Monads for torsion-free sheaves on multi-blow-ups of the projective plane

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Abstract

We construct monads for framed torsion-free sheaves with fixed Chern character on the multi-blow-ups of the complex projective plane. Using these monads we prove that the moduli space of such sheaves is a smooth algebraic variety. Moreover we construct monads for families of such sheaves parametrized by a reduced noetherian scheme S of finite type. A universal monad on the moduli space is introduced and used to prove that the moduli space is fine.

1 Introduction

In this paper we are concerned with the construction of the moduli space $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ of framed torsion-free sheaves of a fixed Chern character on a multi-blow-up of the complex projective plane; $\pi : \tilde{\mathbb{P}} \longrightarrow \mathbb{P}^2$, by using monadic descriptions which lead to an ADHM data. The ADHM data will be useful, at a first step, to give a presentation of the moduli space as a quotient $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k} = P/G$ where P is a space of some matrices satisfying certain conditions and which will be described below. At a second step the monadic description is used to prove that the space $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is a smooth algebraic variety of dimension $2r(k + \frac{|\vec{a}|^2}{2}) - |\vec{a}|^2$. This is done by generalizing Buchdahl construction for holomorphic bundles [1], in order to extend it to torsion-free sheaves. An additional result is the construction of a monad corresponding to a family $\mathcal{F}$ on a product $\tilde{\mathbb{P}} \times S$, where S is a Noetherian reduced scheme of finite type on which $\mathcal{F}$ is flat. In particular, there is a universal monad on $\tilde{\mathbb{P}} \times \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$. Using these monads we construct a natural transformation

$$\Phi : \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(\bullet) \longrightarrow Hom(\bullet, \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k})$$

which is a bijection for every reduced point s. Then using the ADHM presentation of the moduli space $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ and the properties of the universal monads constructed, we prove that the scheme $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is a fine moduli space.

Another way of treating the moduli space is to show that one can choose, on $\tilde{\mathbb{P}}$, a polarization in a such a way that a framed sheaf $(\mathcal{E}, \Phi)$ is stable framed sheaf in the sense of Huybrechts and Lehn, and using their result that the moduli space of such objects is a quasi-projective scheme [4, 14]. The equivalence between the two approaches is established by the fact that in both cases the moduli space is fine as we prove for the moduli space $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ in this work. This generalizes the result by Nakajima [4] and Okonek [6] et al in the cases of Hilbert schemes of points on the projective plane, and rank-2 stable bundles on the projective plane, respectively. This comparison also implies that $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is quasi-projective.

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2 The construction of the monad

Let $\pi : \tilde{\mathbb{P}} \longrightarrow \mathbb{P}^2$ be the blow-up of the projective plane at n distinct points. $\tilde{\mathbb{P}}$ is regular $(H^1(\tilde{\mathbb{P}}, \mathcal{O}) = 0)$ and its Picard group is generated by $n+1$ elements, namely: $Pic(\tilde{\mathbb{P}}) = \oplus_{i=1}^n E_i\mathbb{Z} \oplus l_\infty\mathbb{Z}$, where every E_i is an exceptional divisor with the following intersection numbers: $E_i^2 = -1$, $(E_i \cdot E_j) = 0$ for $i \neq j$, $(E_i \cdot l_\infty) = 0$ and where l_∞ is the divisor which is given by the generic line in $\mathbb{P}^2$ and $l_\infty^2 = 1$. The canonical divisor of the surface $\tilde{\mathbb{P}}$ is given by $K_{\tilde{\mathbb{P}}} = -3l_\infty + \Sigma_{i=1}^n E_i$. The Poincaré duals of these divisors are given by $(L_\infty, e_1, \cdots e_n)$ where $< L_\infty, l_\infty >= 1$, $< e_i, E_j >= \delta_{ij}$ and also $e_i \cdot e_j = -\delta_{ij}$. In terms of line bundles, a divisor of the form $D = pl_\infty + \Sigma_{i=1}^n q_i E_i$ has the associated line bundle $\mathcal{O}(D) = \mathcal{O}(p, \overrightarrow{q}) = \mathcal{O}(pl_\infty) \otimes \mathcal{O}(q_1 E_1) \otimes \cdots \otimes \mathcal{O}(q_n E_n)$ where $\overrightarrow{q} = (q_1, \cdots, q_n)$. Then the canonical divisor is given by $\omega_{\tilde{\mathbb{P}}} = \mathcal{O}(-3, \overrightarrow{1})$, $\overrightarrow{1} = (1, \cdots, 1)$. The Riemann-Roch formula for a line bundle $\mathcal{O}(p, \overrightarrow{q})$ is given by:

$$\chi(\mathcal{O}(p, \overrightarrow{q})) = \frac{1}{2}[(p+1)(p+2) - |\overrightarrow{q}|^2 + \Sigma_{i=1}^n q_i].$$

where $|\overrightarrow{q}|^2 = \Sigma_{i=1}^n q_i^2$. We also use the fact that a line bundle $\mathcal{O}(p, \overrightarrow{q})$ restricts to $\mathcal{O}(p)$ on the linear system $|\mathcal{O}(l_\infty)|$ and to $\mathcal{O}(-q_i)$ when restricted to the linear system $|\mathcal{O}(E_i)|$.

For a torsion-free sheaf $\mathcal{E}$ of Chern character $ch(\mathcal{E}) = r + (al_\infty + \Sigma_{i=1}^n a_i E_i) - (k - \frac{a^2 - |\overrightarrow{a}|^2}{2})\omega$, twisted by a line bundle $\mathcal{O}(p, \overrightarrow{q})$ the Riemann-Roch formula is given by:

$$\chi(\mathcal{E}(p, \overrightarrow{q})) = -[k - \frac{a}{2}(a+3) + \frac{1}{2}\Sigma_{i=1}^n a_i(a_i - 1)] + \frac{r}{2}[(p+1)(p+2) - \Sigma_{i=1}^n q_i(q_i - 1)] + [ap - \Sigma_{i=1}^n a_i q_i].$$

Now we restrict ourselves to the case of torsion-free sheaves $\mathcal{E}$ with Chern character $ch(\mathcal{E}) = r + (\Sigma_{i=1}^n a_i E_i) - (k + \frac{|\overrightarrow{a}|^2}{2})\omega$ which are framed on the generic line l_∞ i.e we have a fixed trivialization $\Phi : \mathcal{E}_{|l_\infty} \longrightarrow \mathcal{O}_\infty^{\oplus r}$. The direct image $\pi_*\mathcal{E}$ of $\mathcal{E}$ is semi-stable and torsion-free. Note that a torsion free on $\mathbb{P}^2$ is semi-stable if and only if $H^0(\mathbb{P}^2, \mathcal{E}(-1)) = H^0(\mathbb{P}^2, \mathcal{E}^*(-1)) = 0$, and this is guarantied, in our case, since $\pi_*\mathcal{E}$ is framed ([6], page 167).

From the natural injection of the sheaf $\mathcal{E}$ in its double dual $\mathcal{E}^{**}$, we have the following exact sequence:

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^{**} \longrightarrow \Delta \longrightarrow 0 \tag{1}$$

where $\mathcal{E}^{**}$ has Chern character $ch(\mathcal{E}^{**}) = r + (\Sigma_{i=1}^n a_i E_i) - (k - l + \frac{|\overrightarrow{a}|^2}{2})\omega$, and l is the length of quotient sheaf Δ supported on finitely many points with $supp\Delta \cap l_\infty = \emptyset$.

Proposition 2.1. $H^0(\tilde{\mathbb{P}}, \mathcal{E}^{**}(p, \overrightarrow{q})) = H^0(\tilde{\mathbb{P}}, \mathcal{E}^*(p, \overrightarrow{q})) = 0 \quad \forall \overrightarrow{q}$ *if* $p < 0$ *and*
$H^2(\tilde{\mathbb{P}}, \mathcal{E}^{**}(p, \overrightarrow{q})) = H^0(\tilde{\mathbb{P}}, \mathcal{E}^*(p, \overrightarrow{q})) = 0 \quad \forall \overrightarrow{q}$ *if* $p = -1, -2$.

Proof. From the following exact sequence:

$$0 \longrightarrow \mathcal{E}^{**}(-(p+1), \overrightarrow{q}) \longrightarrow \mathcal{E}^{**}(-p, \overrightarrow{q}) \longrightarrow \mathcal{E}^{**}_{|l_\infty}(-p) \longrightarrow 0$$

we take the resulting long exact sequence in cohomology. One can see that since $\mathcal{E}^{**}$ is framed then $H^0(l_\infty, \mathcal{E}^{**}_{|l_\infty}(-p)) = H^0(l_\infty, \mathcal{O}^{\oplus r}(-p))$ which vanishes if $p > 0$. By Serre duality also $H^1(l_\infty, \mathcal{E}^{**}_{|l_\infty}(-p)) = H^0(l_\infty, \mathcal{O}^{\oplus r}(p-2)) = 0$ for $p < 2$ in particular for $p = 0, 1$. Then one has $H^0(\tilde{\mathbb{P}}, \mathcal{E}^{**}(-(p+1), \overrightarrow{q})) \cong H^0(\tilde{\mathbb{P}}, \mathcal{E}^{**}(-p, \overrightarrow{q}))$ for $p < 0$ and $H^2(\tilde{\mathbb{P}}, \mathcal{E}^{**}(-(p+1), \overrightarrow{q})) \cong$

$H^2(\tilde{\mathbb{P}}, \mathcal{E}^{**}(-p, \overrightarrow{q}))$ for $p < 2$ and in particular for $p = 0, 1$. But for p large enough $H^0(\tilde{\mathbb{P}}, \mathcal{E}^{**}(-p, \overrightarrow{q})) = 0$, thus the latter vanishes if $p > 0$, $\forall \overrightarrow{q}$. By Serre duality one has $H^2(\tilde{\mathbb{P}}, \mathcal{E}^*(p-3, \overrightarrow{1-q})) = 0$ for $p = 1, 2$.

The proof is the same for $H^0(\tilde{\mathbb{P}}, \mathcal{E}^*(-p, \overrightarrow{q}))$, and by Serre duality also $H^2(\tilde{\mathbb{P}}, \mathcal{E}^{**}(p-3, \overrightarrow{1-q})) = 0$ for $p = 1, 2$.

□

Corollary 2.1. $H^0(\tilde{\mathbb{P}}, \mathcal{E}(p, \overrightarrow{q})) = 0 \quad \forall \overrightarrow{q}$ *if* $p < 0$ *and*
$H^2(\tilde{\mathbb{P}}, \mathcal{E}(p, \overrightarrow{q})) = 0 \quad \forall \overrightarrow{q}$ *if* $p = -1, -2$. □

We also use the following form of Serre-Grothendiek duality for coherent sheaves

Theorem 2.1. *On a smooth algebraic projective variety X of dimension n over an algebraically closed field K, and for every two coherent sheaves $\mathcal{F}$ and $\mathcal{J}$, the following formula holds:*

$$Ext^i(\mathcal{F}, \mathcal{J}) = Ext^{n-i}(\mathcal{J}, \mathcal{F} \otimes \omega_X)^*$$

where ω_X is the canonical sheaf

Let us now start the program to construct the monad which describes the torsion-free sheaf $\mathcal{E}$ as its cohomology. First define the spaces $B_i := Hom(\mathcal{E}, \mathcal{O}_{|E_i}(-1))^*$ and let $\mathcal{B}_1 = \oplus_{i=1}^n B_i(1, -E_i)$. Then the extensions of the form

$$0 \longrightarrow \mathcal{E} \longrightarrow Q_1 \longrightarrow \mathcal{B}_1 \longrightarrow 0$$

are classified by the group $Ext^1(\mathcal{B}_1, \mathcal{E}) = \oplus_{i=1}^n B_i^* \otimes Ext^1(\mathcal{E}, \mathcal{O}(-2, \overrightarrow{1} - E_i))^*$. Applying the functor $Hom(\mathcal{E}, \cdot)$ to the sequence

(2) $$0 \longrightarrow \mathcal{O}(0, -E_i) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{|E_i} \longrightarrow 0$$

after a twisting by $\mathcal{O}(-2, \overrightarrow{1})$ one obtains

$$Hom(\mathcal{E}, \mathcal{O}(-2, \overrightarrow{1})) \xrightarrow{r^*} Hom(\mathcal{E}, \mathcal{O}_{|E_i}(-1)) \longrightarrow Ext^1(\mathcal{E}, \mathcal{O}(-2, \overrightarrow{1} - E_i))$$

but $Hom(\mathcal{E}, \mathcal{O}(-2, \overrightarrow{1})) = H^2(\tilde{\mathbb{P}}, \mathcal{O}(-1, 0))^*$ which vanishes by the corollary above. Then the map

$$B_i^* \xhookrightarrow{r^*} Ext^1(\mathcal{E}, \mathcal{O}(-2, \overrightarrow{1} - E_i))$$

is injective. This implies that the map

$$Ext^1(\mathcal{E}, \mathcal{O}(-2, \overrightarrow{1} - E_i))^* \xrightarrow{r} B_i$$

is surjective. Thus there exists an extension Q_1 which is mapped to the identity in $End(B_i)$ under the composition of the projection on the i^{th} factor and the map r above.

Let us now define $A_i := Ext^1(\mathcal{E}, \mathcal{O}_{|E_i}(-1))^*$ and $\mathcal{A}_1 := \oplus_{i=1}^n A_i(-1, E_i)$. The extensions of the form

$$0 \longrightarrow \mathcal{A}_1 \longrightarrow X_1 \longrightarrow \mathcal{E} \longrightarrow 0$$

are classified by

$$Ext^1(\mathcal{E}, \mathcal{A}_1) = \oplus_{i=1}^n A_i \otimes Ext^1(\mathcal{E}, \mathcal{O}(-1, E_i))$$

Applying this time the functor $Hom(\mathcal{E}, \cdot)$ on (2) twisted by $\mathcal{O}(-1, E_i)$ one have the following exact sequence:

(3) $$0 \longrightarrow Ext^1(\mathcal{E}, \mathcal{O}(-1, E_i)) \longrightarrow \underbrace{Ext^1(\mathcal{E}, \mathcal{O}_{|E_i}(-1))}_{A_i^*} \longrightarrow Ext^2(\mathcal{E}, \mathcal{O}(-1, 0))$$

where $Ext^2(\mathcal{E},\mathcal{O}(-1,0)) = H^0(\tilde{\mathbb{P}},\mathcal{E}(-2,\overrightarrow{1}))^* = 0$ also by the corollary above. Thus there exists an extension in $Ext^1(\mathcal{E},\mathcal{A}_1)$ which maps to the identity in $End(A_i)$.

To construct a display of a monad one may apply proposition 2.3.2 in [7]. In our case one has $Ext^2(\mathcal{B}_1,\mathcal{A}_1) = \oplus_{i,j}^n A_i \otimes B_j^* \otimes H^2(\tilde{\mathbb{P}},\mathcal{O}(-2,E_i+E_j)) = 0$, thus there exists a sheaf W_1 and exact sequences

$$0 \longrightarrow \mathcal{A}_1 \longrightarrow W_1 \longrightarrow Q_1 \longrightarrow 0$$

$$0 \longrightarrow X_1 \longrightarrow W_1 \longrightarrow \mathcal{B}_1 \longrightarrow 0$$

which fit into the following commutative diagram:

(4)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{A}_1 & = & \mathcal{A}_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & X_1 & \to & W_1 & \to & \mathcal{B}_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{E} & \to & Q_1 & \to & \mathcal{B}_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and thus one has a monad

$$M_1: \quad 0 \longrightarrow \mathcal{A}_1 \longrightarrow W_1 \longrightarrow \mathcal{B}_1 \longrightarrow 0$$

with cohomology the torsion free sheaf $\mathcal{E}$.

For further computations one needs to know the Chern characters of the sheaves involved in the display. By standard computations one, first, has $\chi(\mathcal{E},\mathcal{O}_{|E_i}(-1)) = a_i$. If we put $d_i = ext^1(\mathcal{E},\mathcal{O}_{|E_i}(-1)) = dimA_i$ and $d_i' = hom(\mathcal{E},\mathcal{O}_{|E_i}(-1)) = dimB_i$ then $d_i - d_i' = -a_i$. We also put $D = \Sigma_{i=1}^n d_i = rk\mathcal{A}_1$ and $D' = rk\mathcal{B}_1 = \Sigma_{i=1}^n d_i'$, then $D - D' = -\Sigma_{i=1}^n a_i := -\bar{a}$. It follows that:

$$ch(\mathcal{A}_1) = D - [Dl_\infty - \Sigma_{i=1}^n d_i E_i]$$

$$ch(\mathcal{B}_1) = D' + [D'l_\infty - \Sigma_{i=1}^n d_i' E_i]$$

By the additivity of the Chern character on exact sequences one has the following:

$$ch(X_1) = ch(\mathcal{A}_1) + ch(\mathcal{E}) = (r+D) - [Dl_\infty - \Sigma_{i=1}^n (d_i + a_i)E_i] - (k + \frac{|\overrightarrow{a}|^2}{2})\omega$$

$$ch(Q_1) = ch(\mathcal{B}_1) + ch(\mathcal{E}) = (r+D') + [D'l_\infty - \Sigma_{i=1}^n (d_i' - a_i)E_i] - (k + \frac{|\overrightarrow{a}|^2}{2})\omega$$

$$ch(W_1) = ch(\mathcal{A}_1) + ch(Q_1) = ch(\mathcal{B}_1) + ch(X_1) = (r + D + D') + \bar{a}l_\infty - (k + \frac{|\overrightarrow{a}|^2}{2})\omega$$

The monad we got has the disadvantage to have a middle term which is not trivial, this complicates the computations for getting explicit ADHM data. We want to construct another monad, from the one obtained above, with trivial middle term. This will be done in few steps, and first we need the following:

Lemma 2.1. *W_1 is trivial on every exceptional divisor E_i of the blow-up*

Proof. Twisting the monad by $\mathcal{O}(E_i)$ and restricting on the exceptional divisor E_i one has

(5)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A_i(-2) & = & A_i(-2) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_{1|E_i}(-1) & \longrightarrow & W_{1|E_i}(-1) & \longrightarrow & B_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{E}_{|E_i}(-1) & \longrightarrow & Q_{1|E_i}(-1) & \longrightarrow & B_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The long exact sequence in cohomology of the right column in the display above gives:

$$0 \longrightarrow \underbrace{H^0(E_i, A_i(-2))}_{0} \longrightarrow H^0(E_i, W_{1|E_i}(-1)) \longrightarrow H^0(E_i, Q_{1|E_i}(-1))$$

(6)

$$\longrightarrow \underbrace{H^1(E_i, A_i(-2))}_{A_i} \longrightarrow H^1(E_i, W_{1|E_i}(-1)) \longrightarrow H^1(E_i, Q_{1|E_i}(-1)) \longrightarrow 0$$

but from the last row of the display above one has

$$0 \longrightarrow A_i \longrightarrow H^0(E_i, Q_{1|E_i}(-1)) \longrightarrow B_i \xrightarrow{\sim} B_i \longrightarrow H^1(E_i, Q_{1|E_i}(-1)) \longrightarrow 0$$

which means that
$$H^0(E_i, Q_{1|E_i}(-1)) = A_i, \qquad H^1(E_i, Q_{1|E_i}(-1)) = 0$$

and
$$H^0(E_i, W_{1|E_i}(-1)) = 0, \qquad H^1(E_i, W_{1|E_i}(-1)) = 0 \quad \forall i = 1, n.$$

Thus the lemma follows. $\square$

This means that W_1 is the pull back of some sheaf on $\mathbb{P}^2$, namely, it is the pull-back of its direct image: $W_1 = \pi^*(\pi_* W_1)$. As a consequence $\mathcal{R}^1\pi_* W_1 = 0$.

It will be useful to see, from the display, to which kind of sheaves the extensions Q_1 and X_1 correspond. For this let us dualize the last row of the display:

$$0 \longrightarrow \mathcal{B}_1^* \longrightarrow Q_1^* \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{E}xt^1(\mathcal{B}_1, \mathcal{O})$$

where $\mathcal{E}xt^1(\mathcal{B}_1, \mathcal{O})$ vanishes since $\mathcal{B}_1$ is locally free. Dualizing again the resulting short exact sequence one gets:

$$0 \longrightarrow \mathcal{E}^{**} \longrightarrow Q_1^{**} \longrightarrow \mathcal{B}_1 \longrightarrow 0$$

and since we have the natural sequences (1) and

$$0 \longrightarrow T \longrightarrow Q_1 \longrightarrow Q_1^{**} \longrightarrow \Delta' \longrightarrow 0$$

we can construct the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & T & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{E} & \longrightarrow & Q_1 & \longrightarrow & \mathcal{B}_1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \| \\
 & & 0 & \longrightarrow & \mathcal{E}^{**} & \longrightarrow & Q_1^{**} & \longrightarrow & \mathcal{B}_1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & ker\delta & \longrightarrow & \Delta & \xrightarrow{\delta} & \Delta' & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

By the snake lemma one has[2] $T = 0$, $ker\delta = 0$, $\Delta = \Delta'$. Then Q_1 is torsion free with $\Delta' = Q_1^{**}/Q_1 = \mathcal{E}^{**}/\mathcal{E} = \Delta$. Using a similar argument, one can check that X_1 and W_1 are torsion-free and $\Delta_X = \Delta_W$, where $\Delta_X = X_1^{**}/X_1$, $\Delta_W = W_1^{**}/W_1$.

Now we need to go through some intermediate steps to construct the right monad, starting by constructing a monad on $\mathbb{P}^2$. For this we need the following:

Theorem 2.2. *A torsion-free sheaf $\mathcal{F}$ on $\mathbb{P}^2$ is given by the cohomology of a monad with trivial middle term if*

$$H^0(\mathbb{P}^2, \mathcal{F}(-1)) = 0, \qquad \textit{and} \qquad H^0(\mathbb{P}^2, \mathcal{F}^*(-1)) = 0$$

Proof. The Beilinson theorem ([6], 3.1.3 and 3.1.4) extends also to the case of a torsion-free sheaf, hence on $\mathbb{P}^2$ there exists a spectral sequence $E_r^{p,q}$ with second first term: $E_1^{p,q} = H^q(\mathbb{P}^2, \mathcal{F} \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}(-p)$ which converges to :

$$E_\infty^{p,q} = \begin{cases} \mathcal{F} & \text{for p+q=0} \\ 0 & \text{otherwise} \end{cases}$$

We apply this to the sheaf $\mathcal{F}(-1)$ and use the vanishing conditions. Then this leads to a monad, with cohomology $\mathcal{F}(-1)$, given by

$$0 \longrightarrow E_1^{0,1} \xrightarrow{d_1^{0,1}} E_1^{-1,1} \xrightarrow{d_1^{-1,1}} E_1^{-2,1} \longrightarrow 0$$

Twisting the complex by $\mathcal{O}(-1)$ one has the monad:

$$0 \longrightarrow H^1(\mathbb{P}^2, \mathcal{F}(-1)) \otimes \mathcal{O}(-1) \xrightarrow{d_1^{0,1}} H^1(\mathbb{P}^2, \mathcal{F} \otimes \Omega^1) \xrightarrow{d_1^{-1,1}} H^1(\mathbb{P}^2, \mathcal{F}(-2)) \otimes \mathcal{O}(1) \longrightarrow 0$$

with cohomology the sheaf $\mathcal{F}$.

$\square$

Proposition 2.2. *The direct image $\pi_* W_1$ of the sheaf W_1 is given by the cohomology of a monad on $\mathbb{P}^2$ with trivial middle term*

Proof. It suffices to verify the vanishing given in theorem 2.2.

$\underline{H^0(\mathbb{P}^2, \pi_* W_1^*(-1)) = 0}$:

We know that W_1 is trivial on every exceptional divisor. It follows that this is also true for the dual sheaf W_1^*, hence the dual is isomorphic to the pull back of its direct image on $\mathbb{P}^2$, moreover we have $\pi_*(W_1^*) \cong (\pi_* W_1)^*$. We need only to prove that $W_1^*(-1)$ has no global sections.

[2]One can see that Q_1 is torsion free by tensoring the last row of the display by $k_x = \mathcal{O}/\mathfrak{m}_x$. This shows that $Tor^2(Q_1, k_x) = 0$ for $i = 1, 2$ since $\mathcal{E}$ and $\mathcal{B}_1$ are torsion free.

We start by dualizing the first row of the display, which gives the exact sequence:

$$0 \longrightarrow \mathcal{B}_1^* \longrightarrow W^* \longrightarrow X_1^* \longrightarrow 0$$

After twisting by $\mathcal{O}(-1,0)$ we obtain the following sequence in cohomology

$$0 \longrightarrow \oplus_{i=1}^n B_i^* \otimes \underbrace{H^0(\tilde{\mathbb{P}}, \mathcal{O}(-2,-E_i))}_{0}, \longrightarrow H^0(\tilde{\mathbb{P}}, W^*(-1,0)) \longrightarrow H^0(\tilde{\mathbb{P}}, X_1^*(-1,0))$$

Now we want to prove that $H^0(\tilde{\mathbb{P}}, X_1^*(-1,0)) = 0$; The dual sequence of the left column of the display is

$$0 \longrightarrow \mathcal{E}^* \longrightarrow X_1^* \longrightarrow \mathcal{A}_1^* \longrightarrow \mathcal{E}xt^1(\mathcal{E}^*, \mathcal{O}) \longrightarrow \mathcal{E}xt^1(X_1^*, \mathcal{O}) \longrightarrow 0$$

where $\mathcal{E}xt^1(\mathcal{E}, \mathcal{O}) = \Delta$ and $\mathcal{E}xt^1(X_1, \mathcal{O}) = \Delta_X$ and both of them are supported on finite sets of points. We split the above sequence into :

$$0 \longrightarrow \mathcal{E}^* \longrightarrow X_1^* \longrightarrow \mathcal{F} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{A}_1^* \longrightarrow \mathcal{G} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{G} \longrightarrow \Delta \longrightarrow \Delta_X \longrightarrow 0$$

From the third we can see that $\mathcal{G}$ is supported on points and from the second that $\mathcal{F}^* \cong \mathcal{A}_1$. now we take the sequences in cohomology induced by the above exact sequences; from the first one has

$$0 \longrightarrow \underbrace{H^0(\tilde{\mathbb{P}}, \mathcal{E}^*(-1,0))}_{0} \longrightarrow H^0(\tilde{\mathbb{P}}, X^*(-1,0)) \longrightarrow H^0(\tilde{\mathbb{P}}, \mathcal{F}(-1,0))$$

and from the second one has

$$0 \longrightarrow H^0(\tilde{\mathbb{P}}, \mathcal{F}(-1,0)) \longrightarrow H^0(\tilde{\mathbb{P}}, \mathcal{A}_1^*(-1,0))$$

but $H^0(\tilde{\mathbb{P}}, \mathcal{A}_1^*(-1,0)) = \oplus_{i=1}^n A_i^* \otimes H^0(\tilde{\mathbb{P}}, \mathcal{O}(0,-E_i))$ which is zero since $H^0(\tilde{\mathbb{P}}, \mathcal{O}(0,-E_i)) = 0$. Thus $H^0(\tilde{\mathbb{P}}, \mathcal{F}(-1,0)) = 0$ which implies that $W_1^*(-1)$ has no global sections.

$\underline{H^0(\mathbb{P}^2, \pi_* W_1(-1)) = 0}$:
The same argument as above can be used. Hence the torsion free sheaf $\pi_* W_1^*$ is described as the cohomology of a monad with trivial middle term.

□

The monad which has cohomology the sheaf $\pi_* W_1$ is given by

$$M_0' : \quad 0 \longrightarrow K_0(-1) \longrightarrow W \longrightarrow L_0(1) \longrightarrow 0$$

where $K_0 = H^1(\tilde{\mathbb{P}}, W_1(-2, \overrightarrow{1}))$, $L_0 = H^1(\tilde{\mathbb{P}}, W_1(-1,0))$ and W is a trivial bundle, as we shall prove later. To see that the spaces K_0 and L_0 yield the Chern character $ch(W_1)$ we shall compute their dimensions:

Proposition 2.3.

$$H^0(\tilde{\mathbb{P}}, W_1(-2, \overrightarrow{1})) = H^0(\tilde{\mathbb{P}}, W_1(-1,0)) = 0$$

and

$$H^2(\tilde{\mathbb{P}}, W_1(-2, \overrightarrow{1})) = H^2(\tilde{\mathbb{P}}, W_1(-1,0)) = 0$$

Proof. The proof is given by using the display (4) twisted by $\mathcal{O}(-1,0)$ and taking the induced long exact sequences in cohomology. □

Corollary 2.2. *The spaces K_0 and L_0 have, respectively, dimension $k + \frac{|\overrightarrow{a}|^2 - \bar{a}}{2}$ and $k + \frac{|\overrightarrow{a}|^2 + \bar{a}}{2}$.*

Proof. By using the Riemann-Roch formula we compute the Euler characters of $W_1(-2, \overrightarrow{1})$ and $W_1(-1,0)$. This gives

$$\chi(W_1(-2,\overrightarrow{1}) = -(k + \frac{|\overrightarrow{a}|^2 + \bar{a}}{2}) \quad \text{and} \quad \chi(W_1(-1,0)) = -(k + \frac{|\overrightarrow{a}|^2 - \bar{a}}{2}) \tag{7}$$

The corollary follows from the vanishing of the groups in the proposition above. □

Lemma 2.2. *The middle term W of the monad M_0' is trivial.*

Proof. We use the following result:

Lemma 2.3. *On a smooth regular surface rational to a blow-up of $\mathbb{P}^2$ in distinct points, a sheaf $\mathcal{F}$ is trivial if and only if $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = 0$ and $\mathcal{F}_{|l_\infty}$ is trivial*

The proof of the above lemma is similar to the one given in lemma 3.2.2 [7]. We use the display of M_0' and compute the Chern character $ch(W)$. The triviality on the line at infinity can be checked by restricting the above display to this line, from which one has $H^0(l_\infty, W_{|l_\infty}(-1)) = H^1(l_\infty, W_{|l_\infty}(-1)) = 0$. □

Now we want to construct the intermediate monad with trivial middle term and with cohomology the original sheaf $\mathcal{E}$. First we have to pull-back the monad M_0' to a monad M_0 on $\tilde{\mathbb{P}}$:

$$M_0: \quad 0 \longrightarrow K_0(-1,0) \longrightarrow W \longrightarrow L_0(1,0) \longrightarrow 0$$

Then we should lift the morphism $\mathcal{A}_1 \longrightarrow W_1$ to a morphism $\mathcal{A}_1 \longrightarrow X_0'$ where $X_0' = ker(W \longrightarrow L_0(1,0))$ i.e.

$$\begin{array}{ccc} & & X_0' \\ & \nearrow & \downarrow \\ \mathcal{A}_1 & \hookrightarrow & W_1 \end{array}$$

so we want a surjective morphism $Hom(\mathcal{A}_1, X_0') \longrightarrow Hom(\mathcal{A}_1, W_1)$. The obstruction to get this lies in the group $Ext^1(\mathcal{A}_1, K_0(-1,0))$ which is zero since $Ext^1(\mathcal{A}_1, K_0(-1,0)) = \oplus_{i=1}^n A_i^* \otimes K_0 \otimes H^1(\tilde{\mathbb{P}}, \mathcal{O}(0,-E_i)) = 0$. This means that all the extensions

$$0 \longrightarrow K_0(-1,0) \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}_1 \longrightarrow 0$$

split, hence $\mathcal{A} = K_0(-1,0) \oplus \mathcal{A}_1$. Furthermore we have a monomorphism $\mathcal{A} \longrightarrow W$.

Dually we want to lift the morphism $W_1 \longrightarrow \mathcal{B}_1$ to a morphism $Q_0' \longrightarrow \mathcal{B}_1$, where $Q_0' = coker(K_0(-1,0) \longrightarrow W)$ i.e.

$$\begin{array}{ccc} W_1 & \twoheadrightarrow & \mathcal{B}_1 \\ \downarrow & \nearrow & \\ Q_0' & & \end{array}$$

We also want a surjective morphism $Hom(Q_0', \mathcal{B}_1) \longrightarrow Hom(W_1, \mathcal{B}_1)$, and in this case the obstruction is in the group $Ext^1(L_0(1,0), \mathcal{B}_1)$ which also vanishes since $Ext^1(L_0(1,0), \mathcal{B}_1) = \oplus_{i=1}^n B_i \otimes L_0^* \otimes H^1(\tilde{\mathbb{P}}, \mathcal{O}(0,-E_i)) = 0$. This is means that all the extensions

$$0 \longrightarrow \mathcal{B}_1 \longrightarrow \mathcal{B} \longrightarrow L_0(1,0) \longrightarrow 0$$

split, hence $\mathcal{B} = L_0(1,0) \oplus \mathcal{B}_1$. Furthermore we have an epimorphism $W \longrightarrow \mathcal{B}$. Now we have

$$M: \quad 0 \longrightarrow K_0(-1,0) \oplus \mathcal{A}_1 \longrightarrow W \longrightarrow L_0(1,0) \oplus \mathcal{B}_1 \longrightarrow 0$$

which is a monad with the following display

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_0(-1,0)\oplus\mathcal{A}_1 & = & K_0(-1,0)\oplus\mathcal{A}_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \longrightarrow & W & \longrightarrow & L_0(1,0)\oplus\mathcal{B}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & Q & \longrightarrow & L_0(1,0)\oplus\mathcal{B}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{8}
$$

from which we compute the Chern characters, and get

$$
\begin{aligned}
ch(\mathcal{F}) &= ch(Q) - ch(L_0(1,0)) - ch(\mathcal{B}_1) = ch(X) - ch(K_0(-1,0)) - ch(\mathcal{A}_1) \\
&= rk(W) - D - D' - k_0 - l_0 - [(D' - D - k_0 + l_0)l_\infty - \Sigma_{i=1}^n (d_i + d_i')E_i] - \frac{(l_0 + k_0)}{2}\omega
\end{aligned}
$$

and since we have the relations:

$$
\begin{aligned}
rk(W) &= r + D + D' + k_0 + l_0, & k_0 + l_0 &= 2k + |\overrightarrow{a}|^2 \\
\Sigma_{i=1}^n (d_i - d_i') &= -\Sigma_{i=1}^n a_i = -\bar{a} & k_0 - l_0 &= \bar{a} = -(D - D')
\end{aligned} \tag{9}
$$

then

$$
ch(\mathcal{F}) = r + \Sigma_{i=1}^n a_i E_i - (k + \frac{|\overrightarrow{a}|^2}{2})\omega = ch(\mathcal{E})
$$

One can use also the three displays of M_1, M_0 and M to prove that $\mathcal{F} \cong \mathcal{E}$. Thus we have the following :

Theorem 2.3. *Let $\mathcal{E}$ be a torsion free sheaf with Chern character $ch(\mathcal{E}) = r + \Sigma_{i=1}^n a_i E_i - (k + \frac{|\overrightarrow{a}|^2}{2})\omega$ on a multi blow up of $\mathbb{P}^2$ in distinct points. Denoting by $\pi : \tilde{\mathbb{P}} \longrightarrow \mathbb{P}^2$ the blow-down map, the direct image $\pi_*\mathcal{E}$ of $\mathcal{E}$ is semi-stable. Then $\mathcal{E}$ is given by the cohomology of a monad:*

$$
M : \quad 0 \longrightarrow K_0(-1,0) \oplus \mathcal{A}_1 \longrightarrow W \longrightarrow L_0(1,0) \oplus \mathcal{B}_1 \longrightarrow 0
$$

where

$$
A_i := Ext^1(\mathcal{E}, \mathcal{O}_{|E_i}(-1))^*, \qquad B_i := Ext^0(\mathcal{E}, \mathcal{O}_{|E_i}(-1))^*
$$

$$
\mathcal{A}_1 := \oplus_{i=1}^n A_i(-1, E_i), \qquad \mathcal{B}_1 := \oplus_{i=1}^n B_i(1, -E_i)
$$

$$
K_0 := H^1(\tilde{\mathbb{P}}, \mathcal{E}(-2, \overrightarrow{1})), \qquad L_0 := H^1(\tilde{\mathbb{P}}, \mathcal{E}(-1, 0)).
$$

Remark 2.1. ***I****/ Using the display of the monad M_1 we can write*

$$
\begin{aligned}
K_0 &:= H^1(\tilde{\mathbb{P}}, \mathcal{E}(-2, \overrightarrow{1})) \oplus H^1(\tilde{\mathbb{P}}, \mathcal{A}_1(-2, \overrightarrow{1})) \oplus H^1(\tilde{\mathbb{P}}, \mathcal{B}_1(-2, \overrightarrow{1})) \\
L_0 &:= H^1(\tilde{\mathbb{P}}, \mathcal{E}(-1, 0)) \oplus H^1(\tilde{\mathbb{P}}, \mathcal{A}_1(-1, 0)) \oplus H^1(\tilde{\mathbb{P}}, \mathcal{B}_1(-1, 0)).
\end{aligned}
$$

From the Riemann-Roch formula one has $H^1(\tilde{\mathbb{P}}, \mathcal{A}_1(-2, \overrightarrow{1})) = \oplus_{i=1}^n A_i \otimes H^1(\tilde{\mathbb{P}}, \mathcal{O}(-3, \overrightarrow{1} + E_i)) = 0$ since $\chi(\mathcal{O}(-3, \overrightarrow{1} + E_i)) = 0$ and also

$$
H^1(\tilde{\mathbb{P}}, \mathcal{B}_1(-2, \overrightarrow{1})) = \oplus_{i=1}^n B_i \otimes H^1(\tilde{\mathbb{P}}, \mathcal{O}(-1, \overrightarrow{1} - E_i)) = 0
$$

The vanishing holds also for $H^1(\tilde{\mathbb{P}}, \mathcal{A}_1(-1,0))$ and $H^1(\tilde{\mathbb{P}}, \mathcal{B}_1(-1,0))$. Hence we have the forms of K_0 and L_0 given in the theorem above.

II/ *Since the monad in the theorem above has a trivial middle term, the kernel of the second map is locally free. This can be seen from the display by dualizing the sequence $0 \longrightarrow X \longrightarrow W \longrightarrow L_0(1,0) \oplus \mathcal{B}_1 \longrightarrow 0$. As a consequence the first map will vanish exactly on the singularity set of $\mathcal{E}$ since this sheaf is the quotient $(K_0(-1,0) \oplus \mathcal{A}_1)/X$ of two locally free sheaves.*

In order to study the moduli space we are interested in, we need to know if families of such monads behave well in describing families of torsion free sheaves; We start by reminding the following[4]:

Proposition 2.4. *Let be $M : \mathcal{A} \longrightarrow \mathcal{W} \longrightarrow \mathcal{B}$ and $\mathcal{A}' \longrightarrow \mathcal{W}' \longrightarrow \mathcal{B}'$ two monads on a surface X with cohomologies $\mathcal{E}$ and $\mathcal{E}'$ respectively. The morphism $H : Hom(M, M') \longrightarrow Hom(\mathcal{E}, \mathcal{E}')$ is epimorphic if*

$$Ext^1(\mathcal{B}, \mathcal{W}') = Ext^1(\mathcal{W}, \mathcal{A}') = Ext^2(\mathcal{B}, \mathcal{A}') = 0$$

Furthermore the kernel is identified with $Ext^1(\mathcal{B}, \mathcal{A}')$ if $Hom(\mathcal{B}, \mathcal{W}') = Hom(\mathcal{W}, \mathcal{A}') = 0$.

In our case it is easy to check that

$$\begin{array}{lll} Ext^1(\mathcal{B}, W') = 0 & Ext^1(W, \mathcal{A}') = 0 & Ext^2(\mathcal{B}, \mathcal{A}') = 0 \\ Hom(\mathcal{B}, W') = 0 & Hom(W, \mathcal{A}') = & Ext^1(\mathcal{B}, \mathcal{A}') = 0. \end{array}$$

using Riemann-Roch theorem. Hence $H : Hom(M, M') \longrightarrow Hom(\mathcal{E}, \mathcal{E}')$ is an isomorphism.

The monad we constructed describes well families of torsion free sheaves and one can talk about a moduli space of such objects but before we have to fix the problem of the control on the dimensions of A_i and B_i; their difference is constant $dim A_i - dim B_i = a_i$, but each dimension can, a priori, jump. We do as in [1]; We apply the functor $Hom(\mathcal{E}, \cdot)$ to the sequence (2) twisted by $\mathcal{O}(-1, E_i)$:

$$0 \longrightarrow B_i^* \longrightarrow Ext^1(\mathcal{E}, \mathcal{O}(-1,0)) \longrightarrow Ext^1(\mathcal{E}, \mathcal{O}(-1, E_i)) \longrightarrow A_i^* \longrightarrow 0$$

then there exists a splitting through $V_i^* = ker(Ext^1(\mathcal{E}, \mathcal{O}(-1, E_i)) \longrightarrow A_i^*)$ i.e.

$$0 \longrightarrow B_i^* \longrightarrow Ext^1(\mathcal{E}, \mathcal{O}(-1,0)) \longrightarrow V_i^* \longrightarrow 0$$

$$0 \longrightarrow V_i^* \longrightarrow Ext^1(\mathcal{E}, \mathcal{O}(-1, E_i)) \longrightarrow A_i^* \longrightarrow 0$$

dualizing the sequences and using the fact that

$$Ext^1(\mathcal{E}, \mathcal{O}(-1,0))^* = Ext^1(\mathcal{O}(-1,0), \mathcal{E}(-3, \overrightarrow{1})) = H^1(\mathcal{E}(-2, \overrightarrow{1})) \quad \text{and}$$

$$Ext^1(\mathcal{E}, \mathcal{O}(-1, E_i))^* = Ext^1(\mathcal{O}(-1, E_i), \mathcal{E}(-3, \overrightarrow{1})) = H^1(\mathcal{E}(-2, \overrightarrow{1} - E_i))$$

which have dimensions, respectively, $k + \frac{1}{2}\Sigma_{j=1}^n a_j(a_j + 1)$ and $k + \frac{1}{2}\Sigma_{j=1}^n a_j(a_j + 1) - a_i$, one has the isomorphisms :

$$A_i \oplus V_i \cong H^1(\mathcal{E}(-2, \overrightarrow{1} - E_i)), \qquad B_i \oplus V_i \cong H^1(\mathcal{E}(-2, \overrightarrow{1})). \tag{10}$$

On the other hand consider the following extension

$$0 \longrightarrow \mathcal{O}(-1, E_i) \longrightarrow W_0 \longrightarrow \mathcal{O}(1, -E_i) \longrightarrow 0 \tag{11}$$

The Chern character of W_0 is $ch(W_0) = (1 - l_\infty + E_i)(1 + l_\infty - E_i) = 1$ which implies that $c_1(W_0) = 0$ and $c_2(W_0) = 0$. Applying **lemma 2.3** we get:

[4]For the proof see [7] section 2 or [6] 4.1.3 which extends easily to the case of torsion free sheaves

Corollary 2.3. *W_0 is trivial*

Proof. To show this it suffices to show that $H^{0,1}(l_\infty, W_{0|l_\infty}(-1)) = 0$. We restrict the extension (11) twisted by $\mathcal{O}(-1,0)$ to the line l_∞. We have the following sequence:

$$0 \longrightarrow H^0(l_\infty, W_{0|l_\infty}(-1)) \longrightarrow \underbrace{H^0(l_\infty, \mathcal{O}_{|l_\infty})}_{\mathcal{O}_{|l_\infty}} \longrightarrow \underbrace{H^1(l_\infty, \mathcal{O}_{|l_\infty}(-2))}_{\mathcal{O}_{|l_\infty}} \longrightarrow H^1(l_\infty, W_{0|l_\infty}) \longrightarrow 0$$

Thus $H^0(l_\infty, W_{0|l_\infty}(-1)) = H^1(l_\infty, W_{0|l_\infty}) = 0$ □

What we have to do now is to twist again the extension (11) by V_i;

$$0 \longrightarrow V_i(-1, E_i) \longrightarrow V_i \otimes W_0 \longrightarrow V_i(1, -E_i) \longrightarrow 0 \tag{12}$$

and since adding such exact sequences, for every i, to the monad in the theorem (2.2) will not change the cohomology $\mathcal{E}$, then using the isomorphisms (10) one has the following

Proposition 2.5. *Let $\mathcal{E}$ be a torsion free sheaf satisfying the conditions of theorem* (2.2). *There exists a monad M, describing $\mathcal{E}$ as its cohomology, which is of the following form:*

$$M: \quad 0 \longrightarrow \oplus_{i=0}^n K_i(-1, E_i) \longrightarrow W \longrightarrow \oplus_{i=0}^n L_i(1, -E_i) \longrightarrow 0$$

where we put $E_0 := 0$ and

$$\begin{cases} K_i = H^1(\tilde{\mathbb{P}}, \mathcal{E}(-2, \overrightarrow{1} - E_i)) \\ L_i = H^1(\tilde{\mathbb{P}}, \mathcal{E}(-2, \overrightarrow{1})) \end{cases} \quad i \neq 0$$

and where

$$\begin{cases} K_0 := H^1(\tilde{\mathbb{P}}, \mathcal{E}(-2, \overrightarrow{1})) \\ L_0 := H^1(\tilde{\mathbb{P}}, \mathcal{E}(-1, 0)) \end{cases}$$

Remark 2.2. ***I***/ *The reader must be aware of the use, in the proposition above, of the same notation as in theorem 2.3, while the spaces and the monads are different.*

II/ *Since the dimensions of our spaces are as the following:*

$$dimK_0 = k + \frac{|a|^2 + \bar{a}}{2}, \qquad dimL_0 = k + \frac{|a|^2 - \bar{a}}{2}$$
$$dimK_i = k + \frac{|a|^2 + \bar{a}}{2} - a_i, \qquad dimL_i = k + \frac{|a|^2 + \bar{a}}{2}$$

then

$$dimK_0 + \Sigma_{i=1}^n dimK_i = (n+1)k + \frac{n+1}{2}(|a|^2 + \bar{a}) - \bar{a}$$

and

$$dimL_0 + \Sigma_{i=1}^n dimL_i = (n+1)k + \frac{n+1}{2}|a|^2 + \frac{n-1}{2}\bar{a} = (n+1)k + \frac{n+1}{2}(|a|^2 + \bar{a}) - \bar{a}$$

thus the left and right terms in the monad have the same rank $dimL_0 + \Sigma_{i=1}^n dimL_i$. Hence $rkW = 2(dimL_0 + \Sigma_{i=1}^n dimL_i) + r$.

III/ *Using the display of the monad above one can see, straightforwardly, that the kernel K of the second map, that we denote by $\beta(x)$, is a locally free sheaf. Then the fact that the cohomology is a torsion-free sheaf will imply the vanishing of the first map, that we denote by $\alpha(x)$, on finitely many points. These are the singularity set of the torsion free sheaf that we are describing. Vice versa if the map $\alpha(x)$ vanishes on some finite set of points, then the cohomology of the monad would be a torsion free sheaf with singularity set, exactly, the set of the points where $\alpha(x)$ vanishes.*

3 The ADHM data

To describe the ADHM data associated to the constructed monad, we reduce the linear data in this monad by throwing away some of the extra degrees of freedom, using its symmetries (automorphisms). We now summarize Penrose notations introduced by Buchdahl in [1], then review the steps of the reduction.

On $\mathbb{P}^2$ we denote the homogeneous coordinates by (z^0, z^1, z^2). The line at infinity L_∞ is given by the equation $z^2 = 0$. Let p_i be one of the points in $\mathbb{P}^2$ in which we perform a blow-up, and assume $p_i \notin L_\infty$ for all i. So we can consider that all these points are in the chart $[z^0, z^1, 1]$ and denote their inhomogeneous coordinates in this chart by $p_i^a = (p_A^i, 1)$ where $A = 0, 1$ and $a = 0, 1, 2$. Locally, near every blow-up point, $\tilde{\mathbb{P}}$ can be described by $\{([z^0, z^1, z^2], [w_i^0, w_i^1]) \in \mathbb{P}^2 \times \mathbb{P}^1 / (z^0 - p_i^0 z^2) w_i^1 = (z^1 - p_i^1 z^2) w_i^0\}$. If we put now $\lambda_i w_i^1 = z^1 - p_i^1 z^2$ and $\lambda_i w_i^0 = z^0 - p_i^0 z^2$ the equation

$$(z^0 - p_i^0 z^2) w_i^1 = (z^1 - p_i^1 z^2) w_i^0$$

of the blow-up is satisfied. $p_i^A z^2$ is a hyperplane in $\tilde{\mathbb{P}}$ which is in the linear system $|l_\infty|$, while z^A is a divisor in the linear system $|E_i|$. So that all together $(z^A - p_i^A z^2)$ gives a divisor in the linear system $|L_\infty - E_i|$, and we can think of $z^A - p_i^A z^2 = \lambda_i w_i^A$ as a section of $\mathcal{O}(l_\infty - E_i)$, i.e., $w_i^A \in H^0(\tilde{\mathbb{P}}, \mathcal{O}(1, -E_i))$. The restriction of w_i^A to the exceptional divisor E_i gives the homogeneous coordinates (w_i^0, w_i^1) of E_i. Finally we denote

$$\begin{cases} w_{i0} := -w_i^1 \\ w_{i1} := w_i^0 \end{cases}$$

It follows that $w_i^A w_{iA} = 0$.

More generally if we have a pair of matrices as $m_A = (m_0, m_1)$ we use the same two index notations as above, and we write $m^A m_A = m_0 m_1 - m_1 m_0$. For the direct sum $V \oplus V$ and the morphism $(m_0, m_1) : V \longrightarrow V \oplus V$ we use the notation $m_A : V \longrightarrow V_A$. Given a morphism $V \oplus V \longrightarrow V$ where $(v_0, v_1) \longrightarrow m^0 v_0 + m^1 v_1$, this would be written as $m^A : V^A \longrightarrow V$.

Now let us consider a monad M as in the **Proposition 2.5**

$$M : \qquad \oplus_{i=0}^n K_i(-1, E_i) \xrightarrow{\alpha(x)} W \xrightarrow{\beta(x)} \oplus_{i=0}^n L_i(1, -E_i)$$

with cohomology a framed torsion free sheaf $\mathcal{E}$. Following [2] one can choose a basis for W such that $W = \oplus_{i=0}^n L_{iA}$, then the maps $\alpha(x)$ and $\beta(x)$ take the form:

$$\alpha = \begin{bmatrix} a_{00} z_A + a_{00A} z^2 & a_{01} w_{1A} & a_{02} w_{2A} & \cdots & a_{0n} w_{nA} \\ a_{10} \lambda_1 w_{1A} & a_{11} w_{1A} & 0 & \cdots & 0 \\ a_{20} \lambda_2 w_{2A} & 0 & a_{22} w_{2A} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} \lambda_n w_{nA} & 0 & 0 & \cdots & a_{nn} w_{nA} \\ C_0^B z_B + c z^2 & c_1^B w_{1B} & c_2^B w_{2B} & \cdots & c_n^B w_{nB} \end{bmatrix} \tag{13}$$

$$\beta = \begin{bmatrix} z^A + b_{00}^A z^2 & b_{01}^A z^1 & b_{02}^A z^2 & \cdots & b_{0n}^A z^n & d z^2 \\ 0 & w^{1A} & 0 & \cdots & 0 & 0 \\ 0 & 0 & w^{2A} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & w^{nA} & 0 \end{bmatrix} \tag{14}$$

$\alpha(x)$ is of maximal rank except at some finite set of points. $a_{ij} \in Hom(K_i, L_j)$, $a_{0iA} \in Hom(K_0, L_i)$, $c_i^B \in Hom(K_i, \mathbb{C}^r)$, $c \in Hom(K_0, \mathbb{C}^r)$, $b_{0i}^A \in Hom(L_i, L_0)$ and $d \in Hom(\mathbb{C}^r, L_0)$.

The monad condition $\beta \circ \alpha = 0$ will give the following equations:

$$-a_{00}^A + b_{00}^A a_{00} + \Sigma_{i=1}^n b_{0i}^A a_{i0} + dc_0^A = 0 \tag{15}$$

$$b_{00}^A a_{00A} - \Sigma_{i=1}^n b_{0i}^A a_{i0} p_{iA} + dc = 0 \tag{16}$$

$$(p_i^A + b_{00}^A) a_{i0} + b_{0i}^A + dc_i^A = 0. \tag{17}$$

Using the lemma 2.3.4 in [7], the framing condition is equivalent to the non-singularity of the matrix

$$a = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ a_{10} & a_{11} & 0 & \cdots & 0 \\ a_{20} & 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n0} & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \tag{18}$$

To put the equations into a simpler form, we shall use the available symmetry to fix some of the extra degrees of freedom. We use the transformation g_W acting on the space W and thus acting on the monad by $g_W \alpha$, βg_W^{-1} for some g_W of the form $g_W = \begin{bmatrix} \delta_B^A & 0 \\ -c^A a^{-1} & \delta_B^A \end{bmatrix}$, where $c^A := \begin{bmatrix} c_0^A & c_1^A & \cdots & c_n^A \end{bmatrix}$. This transformation replaces all c_i^A by zero for $i \geq 0$ in the equations (15),(16) and (16).

Now let us define the following objects, which will provide useful notation to write the linear data in a more compact way;

$$a_{0\bullet} := \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \end{bmatrix}, a_{\bullet 0} := \begin{bmatrix} a_{00} \\ a_{10} \\ \vdots \\ a_{n0} \end{bmatrix}, a^A := \begin{bmatrix} a_{00}^A & 0 & \cdots & 0 \end{bmatrix}, b^A := \begin{bmatrix} b_{00}^A & b_{01}^A & \cdots & b_{0n}^A \end{bmatrix},$$

and

$$p^A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & p_1^A & 0 & \cdots & 0 \\ 0 & 0 & p_2^A & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_n^A \end{bmatrix} \tag{19}$$

Using the notations above one can write (15) and (17) as

$$b^A a + a_{0\bullet} p^A = a_{00}^A (= a^A) \tag{20}$$

while the second equation (16) takes the form

$$b_{00}^A a_{00A} - b^A p_A a_{\bullet 0} + dc = 0 \tag{21}$$

From (20) we have $b^A = (a^A - a_{0\bullet} p^A) a^{-1}$. By replacing in (22) one can write

$$b_{00}^A a_{00A} - a^A a^{-1} p_A a_{\bullet 0} + a_{0\bullet} p^A a^{-1} p_A a \bullet 0 + dc = 0 \tag{22}$$

Using the commutator $[p^A, a]$ in the ralation (22), the monad condition takes the form

$$(q^A a^{-1} q_A)^{00} + dc = 0 \tag{23}$$

where we defined the matrix q^A as

$$q^A = \begin{bmatrix} -a_{00}^A & p_1^A a_{01} & p_2^A a_{02} & \cdots & p_n^A a_{0n} \\ p_1^A a_{10} & p_1^A a_{11} & 0 & \cdots & 0 \\ p_2^A a_{20} & 0 & p_2^A a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_n^A a_{0n} & 0 & 0 & \cdots & p_n^A a_{nn} \end{bmatrix} \tag{24}$$

If now we consider another monad M' of the same form which satisfies the same condition above with respect to the data (a', q'^A, c', d'), where the associated matrix a' is invertible, and such that its cohomology $\mathcal{E}'$ is isomorphic to $\mathcal{E}$, the cohomology of M, then there exist transformations $g \in \mathcal{A}ut(\oplus_{i=0}^n L_i)$ and $h \in \mathcal{A}ut(\oplus_{i=0}^n K_i)$ of the form:

$$g = \begin{bmatrix} g_{00} & g_{01} & g_{02} & \cdots & g_{0n} \\ 0 & g_{11} & 0 & \cdots & 0 \\ 0 & 0 & g_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_{nn} \end{bmatrix}, \quad h = \begin{bmatrix} h_{00} & 0 & 0 & \cdots & 0 \\ h_{10} & h_{11} & 0 & \cdots & 0 \\ h_{20} & 0 & h_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n0} & 0 & 0 & \cdots & h_{nn} \end{bmatrix} \tag{25}$$

which relate the two configurations by the action $(a', q'^A, c', d') = (g, h)(a, q^A, c, d)$ given by

$$a_{00} \longrightarrow g_{00}a_{00}h_{00} + \Sigma_{i=1}^n g_{0i}a_{i0}h_{00} + g_{00}a_{0i}h_{i0} + g_{0i}a_{ii}h_{i0} \tag{26}$$

$$a_{00}^A \longrightarrow g_{00}a_{00}^A h_{00} - \Sigma_{i=1}^n (g_{0i}a_{i0}h_{00} + g_{00}a_{0i}h_{i0} + g_{0i}a_{ii}h_{i0})p_i^A \tag{27}$$

$$a_{ii} \longrightarrow g_{ii}a_{ii}h_{ii}, \qquad a_{0i} \longrightarrow g_{00}a_{0i}h_{ii} + g_{0i}a_{ii}h_{i0} \tag{28}$$

$$a_{i0} \longrightarrow g_{ii}a_{i0}h_{00} + g_{ii}a_{ii}h_{i0}, \quad c \longrightarrow ch_{00}, \quad d \longrightarrow g_{00}d. \tag{29}$$

The set of transformations that fix the configuration (a, q^A, c, d) are of the form $g = 1 + am$, $h = (1 + ma)^{-1}$, where the matrix m is of the form

$$m = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & m_{11} & 0 & \cdots & 0 \\ 0 & 0 & m_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_{nn} \end{bmatrix} \in Hom(\oplus_{i=0}^n L_i, \oplus_{i=0}^n K_i) \tag{30}$$

Fixing an isomorphism $K_i \cong L_0$, one can choose the matrix a so that $a_{i0} = 1 \quad \forall i > 0$. To preserve this form of a, the transformation h must be of the above form and such that $h_{ii} = (g_{00} + g_{0i}a_{ii})^{-1}$. One checks that with a of the above form, every matrix g such that $g_{00} + g_{01}a_{11}$ is non-singular and there exists a matrix D of the form $D = diag(d_{00}, d_{11}, \cdots, d_{nn})$ such that $g = D(1 + am)$. The explicit form of these matrices is given, by they entries, as the following:

$$\begin{cases} d_{00} = g_{00} \\ m_{ii} = g_{00}^{-1} g_{0i} \\ d_{ii} = g_{ii} g_{i0}^{-1} g_{00}(g_{00} + g_{0i}a_{ii})^{-1} g_{0i}. \end{cases}$$

Finally, the moduli space of framed torsion-free sheaves $\mathcal{E}$, on $\tilde{\mathbb{P}}$, with Chern character $ch(\mathcal{E}) = r + \Sigma_{i=1}^n a_i E_i - (k + \frac{|\vec{a}|^2}{2})$, is identified with the space of configurations (a, a_{00}^A, c, d), such that the elements a_{0i} of the non-singular matrix a are all equal to 1 for $i > 1$ and such that the condition (23) is satisfied, modulo the action of the group of non-singular transformations of the form and

$$g = diag(g_{00}, g_{11}, \cdots, g_{nn}), \qquad h = \begin{bmatrix} h_{00} & 0 & 0 & \cdots & 0 \\ h_{10} & g_{00}^{-1} & 0 & \cdots & 0 \\ h_{20} & 0 & g_{00}^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n0} & 0 & 0 & \cdots & g_{00}^{-1} \end{bmatrix}. \tag{31}$$

Their action is explicitly given by

$$a_{00} \longrightarrow g_{00}(a_{00} + \Sigma_{i=1}^n h_{i0}h_{00}^{-1})h_{00}, \quad a_{00}^A \longrightarrow g_{00}(a_{00}^A - \Sigma_{i=1}^n (h_{i0}h_{00}^{-1}p_i^A)h_{00}$$
$$a_{ii} \longrightarrow g_{ii}a_{ii}g_{00}^{-1}, a_{i0} \longrightarrow g_{ii}(a_{i0} + a_{ii}h_{i0}h_{00}^{-1})h_{00}, \quad i > 0$$
$$c \longrightarrow ch_{00}, \quad d \longrightarrow g_{00}d.$$

The above free action of the group, obtained from the reduction of the monad, with isotropy subgroup $G_{\mathcal{C}} = \{Id\}$ for every configuration $\mathcal{C} = (a, a_{00}^A, c, d)$, will give a nonsingular quotient as we shall prove in the next section. One can check, from the data above, that the dimension of this moduli space is $dim\mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}} = 2(k + \frac{|\vec{a}|^2}{2}) - |\vec{a}|$. The space $\mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}}$ is represented as:

$$\mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}} = P/G$$

where

$$P = \left\{ (a, a_{00}^A, c, d) / \quad \begin{array}{l} (23) \quad \text{is satisfied, } a \text{ is non-singular and such that the associated morphism} \\ \quad \alpha(x) \quad \text{can have a non-maximal rank only on a finite set of points} \end{array} \right\}$$

and the group G is the one defined by (g, h) of the form given in (31).

4 Smoothness of the moduli space $\mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}}$

We want to prove the smoothness of the moduli space $\mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}} = P/G$ where P is the space of the ADHM data $\rho := (a, q^A, c, d)$ and G is the symmetry group acting on this data. First let $\mathcal{E}$ and $\mathcal{E}'$ be two framed torsion-free sheaves with the same fixed Chern class, and let $\alpha : \mathcal{E} \longrightarrow \mathcal{E}'$ be a morphism preserving the framing up to a homothety, i.e., the diagram

$$(32) \qquad \begin{array}{ccccc} \mathcal{E} & \longrightarrow & \mathcal{E}_{|l_\infty} & \xrightarrow{\Phi} & \mathcal{O}_{|l_\infty}^{\oplus r} \\ \downarrow \alpha & & \downarrow \alpha_{|l_\infty} & & \downarrow \lambda \\ \mathcal{E}' & \longrightarrow & \mathcal{E}'_{|l_\infty} & \xrightarrow[\Phi']{} & \mathcal{O}_{|l_\infty}^{\oplus r} \end{array}$$

commutes. $\lambda \in \mathcal{O}_{|l_\infty}^*$ and r is the rank of the sheaves $\mathcal{E}$ and $\mathcal{E}'$. Since Φ and Φ' are isomorphisms, one gets the following relation

$$(33) \qquad \alpha_{|l_\infty} = \lambda \Phi'^{-1}\Phi$$

and since $supp|l_\infty|$ is open dense in $\tilde{\mathbb{P}}$, the morphism α is completely determined by its restriction $\alpha_{|l_\infty}$. Define $Hom^\Phi(\mathcal{E}, \mathcal{E}')$ to be the subgroup of $Hom(\mathcal{E}, \mathcal{E}')$ which contains the morphisms preserving the framing up to a homothety. $\Phi'^{-1}\Phi$ being a fixed element of $End(\mathcal{O}_{|l_\infty}^{\oplus r})$ and $\lambda \in \mathcal{O}_{|l_\infty}$, the dimension of $Hom^\Phi(\mathcal{E}, \mathcal{E}')$ as a subspace of $Hom(\mathcal{E}, \mathcal{E}')$ is 1.

Let us consider the universal monad on $P \times \tilde{\mathbb{P}}$ which will be introduced in section **5**;

$$\mathbb{M}: \qquad \oplus_{i=0}^n \mathcal{O}_{\tilde{\mathbb{P}}}(-1, E_i) \boxtimes K_i \otimes \mathcal{O}_P \longrightarrow \mathcal{O}_{\tilde{\mathbb{P}}} \boxtimes W \otimes \mathcal{O}_P \longrightarrow \oplus_{i=0}^n \mathcal{O}_{\tilde{\mathbb{P}}}(1, -E_i) \boxtimes L_i \otimes \mathcal{O}_P$$

with cohomology the family that we denote by $\mathfrak{F}$; which is P-flat and for every point $\rho \in P$, the fiber is given by $\mathfrak{F}_\rho \cong \mathcal{E}(\rho)$, a framed torsion-free sheaf of . On $P \times P \times \tilde{\mathbb{P}}$ consider the following projections:

$$\begin{array}{ccc} P \times P \times \tilde{\mathbb{P}} & \overset{pr_{13}}{\underset{pr_{23}}{\rightrightarrows}} & P \times \tilde{\mathbb{P}} \\ \downarrow pr_{12} & & \\ P \times P & & \end{array}$$

Consider the sheaf $\mathcal{H}om(pr_{13}^*\mathfrak{F}, pr_{23}^*\mathfrak{F})$ on $P \times P \times \tilde{\mathbb{P}}$. Since the sheaves $pr_{23}^*\mathfrak{F}$ and $pr_{23}^*\mathfrak{F}$ are flat on $P \times P$, then is $\mathcal{H}om(pr_{13}^*\mathfrak{F}, pr_{23}^*\mathfrak{F})$.

Let us use the following notation: We omit the pull-back symbols, and consider $(\mathfrak{F}, \phi) := \mathfrak{F} \xrightarrow{\phi} \mathcal{O}_{l_\infty}^{\oplus r}$ where the morphism ϕ is given by the triangle

$$\begin{array}{ccc} \mathfrak{F} & \xrightarrow{r} & \mathfrak{F}_{l_\infty} \\ & \searrow^{\phi} & \downarrow \Phi \\ & & \mathcal{O}_{l_\infty}^{\oplus r} \end{array}$$

and define

$$Hom((\mathfrak{F}, \phi), (\mathfrak{F}', \psi)) := \left\{ \begin{array}{l} \alpha : \mathfrak{F} \longrightarrow \mathfrak{F}' \\ \lambda \in \mathcal{O}_{l_\infty} \end{array} \middle| \text{such that} \quad \begin{array}{ccc} \mathfrak{F} & \xrightarrow{\alpha} & \mathfrak{F}' \\ \downarrow \phi & \circlearrowleft & \downarrow \psi \\ \mathcal{O}_{l_\infty}^{\oplus r} & \xrightarrow{\lambda} & \mathcal{O}_{l_\infty}^{\oplus r} \end{array} \right\}$$

We know also that $supp|l_\infty|$ is open dense in $\tilde{\mathbb{P}}$, and if we denote it by U_∞. Then for every open set $U \in \tilde{\mathbb{P}}$ has non trivial intersection with U_∞. In this way, for any sheaf $\mathcal{G}$, the restriction on an open set U can be also restricted to $U \cap U_\infty$. Now define the pre-sheaf $\mathcal{H}om((\mathfrak{F}, \phi), (\mathfrak{F}', \psi))$ as

(34) $$U \longrightarrow Hom((\mathfrak{F}_{|U}, \phi_{|U}), (\mathfrak{F}'_{|U}, \psi_{|U})),$$

i.e., to every open subset U we associate the diagram

(35) $$\begin{array}{ccc} \mathfrak{F}_{|U} & \xrightarrow{\alpha_{|U}} & \mathfrak{F}'_{|U} \\ \phi_{|U} \downarrow & \circlearrowleft & \downarrow \psi_{|U} \\ \mathcal{O}_{U \cap l_\infty}^{\oplus r} & \xrightarrow{\lambda} & \mathcal{O}_{U \cap l_\infty}^{\oplus r} \end{array}$$

one can easily show that this is a sheaf. The sheaf axioms for are inherited from the sheaf properties of $\mathfrak{F}$, $\mathfrak{F}'$ and the commutation of the square diagrams.

We want to show that this sheaf defined by (34) is flat over $P \times P$. Since the question is local on $P \times P$, one can consider an open affine $W = SpecA \subset P \times P$ and work with A-modules. Then we can show that $\mathcal{H}om((\mathfrak{F}, \phi), (\mathfrak{F}', \psi))$ is A-flat. Let

(36) $$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules, and let $\mathfrak{F}$ be an A-flat family of framed torsion-free sheaves on $\tilde{\mathbb{P}}$. One has the short exact sequence

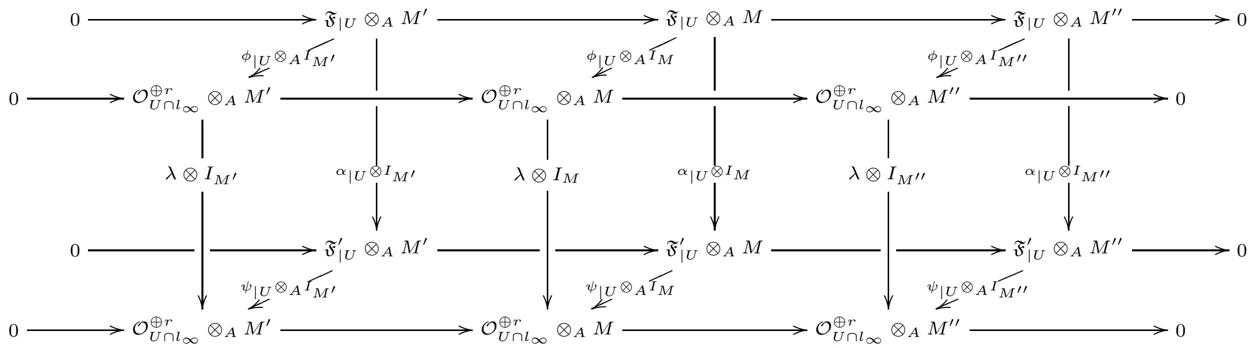
$$0 \longrightarrow \mathfrak{F} \otimes_A M' \longrightarrow \mathfrak{F} \otimes_A M \longrightarrow \mathfrak{F} \otimes_A M'' \longrightarrow 0.$$

The restriction of $\mathfrak{F}$ to any open U being also an A-flat module one has the sequence

$$0 \longrightarrow \mathfrak{F}_{|U} \otimes_A M' \longrightarrow \mathfrak{F}_{|U} \otimes_A M \longrightarrow \mathfrak{F}_{|U} \otimes_A M'' \longrightarrow 0$$

The same situation is true for an other family $\mathfrak{F}'$, and if we consider any morphism $\alpha_{|U} : \mathfrak{F}_{|U} \longrightarrow$

$\mathfrak{F}'_{|U}$ in $Hom((\mathfrak{F}_{|U}, \phi_{|U}), (\mathfrak{F}'_{|U}, \psi_{|U}))$, we get the following diagram:



where by I_M we mean the identity morphism on M and the same is true for $I_{M'}$, $I_{M''}$. Equivalently this means that the sequence

$$0 \longrightarrow Hom((\mathfrak{F}_{|U} \otimes_A M', \phi_{|U} \otimes_A I_{M'}), (\mathfrak{F}'_{|U} \otimes_A M', \psi_{|U} \otimes_A I_{M'})) \longrightarrow Hom((\mathfrak{F}_{|U} \otimes_A M, \phi_{|U} \otimes_A I_M), (\mathfrak{F}'_{|U} \otimes_A M, \psi_{|U} \otimes_A I_M))$$
$$\longrightarrow Hom((\mathfrak{F}_{|U} \otimes_A M'', \phi_{|U} \otimes_A I_{M''}), (\mathfrak{F}'_{|U} \otimes_A M'', \psi_{|U} \otimes_A I_{M''})) \longrightarrow 0$$

is exact. But one has also

$$Hom((\mathfrak{F}_{|U} \otimes_A M, \phi_{|U} \otimes_A I_M), (\mathfrak{F}'_{|U} \otimes_A M, \psi_{|U} \otimes_A I_M)) = Hom((\mathfrak{F}_{|U}, \phi_{|U}), (\mathfrak{F}'_{|U}, \psi_{|U})) \otimes_A M \quad ;$$

this can be seen by twisting the sequence (36). So we get the sequence

$$Hom((\mathfrak{F}_{|U}, \phi_{|U}), (\mathfrak{F}'_{|U}, \psi_{|U})) \otimes_A M' \longrightarrow Hom((\mathfrak{F}_{|U}, \phi_{|U}), (\mathfrak{F}'_{|U}, \psi_{|U})) \otimes_A M \longrightarrow Hom((\mathfrak{F}_{|U}, \phi_{|U}), (\mathfrak{F}'_{|U}, \psi_{|U})) \otimes_A M'' \longrightarrow 0$$

and we have only to prove that the first map, that we call Ξ, is injective. If we denote the first map in (36) by θ, then $\Xi : \Sigma_i \alpha_i \otimes_A m_i \longrightarrow \Sigma_i \alpha_i \otimes_A \theta(m_i)$. To do this we construct the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{F}_{|U} \otimes_A M' & \longrightarrow & \mathfrak{F}_{|U} \otimes_A M & \longrightarrow & \mathfrak{F}_{|U} \otimes_A M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_i \otimes_A I_{M'}} & & \downarrow{\scriptstyle \alpha_i \otimes_A I_M} & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{F}'_{|U} \otimes_A M' & \xrightarrow{\tilde{\theta}} & \mathfrak{F}'_{|U} \otimes_A M & \longrightarrow & \mathfrak{F}'_{|U} \otimes_A M'' & \longrightarrow & 0
\end{array}$$

By commutativity of the diagram one can see that if $\alpha_i \otimes_A I_M = 0$ then $\tilde{\theta} \circ \alpha_i \otimes_A I_{M'} = 0$ but $\tilde{\theta}$ is injective, hence $\alpha_i \otimes_A I_{M'} = 0$. Thus Ξ is injective. After gluing sections of the involved sheaves, one gets an exact sequence

$$0 \longrightarrow \mathcal{H}om((\mathfrak{F}, \phi), (\mathfrak{F}', \psi)) \otimes_A M' \longrightarrow \mathcal{H}om((\mathfrak{F}, \phi), (\mathfrak{F}', \psi)) \otimes_A M$$
$$\longrightarrow \mathcal{H}om((\mathfrak{F}, \phi), (\mathfrak{F}', \psi)) \otimes_A M'' \longrightarrow 0$$

Hence the sheaf $\mathcal{H}om((\mathfrak{F}, \phi), (\mathfrak{F}', \psi))$ is flat over $P \times P$.

Let us remark that since the group G acts freely on the non-singular space P, with trivial stabilizer for any point $\rho \in P$, then the quotient is smooth if the graph of the group action is closed, i.e., the image $\Gamma := Im\gamma$ of the morphism

$$\gamma : G \times P \longrightarrow P \times P$$

is closed (see [18]). In our case the pair (ρ, σ) is in the graph Γ if and only if $dimHom^{\Phi}(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = 1$, in other words

$$\Gamma = \left\{(\rho, \sigma) \in P \times P / h^0(pr_{12}^{-1}(\rho, \sigma), \mathcal{H}om^{\Phi}(pr_{13}^*\mathfrak{F}(\rho), pr_{23}^*\mathfrak{F}(\sigma))) > 0\right\} \tag{37}$$

Since the sheaf $\mathcal{H}om^{\Phi}(pr_{13}^*\mathfrak{F}, pr_{23}^*\mathfrak{F})$ is flat on $P \times P$, then from semicontinuity theorem it follows that Γ is closed analytic space. Hence the quotient P/G is smooth.

5 Moduli functor and universal monads

In the following we want to prove, using the monadic description, that the moduli space $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is fine. This will be done in a few steps by constructing families of such framed torsion free sheaves, parametrized by some scheme S, and building a monad $\mathcal{M}$ on the product $\tilde{\mathbb{P}} \times S$ with cohomology the family we started with. But before starting our program let us give some definitions and set some notations for our moduli functor problem.

Let $\mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k} : \mathfrak{S}ch \longrightarrow \mathfrak{S}et$ be the contravariant functor from the category of noetherian reduced scheme of finite type to the category of sets, which is defined as follows; to every such scheme S we associate

$$\mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(S) = \left\{ [\mathcal{F}] \Big/ \begin{array}{c} \mathcal{F} \text{ is a coherent sheaf on } S \times \tilde{\mathbb{P}} \quad \text{flat on } S \text{ and such that} \\ \mathcal{F} \otimes k(s) \cong \mathcal{E} \text{ is a framed torsion-free sheaf on } \tilde{\mathbb{P}} \text{ with} \\ ch(\mathcal{E}) = r + \Sigma_{i=1}^{n} a_i E_i - (k - \frac{|\vec{a}|^2}{2})\omega \end{array} \right\}$$

where $[\mathcal{F}]$ stands for the class of the sheaf $\mathcal{F}$.

Definition 5.1. *A scheme $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is called a coarse moduli space if the following conditions are satisfied:*

- *There is a natural transformation*

$$\Phi : \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(\bullet) \longrightarrow Hom(\bullet, \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k})$$

which is a bijection for every reduced point s.

- *For every scheme $\mathcal{R}$ and every natural transformation*

$$\Psi : \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(\bullet) \longrightarrow Hom(\bullet, \mathcal{R})$$

there is a unique morphism of schemes $f : \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k} \longrightarrow \mathcal{R}$ such that the diagram

(38)
$$\begin{array}{ccc} \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(\bullet) & \longrightarrow & Hom(\bullet, \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}) \\ & \searrow & \downarrow f_* \\ & & Hom(\bullet, \mathcal{R}) \end{array}$$

commutes.

Definition 5.2. *A scheme $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is called a fine moduli space for the functor $\mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(\bullet)$ if the natural transformation Φ above is an isomorphism.*

Given a noetherian reduced scheme of finite type S, let $\mathcal{F}$ be a coherent sheaf on $\tilde{\mathbb{P}} \times S$ which is flat on S and for every point $s \in S$, $\mathcal{F}_s = \mathcal{F} \otimes k_s$ is a framed torsion-free sheaf $\mathcal{E}$ on $\tilde{\mathbb{P}}$ with Chern character $ch(\mathcal{E}) = r + \Sigma_{i=1}^{n} a_i E_i - (k - \frac{|\vec{a}|^2}{2})\omega$. We want to construct a monad $\mathcal{M}$ on $\tilde{\mathbb{P}} \times S$ which is associated to the sheaf $\mathcal{F}$, i.e., the cohomology $Coh(\mathcal{M}) = \mathcal{F}$. We have

(39)
$$\begin{array}{ccccc} & & \mathcal{F} & & \\ & & \downarrow & & \\ & & \tilde{\mathbb{P}} \times S & & \\ & \swarrow^{\tilde{p}} & & \searrow^{\tilde{q}} & \\ \tilde{\mathbb{P}} & & & & S \end{array}$$

For every two sheaves, $\mathcal{F}$ on $\tilde{\mathbb{P}}$ and $\mathcal{G}$ on S, we use the following notation $\mathcal{F} \boxtimes \mathcal{G} := \tilde{p}^* \mathcal{F} \otimes \tilde{q}^* \mathcal{G}$. We consider also for a morphism $f : X \longrightarrow Y$ of noetherian reduced schemes of finite type, and define the i-th right derived functors of the functor $\mathcal{H}om_f(\mathcal{G}, \bullet) := f_* \circ \mathcal{H}om(\mathcal{G}, \bullet)$, for more details see for example [15]. Before proceeding we want to introduce the following useful result, which is a kind of "relative local-to-global" spectral sequence:

Proposition 5.1. *Let $f : X \longrightarrow Y$ be a morphism of noetherian reduced schemes of finite type such that there is a covering $\mathcal{Y}$ of Y for which $f^{-1}(\mathcal{Y})$ is a covering of X. For a fixed coherent sheaf $\mathcal{G}$ flat on Y we consider the functor $Hom(\mathcal{G}, \bullet)$. Then for any sheaf $\mathcal{J}$ on X flat on Y, there exists a spectral sequence $E_r^{p,q}$ of E_2-term $E_2^{p,q} = H^p(Y, \mathcal{E}xt^q_f(\mathcal{G}, \mathcal{J}))$ which converges to $E_\infty^{p+q} = Ext_X^{p+q}(\mathcal{E}, \mathcal{J})$*

Proof. Let $\mathcal{J} \longrightarrow I^\bullet$ be an injective resolution of the sheaf $\mathcal{J}$, and consider a Cartan-Eilenberg resolution given by the Čech complex associated to the complex of sheaves $\mathcal{H}om_f(\mathcal{G}, I^\bullet)$ for a suitable open cover $\mathcal{Y}$ of Y. This defines a double complex $C^\bullet(\mathcal{Y}, \mathcal{H}om_f(\mathcal{G}, I^\bullet))$ with differentials

$$\begin{cases} \delta_1 = d \text{ the differential associated to the injective resolution of } \mathcal{J} \\ \delta_2 = \delta \text{ the differential associated to the Čech complex.} \end{cases}$$

There are two spectral sequences associated with this double complex [16], with E_2-terms:

$$'E_2^{p,q} = H_d^p[H_\delta^q(C^\bullet(\mathcal{Y}, \mathcal{H}om_f(\mathcal{G}, I^\bullet)))]$$

$$''E_2^{p,q} = H_\delta^p[H_d^q(C^\bullet(\mathcal{Y}, \mathcal{H}om_f(\mathcal{G}, I^\bullet)))]$$

Lemma 5.1. *For an injective object I, the sheaf $\mathcal{H}om(\mathcal{G}, I)$ is flasque.*

The proof is given in ([17], II. 7). The E_1-term of the first spectral sequence is given by:

$$\begin{aligned} 'E_1^{p,q} &= H_\delta^q(C^\bullet(\mathcal{Y}, \mathcal{H}om_f(\mathcal{G}, I^p))) \\ &= H_\delta^q(C^\bullet(\mathcal{Y}, f_* \circ \mathcal{H}om(\mathcal{G}, I^p))) \\ &= H_{\delta'}^q(C^\bullet(f^{-1}(\mathcal{Y}), \mathcal{H}om(\mathcal{G}, I^p))) \end{aligned}$$

$$\text{Then} \quad 'E_1^{p,q} = \begin{cases} Hom(\mathcal{G}, I^p) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

since the sheaf $\mathcal{H}om(\mathcal{G}, I^p)$ is flasque for every term I^p of the injective resolution. We also used the condition on the open covering given in the statement. This spectral sequence degenerates at the second step and converges to $'E^{p+q} = Ext_X^{p+q}(\mathcal{G}, \mathcal{J})$.

The second spectral sequence has E_2-term:

$$\begin{aligned} ''E_2^{p,q} &= H_\delta^p(C^\bullet(\mathcal{Y}, \mathcal{H}_d^q(\mathcal{H}om_f(\mathcal{G}, I^\bullet)))) \\ &= H_\delta^p(C^\bullet(\mathcal{Y}, \mathcal{E}xt_f^q(\mathcal{G}, \mathcal{J}))) \end{aligned}$$

$$\text{Then} \quad ''E_2^{p,q} = H^p(Y, \mathcal{E}xt_f^q(\mathcal{G}, \mathcal{J})).$$

□

This proposition will be used in the next step to prove the existence of a display of a monad on $\tilde{\mathbb{P}} \times S$ associated to every family of framed torsion-free sheaves on $\tilde{\mathbb{P}}$. Consider the following extensions:

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{40}$$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow \mathcal{V} \longrightarrow 0 \tag{41}$$

where $\mathcal{U} = \oplus_{i=0}^n \mathcal{O}(-1, E_i) \boxtimes \mathcal{U}_i$, with

$$\begin{cases} \mathcal{U}_0 = \mathcal{R}^1 \tilde{q}_*(\mathcal{F} \otimes \tilde{p}^* \mathcal{O}(-2, \vec{1})) \\ \mathcal{U}_i = \mathcal{R}^1 \tilde{q}_*(\mathcal{F} \otimes \tilde{p}^* \mathcal{O}(-2, \vec{1} - E_i)) \end{cases}$$

and $\mathcal{V} = \oplus_{i=0}^n \mathcal{O}(1, -E_i) \boxtimes \mathcal{V}_i$ with

$$\begin{cases} \mathcal{V}_0 = \mathcal{R}^1 \tilde{q}_*(\mathcal{F} \otimes \tilde{p}^* \mathcal{O}(-2, 0)) \\ \mathcal{V}_i = \mathcal{R}^1 \tilde{q}_*(\mathcal{F} \otimes \tilde{p}^* \mathcal{O}(-2, \vec{1})) \end{cases}$$

The sheaves $\mathcal{U}$ and $\mathcal{V}$ are locally free on $\tilde{\mathbb{P}} \times S$; To see this let us consider, for the moment, the sheaves $\mathcal{U}_0$ and $\mathcal{U}_i$. By Grauert's theorem ([9], III. 12. 9), for every point $s \in S$, there are maps :

$$\begin{aligned}
&\tilde{q}_*(\mathcal{F} \otimes \tilde{p}^*\mathcal{O}(-2,\vec{1}))) \otimes k(s) \longrightarrow H^0(\tilde{\mathbb{P}},\mathcal{E}(-2,\vec{1})) = 0\\
&\tilde{q}_*(\mathcal{F} \otimes \tilde{p}^*\mathcal{O}(-2,\vec{1}-E_i))) \otimes k(s) \longrightarrow H^0(\tilde{\mathbb{P}},\mathcal{E}(-2,\vec{1}-E_i)) = 0\\
&\mathcal{R}^2\tilde{q}_*(\mathcal{F} \otimes \tilde{p}^*\mathcal{O}(-2,\vec{1}))) \otimes k(s) \longrightarrow H^2(\tilde{\mathbb{P}},\mathcal{E}(-2,\vec{1})) = 0\\
&\mathcal{R}^2\tilde{q}_*(\mathcal{F} \otimes \tilde{p}^*\mathcal{O}(-2,\vec{1}-E_i))) \otimes k(s) \longrightarrow H^2(\tilde{\mathbb{P}},\mathcal{E}(-2,\vec{1}-E_i)) = 0
\end{aligned} \tag{42}$$

Since all of these maps go to zero, the sheaves in the left-hand side have rank zero at all $s \in S$, thus they vanish identically. By the Riemann-Roch theorem, the dimensions of $H^1(\tilde{\mathbb{P}},\mathcal{E}(-2,\vec{1}))$ and $H^1(\tilde{\mathbb{P}},\mathcal{E}(-2,\vec{1}-E_i))$ are constant, hence $\mathcal{U}_0$ and $\mathcal{U}_i$ are locally free. In the same way one can easily see that $\mathcal{V}_0$ and $\mathcal{V}_i$ are locally free.

The two extensions, (40) and (41), fit into the display of a monad on $\tilde{\mathbb{P}} \times S$ if and only if[7] $Ext^2(\mathcal{V},\mathcal{U}) = 0$. To show this vanishing property we use the "relative local to global" spectral sequence that we constructed above; we have

$$E_2^{p,q} = H^p(S,\mathcal{E}xt^q_{\tilde{q}}(\mathcal{V},\mathcal{U})) \Longrightarrow Ext^{p+q}_{\tilde{\mathbb{P}}\times S}(\mathcal{V},\mathcal{U})$$

The spectral sequence terms which contributes to $Ext^2(\mathcal{V},\mathcal{U})$ are

$$H^0(S,\mathcal{E}xt^2_{\tilde{q}}(\mathcal{V},\mathcal{U})), \quad H^1(S,\mathcal{E}xt^1_{\tilde{q}}(\mathcal{V},\mathcal{U})) \quad \text{and} \quad H^2(S,\mathcal{H}om_{\tilde{q}}(\mathcal{V},\mathcal{U}))$$

so we have to prove that all these terms are zero.

$\underline{H^2(S,\mathcal{H}om_{\tilde{q}}(\mathcal{V},\mathcal{U}))}$:

$$\mathcal{H}om_{\tilde{q}}(\mathcal{V},\mathcal{U}) \otimes k(s) = \oplus_{i,j=0}^n \tilde{q}_*[\mathcal{H}om(\tilde{q}^*\mathcal{V}_i,\tilde{q}^*\mathcal{U}_j \otimes \tilde{p}^*\mathcal{O}(-2,E_i+E_j))] \otimes k(s),$$

Again by the Grauert's theorem there is a map

$$\oplus_{i,j=0}^n \tilde{q}_*[\mathcal{H}om(\tilde{q}^*\mathcal{V}_i,\tilde{q}^*\mathcal{U}_j \otimes \tilde{p}^*\mathcal{O}(-2,E_i+E_j))] \otimes k(s) \longrightarrow \oplus_{i,j=0}^n Hom(\tilde{q}^*\mathcal{V}_i(s),\tilde{q}^*\mathcal{U}_j(s) \otimes \mathcal{O}(-2,E_i+E_j))$$

since $\mathcal{V}_i$ are sheaves on S, then the pull-back $\tilde{q}^*\mathcal{V}_i(s)$, of the stalk $\mathcal{V}_i(s)$ at a point s, is just the pull-back associated to the map $\tilde{\mathbb{P}} \longrightarrow s = speck$, hence $\tilde{q}^*\mathcal{V}_i(s)$ is constant on $\tilde{\mathbb{P}}$. This is also true for the pull-backs $\tilde{q}^*\mathcal{U}_j(s)$ of $\mathcal{U}_j(s)$. Thus

$$\oplus_{i,j=0}^n Hom(\tilde{q}^*\mathcal{V}_i(s),\tilde{q}^*\mathcal{U}_j(s) \otimes \mathcal{O}(-2,E_i+E_j)) = \oplus_{i,j=0}^n [\tilde{q}^*\mathcal{V}_j(s)]^* \otimes \tilde{q}^*\mathcal{U}_j(s) \otimes \underbrace{H^0(\tilde{\mathbb{P}},\mathcal{O}(-2,E_i+E_j))}_{0}$$

which implies the vanishing of $\mathcal{H}om_{\tilde{q}}(\mathcal{V},\mathcal{U})$ identically. Hence $H^2(S,\mathcal{H}om_{\tilde{q}}(\mathcal{V},\mathcal{U})) = 0$.

$\underline{H^1(S,\mathcal{E}xt^1_{\tilde{q}}(\mathcal{V},\mathcal{U}))}$:

$$\mathcal{E}xt^1_{\tilde{q}}(\mathcal{V},\mathcal{U}) = \oplus_{i,j=0}^n \mathcal{E}xt^1_{\tilde{q}}(\tilde{q}^*\mathcal{V}_i,\tilde{q}^*\mathcal{U}_j \otimes \tilde{p}^*\mathcal{O}(-2,E_i+E_j)) \otimes k(s)$$

We have a map

$$\oplus_{i,j=0}^n \mathcal{E}xt^1_{\tilde{q}}(\tilde{q}^*\mathcal{V}_i,\tilde{q}^*\mathcal{U}_j \otimes \tilde{p}^*\mathcal{O}(-2,E_i+E_j)) \otimes k(s) \longrightarrow \oplus_{i,j=0}^n Ext^1(\tilde{q}^*\mathcal{V}_i(s),\tilde{q}^*\mathcal{U}_j(s) \otimes \mathcal{O}(-2,E_i+E_j))$$

[7] see [7] section 2

but

$$\oplus_{i,j=0}^{n} Ext^1(\tilde{q}^*\mathcal{V}_i(s), \tilde{q}^*\mathcal{U}_j(s)\otimes\mathcal{O}(-2, E_i+E_j)) = \oplus_{i,j=0}^{n}[\tilde{q}^*\mathcal{V}_i(s)]^*\otimes\tilde{q}^*\mathcal{U}_j(s)\otimes H^1(\tilde{\mathbb{P}}, \mathcal{O}(-2, E_i+E_j))$$

and

$$\begin{cases} \chi(\mathcal{O}(-2, E_i+E_j)) = 0 \\ H^2(\tilde{\mathbb{P}}, \mathcal{O}(-2, E_i+E_j)) = 0 \\ H^2(\tilde{\mathbb{P}}, \mathcal{O}(-2, E_i+E_j)) = 0 \end{cases}$$

Thus $H^1(\tilde{\mathbb{P}}, \mathcal{O}(-2, E_i+E_j)) = 0$, implying $\mathcal{E}xt^1_{\tilde{q}}(\mathcal{V},\mathcal{U})\otimes k(s) = 0$. Hence $H^1(S, \mathcal{E}xt^1_{\tilde{q}}(\mathcal{V},\mathcal{U})) = 0$.

$\underline{H^0(S, \mathcal{E}xt^2_{\tilde{q}}(\mathcal{V},\mathcal{U}))}$:

By the relative Serre duality [15] we have

$$\mathcal{E}xt^2_{\tilde{q}}(\mathcal{V},\mathcal{U}) = \oplus_{i,j=0}^{n}\{\tilde{q}_*[\mathcal{H}om(\tilde{q}^*\mathcal{U}_j, \tilde{q}^*\mathcal{V}_i\otimes\tilde{p}^*\mathcal{O}(-1, \vec{1}-E_i-E_j))]\}^*\otimes k(s)$$

There is a map

$$\oplus_{i,j=0}^{n}\{\tilde{q}_*[\mathcal{H}om(\tilde{q}^*\mathcal{U}_j, \tilde{q}^*\mathcal{V}_i\otimes\tilde{p}^*\mathcal{O}(-1, \vec{1}-E_i-E_j))]\}^*\otimes k(s) \longrightarrow \oplus_{i,j=0}^{n} Hom(\tilde{q}^*\mathcal{U}_j(s), \tilde{q}^*\mathcal{V}_i(s)\otimes\mathcal{O}(-1, \vec{1}-E_i-E_j))^*$$

but

$$\oplus_{i,j=0}^{n} Hom(\tilde{q}^*\mathcal{U}_j(s), \tilde{q}^*\mathcal{V}_i(s)\otimes\mathcal{O}(-1, \vec{1}-E_i-E_j))^* = \oplus_{i,j=0}^{n}[\tilde{q}^*\mathcal{U}_j(s)]^{ast}\otimes\tilde{q}^*\mathcal{V}_i(s)\otimes\underbrace{H^0(\tilde{\mathbb{P}}, \mathcal{O}(-1, \vec{1}-E_i-E_j))^*}_{0}$$

which implies that $\mathcal{E}xt^2_{\tilde{q}}(\mathcal{V},\mathcal{U}) = 0$ and hence $H^0(S, \mathcal{E}xt^2_{\tilde{q}}(\mathcal{V},\mathcal{U})) = 0$.

By this, the vanishing property $Ext^2(\mathcal{V},\mathcal{U}) = 0$ is proved, implying the existence of sequences

(43) $$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{W} \longrightarrow \mathcal{Q} \longrightarrow 0$$

(44) $$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{W} \longrightarrow \mathcal{V} \longrightarrow 0$$

such that the diagram

(45)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{U} & = & \mathcal{U} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{K} & \to & \mathcal{W} & \to & \mathcal{V} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{F} & \to & \mathcal{Q} & \to & \mathcal{V} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

commutes, and we have a monad

(46) $$\mathcal{M}: \quad \oplus_{i=0}^{n}\mathcal{O}_{\tilde{\mathbb{P}}}(-1, E_i)\boxtimes\mathcal{U}_i \longrightarrow \mathcal{W} \longrightarrow \oplus_{i=0}^{n}\mathcal{O}_{\tilde{\mathbb{P}}}(1, -E_i)\boxtimes\mathcal{V}_i$$

associated to the family $\mathcal{F}$ on $\tilde{\mathbb{P}}$. One can show that the restriction to the fibers of $\tilde{q}$ gives a monad isomorphic to the one in **Proposition 2.5** and, using the display, one can show that the second term $\mathcal{W}$ of the monad $\mathcal{M}$ is trivial along the fibers of $\tilde{q}$.

Another useful monad we now introduce is the universal monad on $\tilde{\mathbb{P}}\times P$;

(47)
$$\mathbb{M}: \quad \oplus_{i=0}^{n}\mathcal{O}_{\tilde{\mathbb{P}}}(-1, E_i)\boxtimes K_i\otimes\mathcal{O}_P \longrightarrow \mathcal{O}_{\tilde{\mathbb{P}}}\boxtimes W\otimes\mathcal{O}_P \longrightarrow \oplus_{i=0}^{n}\mathcal{O}_{\tilde{\mathbb{P}}}(1, -E_i)\boxtimes L_i\otimes\mathcal{O}_P$$

which has as cohomology a family that we denote by $\mathfrak{F}$, and where the vector spaces K_i and L_i are given by **Proposition 2.5**.

Using the monads above we now proceed to prove that $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is a coarse moduli. We start by constructing the following natural transformation

$$\Phi : \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(\bullet) \longrightarrow Hom(\bullet, \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k});$$

this goes as follows: for any classifying scheme S, $\xi \in \mathfrak{M}(S)$, and a family $\mathcal{F}$ of torsion-free sheaves on $\tilde{\mathbb{P}}$, with Chern character $r + \Sigma_{i=1}^n a_i E_i - (k - \frac{|\vec{a}|^2}{2})\omega$, classified by S, one has a monad given as in (46) which is canonically associated to $\mathcal{F}$. If we consider an open covering $\{S_j\}_{j\in J}$, then on every open affine S_j the restriction $\mathcal{M}_{|S_j}$ is isomorphic to a monad of the form:

$$\mathbb{M}(\alpha_j, \beta_j) : \qquad \oplus_{i=0}^n \mathcal{O}_{\tilde{\mathbb{P}}}(-1, E_i) \boxtimes K_i \otimes \mathcal{O}_{S_j} \xrightarrow{\alpha_j} \mathcal{O}_{\tilde{\mathbb{P}}} \boxtimes W \otimes \mathcal{O}_{S_j} \xrightarrow{\beta_j} \oplus_{i=0}^n \mathcal{O}_{\tilde{\mathbb{P}}}(1, -E_i) \boxtimes L_i \otimes \mathcal{O}_{S_j}$$

where

$$\alpha_j : S_j \longrightarrow \mathbb{H}, \qquad \beta_j : S_j \longrightarrow \mathbb{F}.$$

The spaces $\mathbb{H}$, $\mathbb{F}$ are defined by

$$\begin{cases} \mathbb{H} = \oplus_{i=0}^n Hom(V_i, Hom(K_i, W)) \\ \mathbb{F} = \oplus_{i=0}^n Hom(V_i, Hom(W, L_i)) \end{cases}$$

and V_i is such that $V_i^* = H^0(\tilde{\mathbb{P}}, \mathcal{O}(1, -E_i))$. Then from the monad condition, we have a map $f_j = (\alpha_j, \beta_j) : S_j \longrightarrow P$ and by construction these morphisms satisfy

$$f_i(s) \sim_G f_j(s)$$

for s in the intersection $S_i \cap S_j$, where G is the group defined in page 14. The maps f_j glue to form a global morphism

$$f : S \longrightarrow \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}.$$

This defines the natural transformation by:

(48)
$$\begin{array}{lccc} \Phi : & \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(\bullet) & \longrightarrow & Hom(\bullet, \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}) \\ & \xi & \longrightarrow & \Phi(\xi) := f \end{array}$$

f depends only on the class $\xi = [\mathcal{F}]$. Taking a reduced point $s \in S$, and using the resulting monad on $\tilde{\mathbb{P}}$ it is easy to see that $\Phi : \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(spec(k(s))) \longrightarrow Hom(spec(k(s)), \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k})$ is a bijection. Now let $\mathcal{R}$ another parametrizing scheme such that there is a natural transformation

$$\Psi : \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(\bullet) \longrightarrow Hom(\bullet, \mathcal{R}).$$

If η is a universal family on $\tilde{\mathbb{P}} \times P$ parametrized by P such that $\Phi(\eta) = \pi : P \longrightarrow \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$, for the natural transformation Ψ we have $\Psi(\eta) : P \longrightarrow R$.

Proposition 5.2. *$\Psi(\eta)$ is constant along the fibers of the projection $\pi : P \longrightarrow \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$*

Proof. Let $p = spec(k(p)) \in P$ and let ρ_1, $\rho_2 \in Hom(p, P)$ such that $\pi(\rho_1) = \pi(\rho_2)$; we consider the pull-back

(49)
$$\begin{array}{ccc} \rho_i^*\eta & \longleftarrow & \eta \\ \downarrow & & \downarrow \\ \tilde{\mathbb{P}} \times p & \xrightarrow{(Id_{\tilde{\mathbb{P}}} \times \rho_i)} & \tilde{\mathbb{P}} \times P \end{array}$$

Then $\Phi(\rho_1^*\eta) = \Phi(\eta)(\rho_1)$, and by definition we have $\Phi(\eta)(\rho_1) = \pi(\rho_1)$. Also by assumption $\pi(\rho_1) = \pi(\rho_2) = \Phi(\eta)(\rho_2) = \Phi(\rho_2^*\eta)$, and since the natural transformation Φ is a bijection for every reduced point, it follows that

$$\rho_1^*\eta = \rho_2^*\eta$$

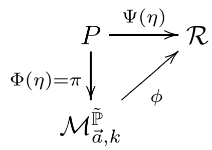
On the other hand we have

$$\Psi(\eta)(\rho_1) = \Psi(\rho_1^*\eta) = \Psi(\rho_2^*\eta) = \Psi(\eta)(\rho_2).$$

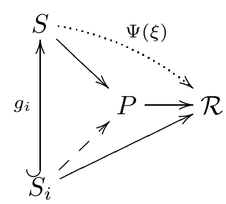
Thus $\Psi(\eta)$ is constant along the fibers of $\pi : P \longrightarrow \mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}}$ ☐

The projection $\pi : P \longrightarrow \mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}}$, locally, has sections, so one can construct local mappings $\bar{\phi} : \mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}} \longrightarrow \mathcal{R}$, but since $\Psi(\eta)$ is constant along the fibers of π, then the map $\bar{\phi}$ can be lifted to a global map ϕ such that the following diagram commutes:

(50)



Now for any parametrizing scheme S and a family ξ on $\tilde{\mathbb{P}} \times S$, one has $\Psi(\xi) : S \longrightarrow \mathcal{R}$. Let $\{S_i\}_{i\in I}$ be an open cover of S. The diagram



commutes, and we have $\xi_{|S_i} = g_i^*(\eta)$. From the commutativity of the diagram, we also have

$$\begin{aligned}\Psi(\xi)_{S_i} &= \Psi(\xi_{|S_i}) \\ &= \Psi(g_i^*(\eta)) \\ &= g_i^*\Psi(\eta)\end{aligned}$$

On the other hand $\Psi(\eta) = \phi \circ \Phi(\eta)$, hence

$$\begin{aligned}\Psi(\xi)_{S_i} &= g_i^*(\phi \circ \Phi(\eta)) \\ &= \phi \circ \Phi(g_i^*(\eta)) \\ &= \phi \circ \Phi(\xi_{|S_i}) \\ \Psi(\xi)_{S_i} &= [\phi \circ \Phi(\xi)]_{|S_i}\end{aligned}$$

These maps glue together to form a global map

$$\Psi(\xi) = \phi \circ \Phi(\xi)$$

on S. Since $\mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}}$ is reduced, the map ϕ is uniquely determined. By this we showed the following:

Theorem 5.1. *The scheme $\mathcal{M}_{\vec{a},k}^{\tilde{\mathbb{P}}}$ is a coarse moduli space.*

The final step is to show that $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is fine. To do this we shall descend the universal monadic description on P to a well behaved monadic description on $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$. This is due to the fact that the space $\tilde{\mathbb{P}} \times P$ is a G-space since there is a natural action

$$G \times \tilde{\mathbb{P}} \times P \longrightarrow \tilde{\mathbb{P}} \times P$$

$$(g, (x, \mathcal{C})) \longrightarrow (x, g \cdot \mathcal{C})$$

This induces a G-action on the universal monad $\mathbb{M}$, in (47), which descends to an action on its cohomology $\mathfrak{F}$, but since the action is free and the isotropy subgroup is trivial at all points, we have a well defined family $\mathfrak{F}/G \longrightarrow \tilde{\mathbb{P}} \times P/G$. We put $\mathfrak{U} := \mathfrak{F}/G$ which is a canonical family

$$\mathfrak{U} \longrightarrow \tilde{\mathbb{P}} \times \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$$

parametrized by $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$.

Proposition 5.3. *For any noetherian reduced scheme S of finite type, the mapping*

$$Hom(S, \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}) \longrightarrow \mathfrak{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}(S)$$

$$\phi \longrightarrow \phi^*[\mathfrak{U}] = [(Id_{\tilde{\mathbb{P}}} \times \phi)^* \mathfrak{U}]$$

is bijective.

Proof. Injectivity:

Let $\phi_1, \phi_2 : S \longrightarrow \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ such that

$$(Id_{\tilde{\mathbb{P}}} \times \phi_1)^* \mathfrak{U} \cong (Id_{\tilde{\mathbb{P}}} \times \phi_2)^* \mathfrak{U}$$

then for every point $s \in S$, one has $\mathfrak{U}(\phi_1(s)) = \mathfrak{U}(\phi_2(s))$. Since the bundle $\mathfrak{U}(\phi_i(s))$ is the one given by the ADHM data associated to the point $\phi_i(s) \in \mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$, then $\phi_1(s) = \phi_2(s)$ for every point $s \in S$, thus $\phi_1 = \phi_2$.

Surjectivity:

For a family $\mathcal{F}$ parametrized by S one can associate the morphism $\phi = \Phi(\mathcal{F})$ given by the natural transformation (48), then by construction it is the pull-back of a family $\mathfrak{U}$ parametrized by $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$. □

This finishes the proof of the following

Theorem 5.2. *The scheme $\mathcal{M}^{\tilde{\mathbb{P}}}_{\vec{a},k}$ is a fine moduli space.*

References

[1] N.P. Buchdahl, "Monads and bundles on rational surfaces". Rocky Mountains J. Math, V 34, no. 2, 2004.

[2] N.P. Buchdahl, "Blowups and gauge fields", Pacific J. Math. 196 (2000), No. 1, 69–111

[3] H. Nakajima, "Lectures on Hilbert schemes of points on a surface", University Lecture Series, 18. Americam Mathematical Society, Providence, RI, 1999

[4] H. Nakajima, K. Yoshioka, "Instanton counting on blow-up I. 4-dimensional pure gauge theory." Invent. Math. 162 (2005), no. 2, 313–355.

[5] V. Baranovsky, "On punctural Quot schemes on Algebraic surfaces", Preprint alg-geom/9703038

[6] C. Okonek, M. Schneider, H. Spindler, "Vector bundles on complex projective spaces", Progress in mathematics 3, Birkhauser, Boston, 1980.

[7] A. King, "Instantons and holomorphic bundles on the blown-up plane", Worcester college, Oxford,Ph. D Thesis 1989.

[8] P. Griffiths and J. Harris, "Principles of algebraic geometry", New York, Wiley-Interscience, 1979

[9] R. Hartshorne, "Algebraic geometry", Springer-Verlag, GTM, 52.

[10] R. Hartshorne, "Residues and duality", LNM 20, Springer, 1966.

[11] D. Huybrechts, M. Lehn, "The geometry of moduli spaces".

[12] D. Huybrechts, M. Lehn, "Framed modules and their moduli", Internat. J. Math. 6 (1995), no. 2, 297–324.

[13] D. Huybrechts, M. Lehn, "Stable pairs on curves and surfaces", J. Algebraic Geom. 4 (1995), no. 1, 67–104.

[14] U. Bruzzo, D. Markushevich, "Moduli of framed sheaves on rational surfaces", SISSA preprint 6/2009/fm.

[15] S. Kleiman, "Relative duality for quasicoherent sheaves", Compositio Math. 41 (1980), no. 1, 39-60.

[16] A. Grothendieck, "Elements de géometrie algébrique III, 1", Publ. Math. IHES, no 11 (1961), 5-165.

[17] R. Godement, "Topologie algébrique et txhéorie des faisceaux", Hermann, Paris (1958).

[18] C. S. Seshadri, "Some results on the quotient space by an algebraic group of automorphisms" Math. Ann. 149 (1963), 286–301.