\tilde{u}_h \to \tilde{u}\neq 0$ then we are done. If $\tilde{u}_h\to 0$, we choose a finite number of points $y_1,...,y_s\in\mathbb{R}^{N-k}$ in such a way that

$$\overline{B_1^{N-k}(0)} \subset \bigcup_{j=1}^{s} B_{1/2}^{N-k}(y_j) \ . \tag{2.5}$$

Let $\psi_1,...\psi_s$ be cut-off functions, such that $\psi_j = \psi_j(y)\in C_c^\infty(B_1^{N-k}(y_j)), \psi_j\equiv 1$ on $B_{1/2}^{N-k}(y_j)$, $0\le\psi_j\le 1$. Also, fix a map $\varphi=\varphi(x)\in C_c^\infty(B_1^k(0))$ s.t. $0\le\varphi\le 1$ and $\varphi\equiv 1$ on $B_{1/2}^k(0)$. Thanks to Lemma 2.5 we can use $\varphi^2\psi_i^2\tilde{u}_h$ as test function in (2.4) to find

$$\int_{\mathbb{R}^N} |x|^a\nabla\tilde{u}_h\cdot\nabla(\varphi^2\psi_j^2\tilde{u}_h)\ d\xi = \int_{\mathbb{R}^N} |x|^{-b_a}|\tilde{u}_h|^{p-2}|\varphi\psi_j\tilde{u}_h|^2\ d\xi + o(1)\ .$$

Notice that

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^a\nabla\tilde{u}_h\cdot\nabla(\varphi^2\psi_j^2\tilde{u}_h)\ d\xi &= \int_{\mathbb{R}^N}|x|^a|\nabla(\varphi\psi_j\tilde{u}_h)|^2\ d\xi \\ &- \int_{\mathbb{R}^N}|x|^a|\tilde{u}_h|^2(|\psi_j\nabla_x\varphi|^2+|\varphi\nabla_y\psi_j|^2)\ d\xi \\ &= \int_{\mathbb{R}^N}|x|^a|\nabla(\varphi\psi_j\tilde{u}_h)|^2\ d\xi + o(1)\end{aligned}$$

by Lemma 2.4. Then, we use Hölder inequality, (2.2), and the definition of $S_{a,p}^D$ to infer that

$$S_{a,p}^D\left(\int_{\mathbb{R}^N}|x|^{-b_a}|\varphi\psi_j\tilde{u}_h|^p\ d\xi\right)^{\frac{2}{p}} \le 2\varepsilon_0\left(\int_{\mathbb{R}^N}|x|^{-b_a}|\varphi\psi_j\tilde{u}_h|^p\ d\xi\right)^{\frac{2}{p}} + o(1)\ .$$

Since $2\varepsilon_0 < S_{a,p}^D$ this implies that $\int_{\mathbb{R}^N}|x|^{-b_a}|\varphi\psi_j\tilde{u}_h|^p\ d\xi = o(1)$, and therefore, by (2.5),

$$\int_{B_{1/2}^k(0)\times B_1^{N-k}(0)}|x|^{-b_a}|\tilde{u}_h|^p\ d\xi \le \sum_{j=1}^{s}\int_{B_{1/2}^k(0)\times B_{1/2}^{N-k}(y_j)}|x|^{-b_a}|\tilde{u}_h|^p\ d\xi = o(1)\ .$$

Finally, from (2.3) we get

$$0<\varepsilon_0^{\frac{p}{p-2}} < \int_{B_1^k(0)\times B_1^{N-k}(0)}|x|^{-b_a}|\tilde{u}_h|^p\ d\xi = \int_K |x|^{-b_a}|\tilde{u}_h|^p\ d\xi + o(1)\ .$$

Lemma 2.6 is completely proved. □

Proof of Theorem 2.1. We notice that entire solutions to equation (0.1) can be found as critical points of the C^1 functional $E : D^1_0(\mathbb{R}^N; |x|^a d\xi) \to \mathbb{R}$ given by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |x|^a |\nabla u|^2 \, d\xi - \frac{1}{p} \int_{\mathbb{R}^N} |x|^{-b_a} |u|^p \, d\xi \ .$$

Using standard arguments one can prove the following statements:

i) the functional E has a mountain pass geometry, with mountain pass level

$$c = \frac{p-2}{2p} \, (S^D_{a,p})^{\frac{p}{p-2}} > 0 \ ;$$

ii) every Palais-Smale sequence for E is bounded in $D^1_0(\mathbb{R}^N; |x|^a d\xi)$;

iii) if u_h is a Palais-Smale sequence for E with $E(u_h) \to c$ and $u_h \to u \neq 0$ weakly in $D^1_0(\mathbb{R}^N; |x|^a d\xi)$, then u is a critical point of E with $E(u) = c$, it achieves $S^D_{a,p}$, it is a ground state solution to (0.1) and it does not change sign.

Let u_h be a Palais Smale sequence at the mountain pass level $c = \frac{p-2}{2p}(S^D_{a,p})^{\frac{p}{p-2}}$. We can assume that $u_h \to u$ weakly in $D^1_0(\mathbb{R}^N; |x|^a d\xi)$ for some $u \in D^1_0(\mathbb{R}^N; |x|^a d\xi)$. Notice that u_h can not converge strongly to 0, since $c > 0$. Thus, by Lemma 2.6 we can assume that

$$\lim_{h \to +\infty} \int_K |x|^{-b_a} |u_h|^p \, d\xi > 0$$

up to a change of coordinates, where $K = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} \mid \frac{1}{2} < |x| < 1 \ , \ \ |y| < 1 \ \}$. If $p < 2^*$, we can use Rellich Theorem to get

$$\int_K |x|^{-b_a} |u|^p \, d\xi = \lim_{h \to +\infty} \int_K |x|^{-b_a} |u_h|^p \, d\xi > 0 \ .$$

This proves that $u \neq 0$ and concludes the proof of *a*) in Theorem 2.1.

Now we assume $N \geq 3$, $p = 2^*$ and $S^D_{a,2^*} < S$. We claim that also in this case $u \neq 0$. By contradiction, assume that $u_h \to 0$ weakly in $D^1_0(\mathbb{R}^N; |x|^a d\xi)$. Choose smooth maps $\varphi \in C^\infty_c(\mathbb{R}^k)$ and $\psi \in C^\infty_c(\mathbb{R}^{N-k})$ in such a way that $\varphi(x) = 0$ for $|x| \leq \frac{1}{4}$, $\varphi(x) = 1$ for $\frac{1}{2} \leq |x| \leq 1$, $\psi(y) = 1$ for $|y| \leq 1$. Notice that $\varphi\psi \equiv 1$ on the rectangle K. Then compute $E'(u_h) \cdot (\varphi^2 \psi^2 u_h) = o(1)$ and argue as in the proof of Lemma 2.6 to get

$$\int_{\mathbb{R}^N} |x|^a |\nabla(\varphi\psi u_h)|^2 \, d\xi \leq S^D_{a,2^*} \left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-2}} |\varphi\psi u_h|^{2^*} \, d\xi \right)^{\frac{2}{2^*}} + o(1) \ . \tag{2.6}$$

Since $\varphi \equiv 0$ in a neighborhood of the singular set $\{x = 0\}$, via direct computation based on the divergence Theorem, one can prove that

$$\int |x|^a |\nabla(\varphi\psi u_h)|^2 = \int |\nabla(|x|^{\frac{a}{2}} \varphi\psi u_h)|^2 - (\lambda_1(0) - \lambda_1(a)) \int |x|^{a-2} |\varphi\psi u_h|^2,$$

where $\lambda_1(0), \lambda_1(a)$ are defined in (1.6). Thus,

$$\int |x|^a |\nabla(\varphi\psi u_h)|^2 = \int |\nabla(|x|^{\frac{a}{2}}\varphi\psi u_h)|^2 + o(1) \geq S\left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-2}} |\varphi\psi u_h|^{2^*}\ d\xi\right)^{\frac{2}{2^*}} + o(1)$$

by Rellich Theorem and Sobolev inequality. Therefore, from (2.6) we finally get

$$S\left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-2}} |\varphi\psi u_h|^{2^*}\ d\xi\right)^{\frac{2}{2^*}} \leq S^D_{a,2^*}\left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-2}} |\varphi\psi u_h|^{2^*}\ d\xi\right)^{\frac{2}{2^*}} + o(1),$$

that implies

$$\int_K |x|^{\frac{Na}{N-2}} |u_h|^{2^*}\ d\xi \leq \int_{\mathbb{R}^N} |x|^{\frac{Na}{N-2}} |\varphi(x)\psi(y)u_h|^{2^*}\ d\xi = o(1)\ ,$$

since $S^D_{a,2^*} < S$ by assumption. This is a contradiction that concludes the proof of b). □

3 Multiplicity for (0.1)

In this Section we deal with entire classical solutions to (0.1). More precisely, we look for multiple smooth maps u on $(\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^{N-k}$, that satisfy (0.1) pointwise, and such that

$$\int_{\mathbb{R}^N} |x|^a |\nabla u|^2\ d\xi = \int_{\mathbb{R}^N} |x|^{-b_a} |u|^p\ d\xi < +\infty\ . \tag{3.1}$$

We recall that problem (0.1) as well as the integrals in (3.1) are invariant with respect to the $(N-k+1)$-dimensional group $G_k = \{T(\tau,\eta) \mid \tau > 0\ ,\ \eta \in \mathbb{R}^{N-k}\ \}$ of transforms given by (1.1) (see however the remarks below for the case $k = 1$). We shall always identify solutions u which belong to the orbit of the same transform T in G_k.

Our strategy to prove multiplicity results for (0.1) is to compare the solution $u^D \in D^1_0(\mathbb{R}^N; |x|^a d\xi)$ of Theorem 2.1 with the solution $u^X \in X^1_0(\mathbb{R}^N; |x|^a d\xi)$ of [15]. More precisely, we look for conditions on a, p that guarantee that

$$S^D_{a,p} < S^X_{a,p}\ .$$

We start with a simple remark.

Lemma 3.1 *1. Let $p \in (2, 2^*]$ with $p < +\infty$ if $N = 2$. Then there exists $\varepsilon > 0$ such that $S^D_{a,p} < S^X_{a,p}$ if*

$$0 < a - 2 + N - \frac{2(N-k)}{p} < \varepsilon\ .$$

2. Let $a \in \left((2-N)\frac{k}{N}, 2-k\right)$. Then exists $\varepsilon > 0$ such that $S^D_{a,p} < S^X_{a,p}$ if

$$0 < p - \frac{2(N-k)}{N-2+a} < \varepsilon\ .$$

Proof. Fix any map $w \in C_c^\infty(\mathbb{R}^N)$ such that $w \equiv 1$ on $\{(x,y) \mid |x| \le 1\ ,\ \ |y| \le 1\ \}$. Then compute

$$S^D_{a,p} \le \frac{\int_{\mathbb{R}^N} |x|^a |\nabla w|^2\ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_a} |w|^p\ d\xi\right)^{2/p}} \le \frac{C}{\left(\int_{\{|x|<1\}} |x|^{-b_a}\ dx\right)^{2/p}}\ ,$$

and notice that the weight $|x|^{-b_a}$ looses its summability at the origin as $b_a \to k$. Therefore $S^D_{a,p} \to 0$ as $(N-2+a)p \to 2(N-k)$. The conclusion readily follows from Lemma 1.3. $\square$

The uniqueness result in the recent paper [12] by Mancini and Sandeep allows us to compute exactly the value of the infimum S^D_{2-k,p_k}, where

$$p_k = \frac{2(N-k+1)}{N-k} \tag{3.2}$$

(compare with (3.8) below). In the next Lemma we use this information to estimate S^D_{a,p_k} from above when $a < 2-k$.

Lemma 3.2 *Assume* $2 \le k < N$, $N > 2(k-1)$ *and* $1 - k + \frac{1}{N-k+1} < a < 2-k$. *Then*

$$S^D_{a,p_k} < S^X_{a,p_k}\ .$$

Proof. Notice that $p_k < 2^*$. Thus, by Theorem 2.1, the infimum S^D_{2-k,p_k} is achieved by a map u that solves, up to a Lagrange multiplier, the degenerate problem

$$\begin{cases} -\mathrm{div}(|x|^{2-k}\nabla u) = |x|^{1-k} u^{p_k-1} & \text{in } \mathbb{R}^N\ , \quad x \ne 0 \\ u > 0 \\ u \in D^1_0(\mathbb{R}^N; |x|^{2-k} d\xi). \end{cases}$$

Set $v(x,y) := |x|^{\frac{2-k}{2}} u(x,y)$. Then by direct computation and by Lemma 1.2 it turns out that v solves the elliptic singular problem

$$\begin{cases} -\Delta v = \left(\frac{k-2}{2}\right)^2 |x|^{-2} + |x|^{-b_0} v^{p_k-1} & \text{in } \mathbb{R}^N \\ v > 0 \\ v \in \hat{X}^1_0(\mathbb{R}^k \times \mathbb{R}^{N-k}), \end{cases} \tag{3.3}$$

where $b_0 = \frac{N-2k-2}{N-k}$. By Theorem 0.3 in [10], v is cylindrically symmetric and therefore, by the uniqueness result in [13], Section 6, it turns out that

$$v(x,y) = c(\lambda, N, k)\ |x|^{\frac{2-k}{2}} \left((1+|x|)^2 + |y|^2\right)^{-\frac{N-k}{2}}\ ,$$

for some constant $c(\lambda, N, k)$ that can be computed explicitly. As a consequence, we have that the map

$$u_M(x,y) := \left((1+|x|)^2 + |y|^2\right)^{-\frac{N-k}{2}}$$

achieves the best constant S^D_{2-k,p_k}. Now we set, for $a \le 2-k$,

$$R_a := \frac{\int_{\mathbb{R}^N} |x|^a |\nabla u_M|^2 \; d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_a} |u_M|^{p_k} \; d\xi\right)^{\frac{2}{p_k}}} \; . \tag{3.4}$$

We are going to prove by direct computation that $R_a < R_{2-k}$ for $a < 2-k$, hence

$$S^D_{a,p_k} \le R_a < R_{2-k} = S^D_{2-k,p_k}.$$

Since $S^D_{2-k,p_k} \le S^X_{2-k,p_k}$, and since the map $a \to S^X_{2-k,p_k}$ decreases for $a \le 2-k$, this will lead to conclude the proof of the Lemma.

To compute R_a we set $r = |x|$ and $s = |y|$ and we notice that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^a |\nabla u_M|^2 \; d\xi &= c_{N,k} \; (N-k)^2 \int_0^{+\infty} r^{a+k-1} dr \int_0^{+\infty} \frac{s^{N-k-1}}{\left((1+r)^2+s^2\right)^{N-k+1}} \; ds \\ &= c_{N,k} \; (N-k)^2 \; \Gamma \int_0^{+\infty} \frac{r^{a+k-1} dr}{(1+r)^{N-k+2}} \; , \end{aligned}$$

where $c_{N,k} = \omega_k \omega_{N-k}$ and

$$\Gamma := \int_0^{+\infty} \frac{t^{N-k-1} dt}{(1+t^2)^{N-k+1}} \; .$$

Therefore one gets, via integration by parts,

$$\int_{\mathbb{R}^N} |x|^a |\nabla u_M|^2 \; d\xi = c_{N,k} \; \Gamma (N-k)^2 \; \frac{a+k-1}{N-2k+2-a} \int_0^{+\infty} \frac{r^{a+k-2} dr}{(1+r)^{N-k+2}} \; . \tag{3.5}$$

Notice that for $p = p_k$ it turns out that $b_a = b_a(p_k) = N - p_k \frac{N-2+a}{2} = -\frac{a p_k}{2} + \frac{N-2k+2}{N-k}$. Therefore we are lead to compute

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{-b_a(p_k)} |u_M|^{p_k} \; d\xi &= c_{N,k} \int_0^{+\infty} r^{\frac{a p_k}{2} + (k-2)\frac{N-k+1}{N-k}} dr \int_0^{+\infty} \frac{s^{N-k-1}}{\left((1+r)^2+s^2\right)^{N-k+1}} ds \\ &= c_{N,k} \; \Gamma \; \Phi_a \; , \end{aligned} \tag{3.6}$$

where

$$\Phi_a := \int_0^{+\infty} \frac{r^{\frac{a+k-2}{2} p_k} dr}{(1+r)^{N-k+2}} \; .$$

Now we use Hölder inequality (with conjugate exponents $p_k/2$ and $N-k+1$) to estimate

$$\Phi_a \le \left(\int_0^{+\infty} \frac{dr}{(1+r)^{N-k+2}} \right)^{\frac{1}{N-k+1}} \left(\int_0^{+\infty} \frac{r^{\frac{a+k-2}{2} p_k} dr}{(1+r)^{N-k+2}} \right)^{\frac{2}{p_k}} \; .$$

Therefore

$$\int_{\mathbb{R}^N} |x|^a |\nabla u_M|^2 \; d\xi \le c_{N,k} \; \Gamma (N-k)^2 \; \frac{a+k-1}{N-2k+2-a} \left(\frac{1}{N-k+1} \right)^{\frac{1}{N-k+1}} \; (\Phi_a)^{\frac{2}{p_k}} \; . \tag{3.7}$$

From (3.5) and (3.6) we infer that

$$S^D_{2-k,p_k} = R_{2-k} = (N-k)\left[\frac{c_{N,k}\ \Gamma}{N-k+1}\right]^{\frac{1}{N-k+1}} \ . \tag{3.8}$$

On the other hand, from (3.7) we get also

$$S^D_{a,p_k} \le R_a \le \left[\frac{c_{N,k}\ \Gamma}{N-k+1}\right]^{\frac{1}{N-k+1}} \frac{(N-k)^2(a+k-1)}{N-2k+2-a} = S^D_{2-k,p_k}\ \frac{(N-k)(a+k-1)}{N-2k+2-a} \ .$$

The conclusion follows from Lemma 1.3, by noticing that the map $a \to \frac{a+k-1}{N-2k+2-a}$ is strictly increasing. □

It is now convenient to distingush the case $p < 2^*$ from the limiting case $p = 2^*$.

3.1 Multiplicity for $p < 2^*$

The next Corollary is an immediate consequence of Lemma 3.1 and of Theorem 2.1.

Corollary 3.3 *Assume $1 \le k < N$ and $p \in (2, 2^*)$. Then there exists $\varepsilon > 0$ such that if*

$$(N-k)\frac{2}{p} - (N-2) < a < (N-k)\frac{2}{p} - (N-2) + \varepsilon \ ,$$

then problem (0.1) has at least two distinct (modulo G_k) entire classical solutions.

Next we point out an immediate Corollary to Lemma 3.2 when $p = p_k$ is given by (3.2).

Corollary 3.4 *Let $2 \le k < N$, $N > 2(k-1)$, $1 - k + \frac{1}{N-k+1} < a < 2-k$, and let p_k be as in (3.2). Then problem (0.1) has at least two distinct (modulo G_2) entire solutions*

$$u^X \in X^1_0(\mathbb{R}^N; |x|^a d\xi) \ , \quad u^D \in D^1_0(\mathbb{R}^N; |x|^a d\xi) \setminus X^1_0(\mathbb{R}^N; |x|^a d\xi) \ .$$

Finally, we focus our attention on the case $k = 1$, when the singular set $\{x = 0\}$ is an hyperplane that disconnects the domain. Notice that indeed a larger noncompact group G_1 of invariances acts on problem (0.1). More precisely, transformations in G_1 depend on $2N$ parameters, and are of the form

$$u(x,y) \longrightarrow (T(\tau_-, \tau_+, \eta_-, \eta_+)u)\,(x,y) := \begin{cases} \tau_-^{\frac{N-2+a}{2}} u(\tau_- x, \tau_- y + \eta_-) & \text{if } x < 0 \ , \\ \tau_+^{\frac{N+2+a}{2}} u(\tau_+ x, \tau_+ y + \eta_+) & \text{if } x > 0 \ , \end{cases}$$

for $\tau_-, \tau_+ \in (0, +\infty)$, and for $\eta_-, \eta_+ \in \mathbb{R}^{N-k}$. In other words, dilations in ξ and translations in y can be made independently for $x < 0$ and $x > 0$, so that the equation in (0.1) is still invariant. Essentially the same remark has been made by Catrina and Wang in [6] for an O.D.E. involving spherically symmetric weights.

By Remark 2.2 we get that $u^D \in D_0^1(\mathbb{R}^N)$ is cylindrically symmetric whenever u^D achieves the best constant $S^D_{0,p}$. On the other hand, by Lemma 1.2 in [15], if $u^X \in X_0^1(\mathbb{R}^N; |x|^0 d\xi)$ achieves $S^X_{0,p}$, then the support of u^X is contained in a half-space, and hence it cannot achieve $S^D_{0,p}$. This proved the next result.

Theorem 3.5 *Assume $k = 1$, $N \geq 3$, and $2_* < p < 2^*$. Then problem*

$$\begin{cases} -\Delta u = |x|^{-b_0}\ u^{p-1} & \text{in } \mathbb{R}^N\ ,\ x \neq 0 \\ u > 0\ , \end{cases}$$

with $b_0 = N - p\frac{N-2}{2}$, has at least two distinct (modulo G_1) entire classical solutions:

$u^X \in D_0^1((0,+\infty) \times \mathbb{R}^{N-1})$, with $u^X(x,y) \equiv 0$ for $x < 0$,

$u^D \in D_0^1(\mathbb{R}^N)$, with $u^D(x,y) = u^D(-x,y)$.

3.2 Existence and multiplicity for $p = 2^*$

In this Section we deal with the limiting case $p = 2^*$, that is, we study problem

$$\begin{cases} -\mathrm{div}(|x|^a \nabla u) = |x|^{\frac{Na}{N-2}} u^{2^*-1} & \text{in } \mathbb{R}^N\ ,\ x \neq 0 \\ u > 0\ . \end{cases} \tag{3.9}$$

Theorem 2.1 provides the existence of a solution to (3.9) whenever

$$S^D_{a,2^*} < S\ . \tag{3.10}$$

Notice that a first set of sufficient conditions for (3.10) can be easily obtained from Lemma 1.4. By this argument and by the symmetry result in [10] one can prove the following result (see also [16] for existence).

Theorem 3.6 *Assume $N \geq 4$ and $k \neq 2$. Then the infimum $S^D_{2-k,2^*}$ is achieved by an entire solution u to*

$$\begin{cases} -\mathrm{div}(|x|^{2-k} \nabla u) = |x|^{-N\frac{k-2}{N-2}}\ u^{2^*-1} & \text{in } \mathbb{R}^N\ ,\ x \neq 0 \\ u > 0\ . \end{cases}$$

Moreover u is symmetric: $u(x,y) = u(|x|,|y|)$, and decreasing in the $|y|$-variable.

From now on we take $a < 2-k$. Recall that $S^X_{a,2^*}$ is achieved by Theorem 1.1 if $S^X_{a,2^*} < S$. Thus, besides (3.10), that gives existence, we are lead to investigate if it may happen that

$$S^D_{a,2^*} < S^X_{a,2^*} < S\ . \tag{3.11}$$

Indeed, (3.11) would give multiplicity for (3.9). The aim of this Section is to estimate from above $S^D_{a,2^*}$ in order to find sufficient conditions for (3.10) or for (3.11). This will be done trough two Lemmata.

Lemma 3.7 *Inequality (3.10) holds true if*

$$(2-N)\frac{k}{N} < a \le 2-k \quad and \quad a < 0 \ .$$

Proof. We are going to estimate the infimum $S^D_{a,p}$ by using the map

$$v_T(x,y) = \left(1+|x|^2+|y|^2\right)^{-\frac{N-2}{2}}$$

already defined in (0.9). In order to simplify notations we set $r=|x|$, $s=|y|$, $\eta := \frac{N+k}{2}$ and

$$\Gamma := \int_0^{+\infty} \frac{t^{N-k-1}}{(1+t^2)^{N-1}}\ dt\ , \quad \Phi_a := \int_0^{+\infty} \frac{r^{a+k-1}}{(1+r^2)^{\eta}}\ dr\ , \quad \Psi_a := \int_0^{+\infty} \frac{r^{\frac{Na}{N-2}+k-1}}{(1+r^2)^{\eta}}\ dr\ .$$

We start with two identities, that can be proved via simple computations:

$$\int_0^{+\infty} \frac{t^{N-k-1}}{(1+t^2)^{N}}\ dt = \frac{N+k-2}{2(N-1)}\ \Gamma \tag{3.12}$$

$$\int_0^{+\infty} \frac{r^{a+k-1}}{(1+r^2)^{\eta-1}}\ dr = \frac{N+k-2}{N-2-a}\ \Phi_a \tag{3.13}$$

It turns out that

$$|\nabla v_T|^2 = (N-2)^2\left[\frac{1}{(1+r^2+s^2)^{N-1}} - \frac{1}{(1+r^2+s^2)^{N}}\right]\ .$$

First we compute, for every r,

$$\begin{aligned}\int_0^{+\infty} \frac{s^{N-k-1}}{(1+r^2+s^2)^{N}}\ ds &= \frac{1}{(1+r^2)^N}\int_0^{+\infty} \frac{s^{N-k-1}}{\left(1+\left(s(1+r^2)^{-\frac{1}{2}}\right)^2\right)^N}\ ds \\ &= \frac{1}{(1+r^2)^{\eta}}\int_0^{+\infty} \frac{t^{N-k-1}dt}{(1+t^2)^N} = \Gamma\ \frac{N+k-2}{2(N-1)}\ \frac{1}{(1+r^2)^{\eta}}\end{aligned}$$

by (3.12). In the same way one gets

$$\int_0^{+\infty} \frac{s^{N-k-1}}{(1+r^2+s^2)^{N-1}}\ ds = \Gamma\ \frac{1}{(1+r^2)^{\eta-1}}\ .$$

Set $c_{N,k} := \omega_k\omega_{N-k}$, and compute

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^a|\nabla v_T|^2\ d\xi &= c_{N,k}(N-2)^2\int_0^{+\infty} r^{a+k-1}\ dr \int_0^{+\infty} |\nabla v_T|^2 s^{N-k-1}\ ds \\ &= c_{N,k}\Gamma(N-2)^2\left(\int_0^{+\infty}\frac{r^{a+k-1}\ dr}{(1+r^2)^{\eta-1}} - \frac{N+k-2}{2(N-1)}\int_0^{+\infty}\frac{r^{a+k-1}\ dr}{(1+r^2)^{\eta}}\right) \\ &= c_{N,k}\Gamma(N-2)^2\ \frac{N+k-2}{2(N-1)}\ \frac{N+a}{N-2-a}\ \Phi_a\end{aligned}$$

by (3.13). We also compute

$$\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-2}} |v_T|^{2^*}\ d\xi \ = \ c_{N,k} \int_0^{+\infty} r^{\frac{Na}{N-2}+k-1}\ dr \int_0^{+\infty} \frac{s^{N-k-1}}{(1+r^2+s^2)^N}\ ds$$
$$= \ c_{N,k} \Gamma \frac{N+k-2}{2(N-1)} \Psi_a\ .$$

Next, we use Hölder inequality (with conjugate exponents $\frac{N}{N-2}$ and $\frac{N}{2}$) to estimate

$$\Phi_a = \int_0^{+\infty} \frac{r^{a+k-1}\ dr}{(1+r^2)^\eta} \le \Psi_a^{\frac{N-2}{N}}\ \ \Phi_0^{\frac{2}{N}}\ . \tag{3.14}$$

and we set

$$R_a := \frac{\int_{\mathbb{R}^N} |x|^a |\nabla v_T|^2\ d\xi}{\left(\int_{\mathbb{R}^N} |x|^{\frac{Na}{N-2}} |v_T|^{2^*}\ d\xi\right)^{\frac{N-2}{N}}}\ ,$$

in such a way that $S = R_0$ and $S^D_{a,2^*} \le R_a$. From the above equality we get

$$S = \left(c_{N,k} \Gamma \frac{N+k-2}{2(N-1)} \Phi_0 \right)^{\frac{2}{N}} N(N-2) \tag{3.15}$$

since $\Psi_0 = \Phi_0$. Therefore, using also (3.14) and (3.15), we can estimate

$$S^D_{a,2^*} \le R_a \le \left(c_{N,k} \Gamma \frac{N+k-2}{2(N-1)} \Phi_0 \right)^{\frac{2}{N}} (N-2)^2\ \frac{N+a}{N-2-a} = S\ \frac{N-2}{N}\ \frac{N+a}{N-2-a}\ .$$

The conclusion readily follows. □

We can get new sufficient conditions for (3.11) in the special case $N = 2(k-1)$. Indeed, in this case the exponent p_k defined in (3.2) coincides with the critical exponent 2^*. The arguments and the computations of Lemma 3.2, together with the existence Theorem 3.6 and the uniqueness result in [13] lead to the following result.

Lemma 3.8 *Inequality (3.11) holds true if* $2 < k < N$, $N = 2(k-1)$ *and* $-\frac{N+2}{2^*} < a < -\frac{N}{2^*}$.

We summarize here the existence/multiplicity results known up to now in the critical case.

Theorem 3.9 (existence) *Assume* $(2-N)\frac{k}{N} < a \le 2-k$. *If* $k = 1$ *and* $N = 3$ *assume in addition that* $a \le 0$. *Then problem (3.9) has at least an entire solutions* u^D *that achieves the best constant* $S^D_{a,2^*}$.

Theorem 3.10 (multiplicity) *Assume* $(2-N)\frac{k}{N} < a < 2-k$. *Then problem (3.9) has two distinct entire solutions:*

$$u^D \in D^1_0(\mathbb{R}^N; |x|^a d\xi) \setminus L^2(\mathbb{R}^N; |x|^{a-2} d\xi)\ , \quad u^X \in D^1_0(\mathbb{R}^N; |x|^a d\xi) \cap L^2(\mathbb{R}^N; |x|^{a-2} d\xi)$$

if one of the following conditions is satisfied:

- $k = \frac{N+2}{2}$;
- $k = 3$, $N \geq 7$ and $a = -2$;
- $k = 3$, $N = 5, 6$, or $k \geq 4$ and a is close enough to $(2 - N)k/N$.

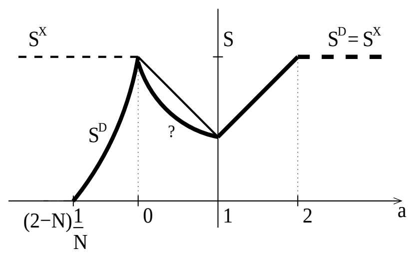

Fig.2 $k = 1$ and $N \geq 4$.

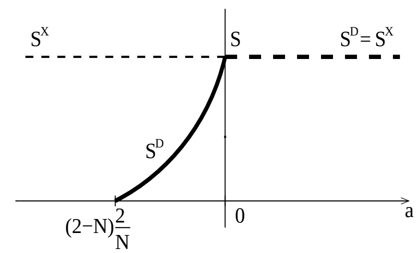

Fig.3 $k = 2$.

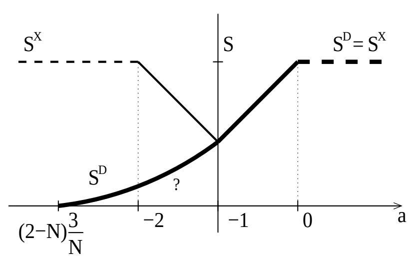

Fig.4 $k = \frac{N+2}{2}$.

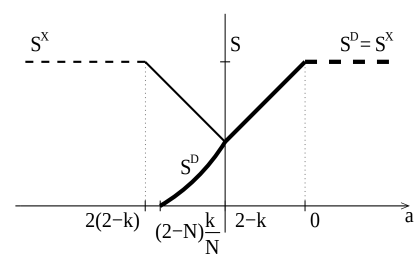

Fig.5 $k = 3$ and $N \geq 7$.

Remark 3.11 We do not know whether $S^D_{a,2^*} < S^X_{a,2^*} = S$ holds true if $k = 1$, $N = 3$ and $a \in (0, 1)$. If $k = 1$, $N \geq 4$ and $a \in (0, 1)$ we know that $S^D_{a,2^*}, S^X_{a,2^*}$ are both achieved, but we do not know if $S^D_{a,2^*} < S^X_{a,2^*}$. The same question is still open if $k \geq 3$ and $a \in (2(2-k), 2-k)$, unless $N = 2(k + 1)$.

4 Existence results for (0.7)

In this section we deal with classical solutions to problem (0.7) under the assumption

$$\lambda \leq \lambda_1(0) = \left(\frac{k-2}{2}\right)^2 .$$

In particular, solutions u we are looking for satisfy $\int_{\mathbb{R}^N} |x|^{-b_0}|u|^p \; d\xi < +\infty$, even if they might be singular, in the sense that one or both of the integrals

$$\int_{\mathbb{R}^N} |\nabla u|^2 \; d\xi \; , \quad \int_{\mathbb{R}^N} |x|^{-2}|u|^2 \; d\xi$$

might be unbounded. We distinguish the cases $\lambda = \lambda_1(0)$ and $\lambda < \lambda_1(0)$.

4.1 Existence for $\lambda = \lambda_1(0)$

In this Subsection we study problem

$$\begin{cases} -\Delta v = \left(\frac{k-2}{2}\right)^2 |x|^{-2} \; v + |x|^{-b_0} \; v^{p-1} & \text{in } \mathbb{R}^N \; , \; x \neq 0 \\ v > 0, \end{cases} \tag{4.1}$$

where $p > 2$, $p \leq 2^*$ if $N \geq 3$, and $b_0 = N - p\frac{N-2}{2}$. We are going to give an alternative proof of a result by Tertikas and Tintarev [16], that works also for $p \in (2, 2^*)$ and $N \geq 2$.

Theorem 4.1 *Problem (4.1) has a cylindrically symmetric classical solution v such that*

$$\int_{\mathbb{R}^N} |x|^{-b_0}|v|^p \; d\xi \; = \int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0)|x|^{-2}|v|^2 \; \right] \; d\xi < +\infty \; ,$$

provided that one of the following conditions are satisfied:

a) $N = 2$ *or* $p \in (2, 2^*)$;

b) $p = 2^*$, $N \geq 4$ *and* $k \neq 2$.

Proof. Our aim is to prove that the infimum

$$\hat{S}_p := \inf_{v \in \hat{X}^1_0(\mathbb{R}^k \times \mathbb{R}^{N-k})} \frac{\int_{\mathbb{R}^N} \left[|\nabla v|^2 - \lambda_1(0)|x|^{-2}|v|^2 \; \right] \; d\xi}{\left(\int_{\mathbb{R}^N} |x|^{-b_0}|v|^p \; d\xi\right)^{2/p}}$$

is achieved by a map $v \in \hat{X}^1_0(\mathbb{R}^k \times \mathbb{R}^{N-k})$ that solves problem (4.1). This can be done directly, arguing as in Section 2, or it can be obtained as a Corollary to Theorem 2.1. Indeed, by Lemma 1.5 it is clear that the minimization problems $\hat{S}_p$ and $S^D_{2-k,p}$ are completely equivalent, in the sense that $u \in D^1_0(\mathbb{R}^N; |x|^{2-k}d\xi)$ achieves $S^D_{2-k,p}$ if and only if $v = L_{2-k}u$ achieves $\hat{S}_p$. Then Theorem 2.1 easily leads to existence. The symmetry follows from Theorem 0.3 in [10]. □

Remark 4.2 We remark that the existence result in Theorem 4.1 does not hold for spherically symmetric weights (case $k = N$). Indeed, it has been proved in [2] that for $k = N$ and $\lambda = \left(\frac{N-2}{2}\right)^2$ problem (0.7) has no distributional solutions in $L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$.

4.2 Existence of singular solutions for $\lambda < \lambda_1(0)$

Here we study problem (0.7) in case $\lambda < \lambda_1(0) = \left(\frac{k-2}{2}\right)^2$. A first solution $v^X \in D^1_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N; |x|^{-2}d\xi)$ can be find by studying the minimization problem for the the infimum $S^X_{a,p}$ in (1.9). This was done in [15], Theorems 1 and 2. More precisely, $S^X_{a,p}$ is achieved, provided that $p < 2^*$; if $p = 2^*$ then existence is proved if in addition $\lambda > 0$, $N \ge 4$ and $k \ne 2$.

Our aim is to use here the results in Sections 2 and 3 to find new classical solutions $v \in L^p(\mathbb{R}^N; |x|^{-b_0}d\xi)$ to (0.7) that are *singular* in the sense that $\int_{\mathbb{R}^N} |x|^{-2}|v|^2 \, d\xi$ diverges. Notice that Theorem 3.5 already provides the existence of a solution $u \notin L^2(\mathbb{R}^N; |x|^{-2}d\xi)$ when $k = 1$, $\lambda = 0$ and $p \in (2_*, 2^*)$.

To handle the case $k \ge 1$ we use a simple trick: assume

$$\lambda_1(0) - \left(\frac{N-k}{N}\right)^2 < \lambda < \lambda_1(0) \ , \quad \frac{2(N-k)}{N-k-2\sqrt{\lambda_1(0)-\lambda}} < p \le 2^* \ , \tag{4.2}$$

and define

$$a = a_{k,\lambda} := 2 - k - 2\sqrt{\lambda_1(0) - \lambda} \ .$$

Notice that with this choice, assumptions (0.3) on a and p are satisfied by (4.2). Moreover, it turns out that $b_a = N - p\frac{N-2+a}{2} = b - \frac{pa}{2}$, accordingly with (0.2). Assume that u is a solution to (0.1) with respect to this choice of the parameters a, b and with respect to the same p. Then the map

$$v = L_a u := |x|^{\frac{a}{2}} u$$

is a $C^\infty(\mathbb{R}^N_0)$- solution to (0.7). In addition, if for example $u \in D^1_0(\mathbb{R}^N; |x|^a d\xi)$, then

$$\int_{\mathbb{R}^N} |x|^{-b_0}|v|^p \ d\xi = \int_{\mathbb{R}^N} |x|^{-b_a}|u|^p \ d\xi < +\infty \ .$$

On the other hand, since $a < 2-k$, then it might happen that $u \notin L^2(\mathbb{R}^N; |x|^{a-2}d\xi)$. If this is the case then $\int |x|^{-2}v^2 = +\infty$. By Hardy inequality, in case $k > 2$ we have also that

$$\int_{\mathbb{R}^N} |\nabla v|^2 \ d\xi = +\infty \ .$$

In conclusion, trough the functional change L_a, we can construct a singular solution to (0.7) starting from any solution $u \in D^1_0(\mathbb{R}^N; |x|^a d\xi) \setminus X^1_0(\mathbb{R}^N; |x|^a d\xi)$ of problem (0.1). Thus, the results in Section 3.1 lead to the following existence result.

Theorem 4.3 *Assume $1 \le k < N$, $p > 2$ and $p \le 2^*$ if $N \ge 3$. Then problem (0.7) has a solution $v_\infty \in L^p(\mathbb{R}^N; |x|^{-b_0}d\xi)$ that satisfies*

$$\int_{\mathbb{R}^N} |x|^{-2}v^2 \ d\xi = +\infty \ ,$$

provided that one of the following conditions is satisfied:

i) $p \le 2^*$ *and* $p\left(N-k-2\sqrt{\lambda_1(0)-\lambda}\right)$ *is close enough to* $2(N-k)$*;*

ii) $p = p_k = \frac{2(N-k+1)}{N-k}$, $2 \le k \le \frac{N+2}{2}$ *and* $\lambda_1(0) - p_k^{-2} < \lambda < \lambda_1(0)$*;*

iii) $p = 2^*$, $k = 1, 2$ *and* $\lambda_1(0) - \left(\frac{N-k}{N}\right)^2 < \lambda < 0$*;*

iv) $p = 2^*$, $k = 3$, $N \ge 7$ *and* $\frac{1}{4} - \left(\frac{N-3}{N}\right)^2 < \lambda \le 0$*;*

We remark also the following immediate consequence to *iv)* of Theorem 4.3, and to Theorem 0.3 in [10].

Corollary 4.4 *Assume* $N \ge 7$*. Then the equation*

$$-\Delta v = v^{2^*-1} \qquad \text{on } (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^{N-3}$$

has a positive smooth cylindrically symmetric solution v_∞ *such that*

$$\int_{\mathbb{R}^N} |v_\infty|^{2^*} \, d\xi < S^{N/2} \ , \quad \int_{\mathbb{R}^N} |\nabla v_\infty|^2 \, d\xi = +\infty \ .$$

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