Abnormal Hamilton-Jacobi Equations arising in Infinite Horizon Problems

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Calculus of Variations and Applications

for 65th birthday of Gianni Dal Maso

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My Three Papers with Gianni

- (2000) *Value function for Bolza problem with discontinuous Lagrangian and Hamilton-Jacobi inequalities*, ESAIM-COCV
- (2003) *Autonomous integral functionals with discontinuous nonconvex integrands : Lipschitz regularity of minimizers, Du Bois-Reymond necessary conditions and Hamilton-Jacobi equations*, Applied Mathematics and Optimization
- (2001) *Uniqueness of solutions to Hamilton-Jacobi equations arising in calculus of variations*, in Optimal Control and Partial Differential Equations,
  In honour of Professor Alain Bensoussan 60th Birthday, J.Menaldi and al. Eds., IOS Press

Finite Horizon Calculus of Variation Problem

Value function $V(t,x) :=$

$$\min\{\varphi(y(t)) + \int_0^t L(y(s), y'(s))ds : y(0) = x,\ y \in W^{1,1}(0,t;\mathbb{R}^n)\}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is lsc and not identically $+\infty$, $L : \mathbb{R}^n \times \mathbb{R}^n \to R_+$ is locally bounded, lsc, convex in the second variable and $L(x,u) \geq \Theta(u)$ with

$$\lim_{|u|\to\infty} \frac{\Theta(u)}{|u|} = +\infty$$

Lemma (Dal Maso, HF 2000)

V is lower semicontinuous on $[0,\infty) \times \mathbb{R}^n$ and locally Lipschitz on $(0,\infty) \times \mathbb{R}^n$.

Outline

Infinite Horizon Optimal Control Problem

$$V(t_0, x_0) = \inf \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt$$

over **(viable)** trajectory-control pairs (x, u) subject to the state equation and state constraint

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U(t) \quad \text{for a.e. } t \geq t_0 \\ x(t_0) = x_0, & x(t) \in K \quad \text{for all } t \geq t_0 \end{cases}$$

$U : \mathbb{R}_+ \rightsquigarrow \mathbb{R}^m$ is measurable with closed $\neq \emptyset$ values,
$L : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$,
$K = \overline{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is open, $x_0 \in K$
$L(t, x, u) \geq \alpha(t) \quad \forall\, (t, x, u)$ and an integrable $\alpha : \mathbb{R}_+ \to \mathbb{R}$.

Controls $u(t) \in U(t)$ are **Lebesgue measurable** selections.
Set $V(t_0, x_0) = +\infty$ if there is no viable (feasible) trajectory.

Classical Infinite Horizon Problem

A discounted infinite horizon optimal control problem

$$W(x_0) = \mathrm{minimize} \int_0^{\infty} e^{-\lambda t} \ell(x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u) subject to

$$\begin{cases} x'(t) = f(x(t), u(t)), & u(t) \in U \quad \text{for a.e. } t \geq 0 \\ x(0) = x_0, & x(t) \in K \quad \text{for all } t \geq 0 \end{cases}$$

Controls $u(\cdot)$ are Lebesgue measurable, $\lambda > 0$.

The economic literature addressing this problem deals with traditional questions of **existence** of optimal solutions, **regularity** of W, **necessary and sufficient** optimality conditions.
A. Seierstad and K. Sydsaeter. Optimal control theory with economic applications, 1986.

Stationary Hamilton-Jacobi Equation

Under some technical assumptions W is the unique **bounded lower semicontinuous solution** of the Hamilton-Jacobi equation

$$\lambda W(x) + H(x, -\nabla W(x)) = 0,$$

where $H(x,p) = \sup_{u\in U}(\langle p, f(x,u)\rangle - \ell(x,u))$ in the sense:

$$\lambda W(x) + H(x,-p) = 0 \quad \forall\, p \in \partial^- W(x), \;\; x \in \mathrm{Int}\, K$$

$$\lambda W(x) + H(x,-p) \geq 0 \quad \forall\, p \in \partial^- W(x), \;\; x \in \partial K$$

$$\lambda W(x) + \sup_{-f(x,u)\in \mathrm{Int}\, C_K(x), u\in U} (\langle -p, f(x,u)\rangle - \ell(x,u)) \leq 0, \; x \in \partial K$$

for all $p \in \partial^- W(x)$, where $\partial^- W(x)$ denotes the **subdifferential** of W at x and $C_K(x)$ - Clarke tangent cone. HF and Plaskacz 1999. Earlier results by Soner 1986, with smooth compact **state constraint** and BUC solutions.

Fréchet Subdifferential

Let $W : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $x \in \mathbb{R}^n$, $W(x) \neq +\infty$.
$\partial^- W(x)$ - **Fréchet subdifferential** of W at $x \in dom(W)$.

$$p \in \partial^- W(x) \iff \lim_{y \to x} \frac{W(y) - W(x) - \langle p, y - x \rangle}{|y - x|} \geq 0$$

For $K \subset \mathbb{R}^n$ and $x \in K$ the **contingent** (Peano) cone to K at x

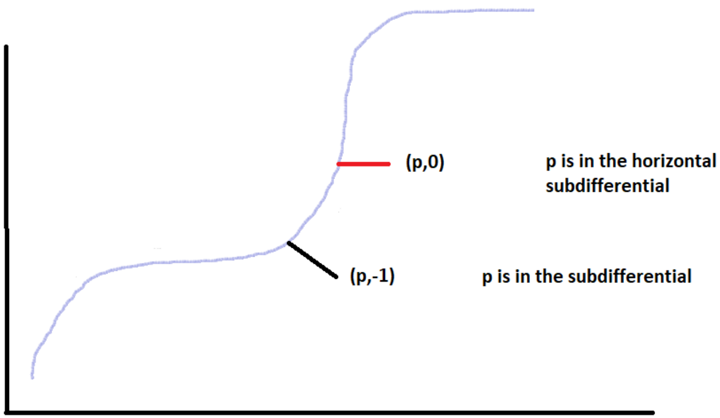
$$T_K(x) := \left\{ u \in X \mid \liminf_{\varepsilon \to 0+} \frac{\mathrm{dist}(x + \varepsilon u, K)}{\varepsilon} = 0 \right\}$$

$$p \in \partial^- W(x) \iff (p, -1) \in \left[T_{epi(W)}(x, W(x)) \right]^-$$

epi (W) – epigraph of W

If $(p, q) \in \left[T_{epi(W)}(x, W(x)) \right]^-$ and $q \neq 0$, then $\frac{p}{|q|} \in \partial^- W(x)$.

Horizontal Subdifferentials



Infinite Horizon Optimal Control Problem

$$V(t_0, x_0) = \inf \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt$$

over **(viable)** trajectory-control pairs (x, u) subject to the state equation and state constraint

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U(t) \quad \text{for a.e. } t \geq t_0 \\ x(t_0) = x_0, & x(t) \in K \quad \text{for all } t \geq t_0 \end{cases}$$

In general V is not locally Lipschitz and very strong assumptions are needed to guarantee its local Lipschitz continuity.
The **Hamilton-Jacobi** equation is no longer stationary :

$$-\frac{\partial V}{\partial t}(t, x) + H(t, x, -\frac{\partial V}{\partial x}(t, x)) = 0$$

Uniqueness of Locally Lipschitz Solutions for HJ

Under some very restrictive technical assumptions V is the unique **locally Lipschitz solution** of the Hamilton-Jacobi equation

$$-\frac{\partial V}{\partial t}(t,x) + H(t,x,-\frac{\partial V}{\partial x}(t,x)) = 0,$$

where $H(t,x,p) = \sup_{u \in U(t)} (\langle p, f(t,x,u)\rangle - L(t,x,u))$, satisfying the final condition

$$\lim_{t\to\infty} \sup_{y\in K} |V(t,y)| = 0$$

in the following sense: for a.e. $t > 0$ and for all $x \in \text{Int } K$

$$-p_t + H(t,x,-p_x) = 0 \quad \forall\, (p_t,p_x) \in \partial^- V(t,x)$$

and for all $x \in \partial K$ (boundary of K)

$$-p_t + H(t,x,-p_x) \geq 0 \quad \forall\, (p_t,p_x) \in \partial^- V(t,x)$$

Finite Horizon Bolza Problem

Question: Can the infinite horizon problem be seen as limit of finite horizon **Bolza** type optimal control problems when $T \to \infty$

$$\inf \int_0^T L(t, x(t), u(t))\, dt$$

over all trajectory-control pairs (x, u) subject to the state equation

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U(t) \quad \text{for a.e. } t \in [0, T] \\ x(0) = x_0 & x(t) \in K \quad \forall\, t \in [0, T] \end{cases}$$

If $(\bar{x}, \bar{u})$ is optimal for the infinite horizon problem with $x(0) = x_0$, then, in general, its restriction to the time interval $[0, T]$ is not optimal for the above Bolza problem.

Maximum Principle for the Bolza Problem

Case without state constraints.

If $(\bar{x}, \bar{u})$ is optimal, then, under mild assumptions, the solution $p : [0, T] \to \mathbb{R}^n$ of the **adjoint system**

$$-p'(t) = p(t) f_x(t, \bar{x}(t), \bar{u}(t)) - L_x(t, \bar{x}(t), \bar{u}(t)), \quad p(T) = 0$$

satisfies the **maximality condition** for a.e. $t \in [0, T]$

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p(t))$$

If restrictions of optimal pairs were optimal, then we could try to pass to the limit when $T \to \infty$ and get the maximum principle also for the **infinite horizon problem**.

Maximum Principle for the Infinite Horizon Problem

If $(\bar{x}, \bar{u})$ is optimal, then $\exists\ p_0 \in \{0, 1\}$ and a locally absolutely continuous $p : [0, \infty[\to \mathbb{R}^n$ with $(p_0, p) \neq 0$, solving the **adjoint system**

$$-p'(t) = p(t) f_x(t, \bar{x}(t), \bar{u}(t)) - p_0 L_x(t, \bar{x}(t), \bar{u}(t)) \quad \text{for a.e.} \quad t \geq 0$$

and satisfying the **maximality condition**

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - p_0 L(t, \bar{x}(t), \bar{u}(t)) =$$

$$\max_{u \in U(t)}(\langle p(t), f(t, \bar{x}(t), u)\rangle - p_0 L(t, \bar{x}(t), u)) \quad \text{for a.e.} \quad t \geq 0$$

If $p_0 = 0$ this maximum principle (MP) is called abnormal.

Transversality condition like $\lim_{t\to\infty} p(t) = 0$ is, in general, absent, cf. Halkin 1974 for a counterexample.

Main Differences with the Finite Horizon Case

Even in the absence of state constraint

- The maximum principle may be **abnormal**
- Transversality conditions are absent :
  some authors, under appropriate assumptions, obtain a **transversality condition** at infinity in the form

$$\lim_{t\to\infty} p(t) = 0 \quad \text{or} \quad \lim_{t\to\infty} \langle p(t), \bar{x}(t)\rangle = 0$$

However they are a consequence of the growth assumptions on f, L.

Reduction to the Bolza Problem with Finite Horizon

Introducing $g_T(y) := V(T, y)$ we get, using the **dynamic programming principle**, the **Bolza** type problem

$$V^B(t_0, x_0) := \inf \left(g_T(x(T)) + \int_{t_0}^{T} L(t, x(t), u(t))\, dt \right)$$

over all trajectory-control pairs (x, u) subject to the state equation

$$\begin{cases} x'(t) = f(t, x(t), u(t)), & u(t) \in U(t) \quad \text{for a.e. } t \in [t_0, T] \\ x(t_0) = x_0, & x(t) \in K \quad \text{for all } t \in [t_0, T] \end{cases}$$

Under assumptions (H1) $i) - iv)$ below, $V^B(s_0, y_0) = V(s_0, y_0)$ for all $s_0 \in [0, T]$, $y_0 \in K$. Furthermore, if $(\bar{x}, \bar{u})$ is optimal for the infinite horizon problem at (t_0, x_0) then the restriction of $(\bar{x}, \bar{u})$ to $[t_0, T]$ is optimal for the above Bolza problem.

Assumptions (H1)

i) $\exists$ locally integrable $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \geq 0$

$$|f(t,x,u)| \leq c(t)(|x|+1), \quad \forall\, x \in \mathbb{R}^n,\ u \in U(t);$$

ii) $\forall\, R > 0$, $\exists$ a locally integrable $c_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \geq 0$, $\forall\, x, y \in B(0,R)$, $\forall\, u \in U(t)$

$$|f(t,x,u) - f(t,y,u)| + |L(t,x,u) - L(t,y,u)| \leq c_R(t)|x-y|;$$

iii) $\forall\, x \in \mathbb{R}^n$, $f(\cdot,x,\cdot)$, $L(\cdot,x,\cdot)$ are Lebesgue-Borel measurable ;

iv) $\exists$ a locally integrable $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and a locally bounded nondecreasing $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \geq 0$,

$$L(t,x,u) \leq \beta(t)\phi(|x|), \quad \forall\, x \in \mathbb{R}^n,\ u \in U(t);$$

v) For a.e. $t \geq 0$, $\forall\, x \in \mathbb{R}^n$ the set $F(t,x)$ is closed and convex

$$F(t,x) := \{(f(t,x,u), L(t,x,u) + r) : u \in U(t) \text{ and } r \geq 0\}$$

Absolute Continuity of Maps

Proposition

Assume (H1). Then V is lower semicontinuous and for every $(t_0, x_0) \in dom\ V$, there exists a viable in K trajectory-control pair $(\bar{x}, \bar{u})$ satisfying $V(t_0, x_0) = \int_{t_0}^{\infty} L(t, \bar{x}(t), \bar{u}(t))\, dt$.

A set-valued map $P : \mathbb{R}_+ \rightsquigarrow \mathbb{R}^k$ is **locally absolutely continuous** if it takes nonempty closed images and for any $[S, T] \subset \mathbb{R}_+$, $\varepsilon > 0$, and any compact $K \subset \mathbb{R}^k$, $\exists\, \delta > 0$ such that for any finite partition $S \leq t_1 < \tau_1 \leq t_2 < \tau_2 \leq \ldots \leq t_m < \tau_m \leq T$ of $[S, T]$,

$$\sum_{i=1}^{m} (\tau_i - t_i) < \delta \quad \Longrightarrow \quad \sum_{i=1}^{m} \max\{d_{P(t_i)}(P(\tau_i) \cap K), d_{P(\tau_i)}(P(t_i) \cap K)\} < \varepsilon,$$

where $d_E(E') := \inf\{r > 0 \ :\ E' \subset E + rB\}$ for any $E, E' \subset \mathbb{R}^k$ (the infimum over an empty set is $= +\infty$).

Absolute Continuity of the Epigraph of Value

Lemma

If dom$(V) \neq \emptyset$*, (H1) holds and for a.e.* $t \geq 0$

$$-f(t,x,U(t)) \cap \overline{co}\, T_K(x) \neq \emptyset \quad \forall\, x \in \partial K,$$

then $t \rightsquigarrow$ *epi* $V(t,\cdot)$ *is locally absolutely continuous.*

Define the abnormal Hamiltonian

$$\mathcal{H}(t,x,p,q) := \sup_{u \in U(t)} \left(\langle p, f(t,x,u) \rangle - qL(t,x,u) \right)$$

Weak Solutions to HJ equation

A function $W : \mathbb{R}_+ \times K \to \mathbb{R} \cup \{+\infty\}$ is called a weak solution of HJ equation on $(0, \infty) \times K$ if $t \rightsquigarrow epi\, W(t, \cdot)$ is locally absolutely continuous and there exists a set $A \subset (0, \infty)$, with $\mu(A) = 0$ such that for all $(t, x) \in \text{dom}(W) \cap ((0, \infty) \times \text{Int } K)$, $t \notin A$

$$-p_t + \mathcal{H}(t, x, -p_x, -q) = 0 \quad \forall (p_t, p_x, q) \in \left[T_{epi(W)}(t, x, W(t, x)) \right]^-$$

and $\forall (t, x) \in \text{dom}(W) \cap ((0, \infty) \times \partial K)$, $t \notin A$

$$-p_t + \mathcal{H}(t, x, -p_x, -q) \geq 0 \quad \forall (p_t, p_x, q) \in \left[T_{epi(W)}(t, x, W(t, x)) \right]^-$$

Outward Pointing Condition

If f, U are continuous and bounded, K is bounded and $\partial K \in C^1$, then (OPC) : $\exists\, r > 0\ \forall\, t \in \mathbb{R}_+, \forall\, x \in \partial K, \exists\, u \in U(t)$

$$\langle n_x, f(t,x,u)\rangle \geq r$$

where n_x is the unit outward normal to K at x.

In the general case (OPC) becomes: $\exists\, \eta > 0$, $r > 0$, $M \geq 0$ such that for a.e. $t > 0$ and any $y \in \partial K + \eta B$, and any $v \in f(t,y,U(t))$, with $\min_{n \in N^1_{y,\eta}} \langle n, v\rangle \leq 0$, we can find $w \in f(t,y,U(t)) \cap B(v,M)$ satisfying

$$\min_{n \in N^1_{y,\eta}} \{\langle n, w\rangle, \langle n, w - v\rangle\} \geq r$$

where $N^1_{y,\eta} := \{n \in N^1_K(x) \ :\ x \in \partial K \cap B(y,\eta)\}$,
$N^1_K(x) := N_K(x) \cap S^{n-1}$
and $N_K(x)$ denotes the Clarke normal cone to K at x.

Uniqueness of Weak Solutions to HJ Equation

Theorem (V. Basco, HF. 2019) Assume (OPC) and (H1) with $c(t)$, $c_R(t)$, independent from t, R and that for an uniformly integrable $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and a.e. $t > 0$

$$\sup_{u \in U(t)} (|f(t,x,u)| + |L(t,x,u)|) \leq \gamma(t) \qquad \forall\, x \in \partial K.$$

Let $W : \mathbb{R}_+ \times K \to \mathbb{R} \cup \{+\infty\}$ be lsc such that for all large $t > 0$, $\operatorname{dom} V(t,\cdot) \subset \operatorname{dom} W(t,\cdot) \neq \emptyset$ and

$$\lim_{t \to \infty} \sup_{y \in domW(t,\cdot)} |W(t,y)| = 0. \qquad (*)$$

Then the following statements are equivalent:
(i) W is a weak solution of (HJ) equation on $(0,\infty) \times K$;
(ii) W=V.

Maximum Principle for LSC Value Function

Theorem (Cannarsa, HF, 2018)

Let $K = \mathbb{R}^n$, $(\bar{x}, \bar{u})$ be optimal at (t_0, x_0) and $\partial_x^- V(t_0, x_0) \neq \varnothing$. If $f(t, \cdot, u)$ and $L(t, \cdot, u)$ are differentiable, then $\forall\, p_0 \in \partial_x^- V(t_0, x_0)$ the solution $p(\cdot)$ of the adjoint system

$$-p'(t) = p(t) f_x(t, \bar{x}(t), \bar{u}(t)) - L_x(t, \bar{x}(t), \bar{u}(t)), \quad p(t_0) = -p_0$$

*satisfies for a.e. $t \geq t_0$ the **maximality condition***

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p(t))$$

*and the **sensitivity relation***

$$-p(t) \in \partial_x^- V(t, \bar{x}(t)) \quad \forall\, t \geq t_0.$$

Second Order Subjets

$\mathbf{S}(\mathrm{n})$ - symmetric $(n \times n)-$matrices.
Let $\varphi \colon \mathbb{R}^n \to [-\infty, \infty]$ and $x \in dom(\varphi)$.
A pair $(q, Q) \in \mathbb{R}^n \times \mathbf{S}(\mathrm{n})$ is a **subjet** of φ at x if

$$\varphi(x) + \langle q, y - x\rangle + \frac{1}{2}\langle Q(y-x), y-x\rangle \leq \varphi(y) + o(|y-x|^2)$$

for some $\delta > 0$ and for all $y \in x + \delta B$.
The set of all subjets of φ at x is denoted by $J^{2,-}\varphi(x)$.

We assume next that $H(t, \cdot, \cdot) \in C^{2,1}_{loc}$, that f, L are differentiable with respect to x and consider an optimal trajectory-control pair $(\bar{x}, \bar{u})$ starting at (t_0, x_0).

Riccati Equation, NO State Constraints

Let $(p_0, R_0) \in J_x^{2,-} V(t_0, x_0)$ and $\bar{p}(\cdot)$ solve the adjoint system

$$-p'(t) = p(t)f_x(t, \bar{x}(t), \bar{u}(t)) - L_x(t, \bar{x}(t), \bar{u}(t)), \quad p(t_0) = -p_0.$$

If for some $T > t_0$, $V(t, \cdot)$ is twice Fréchet differentiable at $\bar{x}(t)$ for all $t \in [t_0, T]$, then the Hessian $t \mapsto -V_{xx}(t, \bar{x}(t))$ solves the celebrated **matrix Riccati equation**:

$$\dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0$$

where $H_{px}[t]$ abbreviates $H_{px}(t, \overline{x}(t), \overline{p}(t))$, and similarly for $H_{xp}[t], H_{pp}[t], H_{xx}[t]$.

Forward Propagation of Subjets

Theorem (corollary of Cannarsa, HF, Scarinci, SICON, 2015)

Assume $(p_0, R_0) \in J_x^{2,-}V(t_0, x_0)$. If the solution R of the matrix Riccati equation

$$\dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0$$

with $R(t_0) = -R_0$ is defined on $[t_0, T]$, $T > t_0$, then the following ***second order sensitivity relation*** *holds true:*

$$(-\bar{p}(t), -R(t)) \in J_x^{2,-}V(t, \bar{x}(t)), \; \forall\, t \in [t_0, T].$$

Assumptions for Lipschitz Continuity of $V(t,\cdot)$

We denote by (H2) the following assumptions

- for some $\lambda > 0$, $L(t,x,u) = e^{-\lambda t}\ell(t,x,u)$
- $f(\cdot,x,\cdot)$, $\ell(\cdot,x,\cdot)$ is Lebesgue-Borel measurable $\forall\, x \in \mathbb{R}^n$
- $\{(f(t,x,u),\ell(t,x,u)) \,:\, u \in U(t)\}$ is closed $\forall\,(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$
- $\sup\{|f(t,x,u)| + |\ell(t,x,u)| : u \in U(t), (t,x) \in \mathbb{R}_+ \times \partial K\} < \infty$
- for some uniformly integrable $k : \mathbb{R}_+ \to \mathbb{R}$ and a.e. $t \in \mathbb{R}_+$, $(f(t,\cdot,u),\ell(t,\cdot,u))$ is $k(t)$-Lipschitz $\forall\, u \in U(t)$
- for some locally integrable $c : \mathbb{R}_+ \to \mathbb{R}_+$ and all $x \in \mathbb{R}^n$

$$\sup\{|f(t,x,u)| + |\ell(t,x,u)| \,:\, u \in U(t)\} \le c(t)(1+|x|)$$

- $\limsup_{t\to\infty} \frac{1}{t}\int_0^t (c(s)+k(s))\,ds < \infty$

Lipschitz Continuity of $V(t,\cdot)$

Theorem

If (H2) and (IPC)′ hold, then there exist $b > 1$, $C > 0$ such that for all $\lambda > C$ and every $t \geq 0$ the function $V(t,\cdot)$ is $\gamma(t)$-Lipschitz continuous on K with $\gamma(t) = be^{-(\lambda - C)t}$

(IPC)′ $\exists\, \eta > 0$, $r > 0$ such that for a.e. $t \in \mathbb{R}_+$,
$\forall\, y \in \partial K + \eta B$, $\forall\, v \in f(t, y, U(t))$ with $\max_{n \in N^1_{y,\eta}} \langle n, v\rangle \geq 0$,
there exists $w \in f(t, y, U(t))$ such that

$$\max_{n \in N^1_{y,\eta}} \{\langle n, w\rangle, \langle n, w - v\rangle\} \leq -r,$$

where $N^1_{y,\eta} := \{n \in N^1_K(x) \;:\; x \in \partial K \cap B(y,\eta)\}$.

Generalized Differentials and Limiting Normals

For $K \subset \mathbb{R}^n$ and $x \in K$, $N_K^L(x)$ - limiting normal cone to K at x.
$N_K(x) = \overline{co}\, N_K^L(x)$ - the Clarke normal cone to K at x.

$$N_K^1(x) := N_K(x) \cap S^{n-1}$$

$\text{hyp}(\varphi)$ - **hypograph** of $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$.

Limiting superdifferential at $x \in \text{dom}(\varphi)$:

$$\partial^{L,+}\varphi(x) := \{p \mid (-p, 1) \in N^L_{hyp(\varphi)}(x, \varphi(x))\}$$

For a locally Lipschitz $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$\partial\varphi(x) := co\, \partial^{L,+}\varphi(x)$ – the **generalized gradient** of φ at x.

Generalized Gradients of V on $\mathbb{R}_+ \times K$

If V is locally Lipschitz, then consider **generalized gradients**:
For $(t, x) \in \mathbb{R}_+ \times K$ and the **Peano-Kuratowski limits** Limsup

$$\partial_x^0 V(t,x) := \operatorname*{Limsup}_{\substack{y \to x \\ \text{Int } K}} \partial_x V(t,y)$$

$$\partial^0 V(t,x) := \operatorname*{Limsup}_{\substack{(s,y)\to(t,x) \\ (0,\infty)\times \text{Int } K}} \partial V(s,y).$$

Note $\partial_x^0 V(t,x) = \partial_x V(t,x)$ whenever $x \in \text{Int } K$.

Inward Pointing Condition

If f, U are continuous and bounded, K is bounded and $\partial K \in C^1$, then (IPC) : $\exists\, r > 0$, $\forall\, t \in \mathbb{R}_+$, $\forall\, x \in \partial K$, $\exists\, u \in U(t)$

$$\langle n_x, f(t,x,u)\rangle < -r$$

where n_x is the unit outward normal to K at x.

In the general case (IPC) becomes :
$\forall\, t \in \mathbb{R}_+$, $\forall\, x \in \partial K$,
$\forall\, v \in \text{Limsup}_{(s,y)\to(t,x)} f(s,y,U(s))$ with $\max_{n\in N_K^1(x)} \langle n, v\rangle \geq 0$,
$\exists\, w \in \text{Liminf}_{(s,y)\to(t,x)} co\, f(s,y,U(s))$ such that

$$\max_{n\in N_K^1(x)} \langle n, w - v\rangle < 0$$

Maximum Principle and Sensitivity Relations

Theorem. Assume (H1), (IPC), that $V(T,\cdot)$ is **locally Lipschitz** on K for all large T. Then V is locally Lipschitz on $\mathbb{R}_+ \times K$.
If $(\bar{x},\bar{u})$ is optimal at $(t_0,x_0) \in \mathbb{R}_+ \times K$, then there exist a locally absolutely continuous $p : [t_0,\infty[\to \mathbb{R}^n$ with $-p(t_0) \in \partial_x^{L,+} V(t_0,x_0)$, a positive Borel measure μ on $[t_0,\infty[$ and a Borel measurable $\nu(t) \in N_K(\bar{x}(t)) \cap B$ a.e. $t \geq t_0$ such that for $q(t) = p(t) + \eta(t)$, where

$$\eta(t) := \int_{[t_0,t]} \nu(s) d\mu(s) \ \ \forall\, t > t_0 \quad \& \quad \eta(t_0) = 0$$

and for a.e. $t \geq t_0$, we have

$$(-p'(t), \bar{x}'(t)) \in \partial_{x,p} H(t, \bar{x}(t), q(t))$$

$$-q(t) \in \partial_x^0 V(t,\bar{x}(t)), \ \ (H(t,\bar{x}(t),q(t)), -q(t)) \in \partial^0\, V(t,\bar{x}(t))$$

Maximality Condition

The inclusion

$$(-p'(t), \bar{x}'(t)) \in \partial_{x,p} H(t, \bar{x}(t), q(t))$$

contains the maximality condition: for a.e. $t \geq t_0$

$$\langle q(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), q(t))$$

If the Lipschitz constants of $V(t, \cdot)$ at $\bar{x}(t)$ converge to zero when $t \to \infty$, then $\lim_{t\to\infty} q(t) = 0$ (**the transversality condition**).


THE END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