Dissipative solutions to Hamiltonian systems

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Abstract

We extend the notion of dissipative particle solutions [5] to the case of Hamiltonian flow in the space of probability measures $\mu \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d)$ in the sense of [4]. The Hamiltonian is of the form

$$H(\mu) = \int V(q,p)\mu(dqdp) + \frac{1}{2}\int\int W(q,p,q',p')\mu(dqdp)\mu(dq'dp'),$$

with at most quadratic growth, so that a conservative flow

$$\dot{q} = \nabla_p V + \int \nabla_p W \mu, \quad \dot{p} = -\nabla_q V - \int \nabla_q W \mu$$

is uniquely defined.

The dissipative solution is defined by requiring that the equation of p is replaced by

$$p(t) = \mathbb{P}_t\bigg(p(0) + \int_0^t \bigg[-\nabla_q V - \int \nabla_q W \mu\bigg] ds\bigg).$$

where $\mathbb{P}_t$ is the projection on the space of functions corresponding to the restriction map

$$\mathbb{T}_t \gamma = \gamma \mathbf{I}_{s>t}.$$

Equivalently the particle merge preserving the average momentum p.

We obtain several results on the structure of dissipative solutions; among them, regularity, dissipation of energy, approximations with finite particles solutions, density of conservative solutions. The proofs require additional technical difficulties, not present in the analysis of [5] where $H(q,p) = p^2/2$.

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Contents

1 Introduction

We consider the Hamiltonian function

$$H(\mu) = \int V(q,p)\mu(dqdp) + \frac{1}{2}\int\int W(q,p,q',p')\mu(dqdp)\mu(dq'dp'),$$

where V, W are smooth semiconvex functions with quadratic growth, W symmetric, and $\mu \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ is a probability measure with finite quadratic moments. It is known [4] that for every $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ there is a unique solution $t \mapsto \mu(t)$ to the Hamiltonian flow

$$\partial_t \mu(t) + \mathrm{div}_{p,q}\big(\mathrm{J}\nabla H(\mu(t))\mu(t)\big) = 0, \quad \mu(0) = \mu_0.$$

where J is the symplectic matrix

$$\mathrm{J} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}, \qquad \mathbb{I} \ \ d \times d\text{-identity matrix},$$

and

$$\nabla H(\mu) = \nabla H(q,p;\mu) = \nabla V(q,p) + \int \nabla W(q,p,q',p')\mu(dq'dp').$$

At the level of trajectories (Q,P) in the phase space, these are solutions to

$$\begin{aligned} Q(t,q,p;\mu_0) = q &+ \int_0^t \nabla_v V(Q(s,q,p;\mu_0),P(s,q,p;\mu_0))ds \\ &+ \int_0^t \int \nabla_v W\big(Q(s,q,p;\mu_0),P(s,q,p;\mu_0),Q(s,q',p';\mu_0),P(s,q',p';\mu_0)\big)\mu_0(dq'dp')ds, \end{aligned} \tag{1a}$$

$$\begin{aligned} P(t,q,p;\mu_0) = p &- \int_0^t \nabla_q V(Q(s,q,p;\mu_0),P(s,q,p;\mu_0))ds \\ &- \int_0^t \int \nabla_q W\big(Q(s,q,p;\mu_0),P(s,q,p;\mu_0),Q(s,q',p';\mu_0),P(s,q',p';\mu_0)\big)\mu_0(dq'dp')ds. \end{aligned} \tag{1b}$$

In this paper we study the existence and stability of solutions when the particles are "sticky", i.e. they are allowed to merge (thus dissipating energy but preserving momentum) if they occupy the same position q: this leads to the notion of dissipative solution.

The prototype example is the sticky particle system, where the Hamiltonian is simply

$$H(\mu) = \int \frac{p^2}{2}\mu(dqdp). \tag{2}$$

In this case the conservative solution is made of straight lines

$$\dot q(t) = p(t) + q(0), \quad p(t) = p(0).$$

The condition for sticky particle solution is that the momentum p is conserved when particles merge, i.e. in the case of finitely many particles with mass m_i colliding at time $\bar t$

$$q_i(\bar t) = q,\ i = i_1,\dots,i_n \quad \Rightarrow \quad \sum_{j=i_1}^{i_n} m_j p_i(\bar t -) = p_i(\bar t +) \sum_{j=i_1}^{i_n} m_j,\ i = i_1,\dots,i_n.$$

In one space dimension, the above condition is suitable to single out a unique sticky particle solution, see [8, 7, 10, 11, 14] and the references therein for an overview of the results.
In [9], it is shown that when the space dimension d is strictly greater than 1 the existence and uniqueness of a sticky particle solution is in general false: the correct notion which preserves the compactness of solutions is the notion of *dissipative solutions* introduced in [5]. The main results of [5] is that dissipative solutions form a weakly compact sets w.r.t. the narrow convergence, and the study of the genericity of dissipative solutions.
The fundamental difference between a sticky particle solution and a dissipative solution is that in the first case, if particles occupy the same point in space-time, then they are forced to merge into a single particle; for dissipative solutions, instead, particles do not need to merge, but if they do the conservation of momentum is required. Figure 1 shows the different behavior to the three family of solutions (conservative, sticky and dissipative) in the case of initial data made of 2 particles: observe

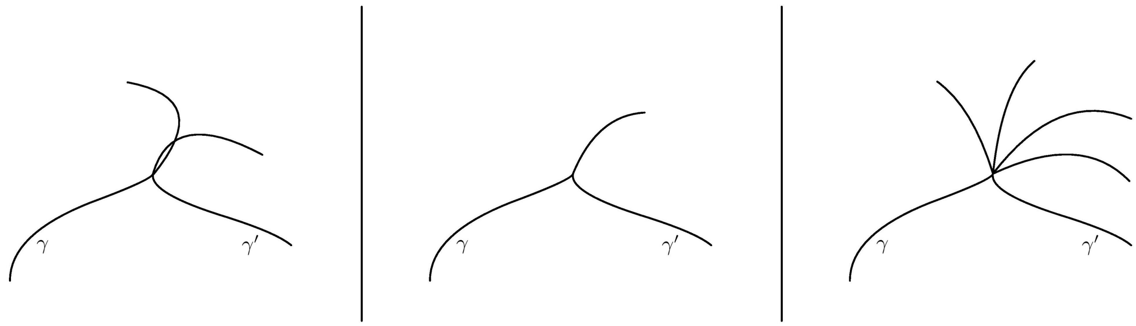

Figure 1: from left to right, examples of a conservative solution, a sticky solution and a dissipative solution with the same initial data.

that dissipative solutions allows interactions of fraction of the initial particles.
A simple example of dissipative solutions in 1d is

$$\mu(t,dqdp)=\begin{cases}\frac{\delta_{(t,1)}(dqdp)+\delta_{(-t,-1)}(dqdp)}{2} & t<0,\\ \frac{1-\alpha}{2}\delta_{(t-,1)}(dqdp)+\frac{\alpha+\beta}{2}\delta_{(\frac{\beta-\alpha}{\beta+\alpha}t,\frac{\beta-\alpha}{\beta+\alpha})}(dqdp)+\frac{1-\beta}{2}\delta_{(t,1)}(dqdp) & t\geq 0,\end{cases}$$

with $\alpha,\beta\in[0,1]$. One can think that only a fraction α of the first particle decides to merge with a fraction β of the second, resulting in a particle traveling with speed $\frac{\beta-\alpha}{\alpha+\beta}$: the sticky particle solution is obtained when $\alpha=\beta=1$, and the conservative for $\alpha=\beta=0$.

The main result of this paper is the extension of the results of [5] to a general semiconvex Hamiltonian case with quadratic growth.

Theorem 1.1. *Assume that $V(q,p)$ is semiconcave and uniformly convex in p, $W(q,p,q',p')$ is convex and symmetric, i.e. $W(q,p,q',p')=W(q',p',q,p)$. Then the following holds.*

1. *The set of dissipative solutions is not empty, and contains all conservative solutions.*
2. *The set of dissipative solutions is weakly compact, and it coincides with the weak closure of the set of sticky particle solutions made of finitely many particles (Theorem 5.2).*
3. *There is a G_δ-set of initial data such that the unique dissipative solution is the conservative one (Theorem 7.1).*

We remark that the first point follows immediately from the very definition of dissipative solutions, Definition 4.1.

These results are the exact extension of the results for the sticky particle case. Nevertheless their proof requires much more effort for the following reason.
For the Hamiltonian (2) (and more in general for purely quadratic Hamiltonian, i.e. when the ODEs (1) are linear), the dissipative solution at time t can be computed directly from the conservative solution as follows: the position and momentum of merged particles can be obtained by taking the average position and average momentum (i.e. applying the projection $\mathbb{P}_t$ defined below at the conservative flow, Proposition 4.17).
For the nonlinear case the above property is clearly false, and then one has to rely on the integral

formulation of the dissipative solutions (Definition 4.1): the correct definition of dissipative solution is actually the central point of the paper, and we give it here below.

Consider the space of $W^{1,2}$-curves in $\mathbb{R}^d$, and define the family of projections

$$W^{1,2}([0,T],\mathbb{R}^d) \ni \gamma \mapsto \hat{\mathbb{P}}_t \gamma(\tau) = \gamma(t)\mathbb{1}_{\tau<t} + \gamma(\tau)\mathbb{1}_{\tau\geq t}.$$

Let Ω_t be the smallest σ-algebra such that $\hat{\mathbb{P}}_t$ is measurable, and let $\mathbb{P}_t$ be the projection acting on L^2_η. Then $\eta \in \mathbb{P}_2(\Gamma)$ is a dissipative solutions if there is a function $v \in L^2_{\mathscr{L}^1\times\eta}((0,T)\times\Gamma,\mathbb{R}^d)$ such that for $\mathscr{L}^1$-a.e. t

$$\dot{\gamma}(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t),\gamma'(t),v(t,\gamma),v(t,\gamma'))\eta(d\gamma'), \tag{3a}$$

$$v(t,\gamma) = \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s),v(s,\gamma))ds - \int_0^t\int \nabla_q W(\gamma(s),\gamma'(s),v(s,\gamma),v(s,\gamma'))\eta(d\gamma')ds\bigg). \tag{3b}$$

The uniform convexity of V in p implies that the first equation, Equation (3a), is the graph of Lipschitz function, relating $\dot{\gamma}(t)$ uniquely with $v(t,\gamma)$. The evolution is then described by the second equation, Equation (3b), and the projection acts only on this one: it describes the conservation of momentum.

We end this introduction by listing some additional technical results, which have an interest on their own.

- The map $t \mapsto v(t,\gamma)$ belongs to $\mathrm{BV}^{1/2}_t L^2_\eta$ (Lemma 4.6): note that the conservative solution has $t \mapsto p(t)$ in $W^{1,2}_\mu$, i.e. the projection reduces the regularity but it still preserves some regularity w.r.t. t. In particular, the incremental ratio $\frac{\gamma(t+s)-\gamma(t)}{s}$ converges to $\dot{\gamma}(t)$ in $L^2_{\mathscr{L}^1\times\eta}$ (Lemma 4.7).

- It is possible to approximate a dissipative solution by alternating the conservative flow and the projection $\mathbb{P}_t$ (Proposition 4.12 of Section 4.1). This is in some sense the natural idea of a dissipative solution: the particles travel by the conservative flow, and then some of them interact and merge. This approximation collects the interaction at a finite number of times t_i. It is important to remark here that by just trowing in a family of projection, one is not going to construct an approximation of a dissipative solution: indeed the limit is not in general conservative, since the approximation we construct needs to project also the positions $\gamma(t)$. Another approximation we present below will actually construct dissipative solutions.

- In the sticky particle model (or in linear case i.e. H purely quadratic), the dissipation of energy $E(t) = H((\gamma(t),v(t))_\sharp\eta)$ is immediate since the projection commutes with the conservative flow. Here the same result holds, together with the fact that the distance of the conservative solution and the dissipative solution is controlled by $\sqrt{E(s)-E(t)}$; the analysis is however more complicated and requires some preliminary estimates (Section 4.2 and Proposition 4.15).

- As for the sticky particle systems, in the linear case requiring that the trajectories of the conservative solution are disjoint is a necessary condition in order to have that the only dissipative solution for a given initial data is the conservative one (Proposition 4.18). This is however false for the general case, and in Appendix C an explicit example is worked out. We observe here that the first example in [9] shows that the non crossing of trajectories is not sufficient, if the number of particles is not finite (in the latter case the analysis become trivial).

- The compactness of dissipative solutions is exactly the same result as for the sticky particle case [5]. Here (Section 5), the proof is slightly simplified: it is based on the use of the $\mathrm{BV}^{1/2}_t L^2_\eta$ compactness of $v(t)$ to prove the weak convergence of $v_n(t)_\sharp \eta_n$, where $\eta_n, v_n(t)$ is a sequence of solutions converging to η (Proposition 5.1).
- The construction of a backward dissipative approximation (made only of finitely many particles and also being a sticky particle solution, not just a dissipative solution) to any given dissipative solution is the same as in [5]: the only variation is that since the conservative flow is nonlinear, some additional estimates are needed to assure that there is an arbitrarily small perturbation such that the trajectories are now disjoint in some time interval. The proof of the main result here, Proposition 6.8, is indeed based on some elementary properties of the conservative flow, which we present in Section 3.1.
- The last result on the genericity of initial data such that the only dissipative solution is the conservative one (Theorem 7.1) is exactly as in [5], and follows easily from the analysis of the previous sections.

Finally, it is not clear to us how much of this theory can be extended to convex Hamiltonian with superquadratic growth: at least we would like to have that the conservative flow is unique and conserves energy. A couterexample to the uniqueness of the dissipative flow is presented in Appendix B, but the Hamiltonian is not even semiconvex.

1.1 Plan of the paper

The paper is organized as follows.

In Section 2 we list the notation we will use throughout the paper: it is in general standard notation in analysis. Section 2.1 collects the properties of the Hamiltonian function we require in this paper, while Section 2.2 lists some elementary properties of the space of curves used here.

The conservative flow is studied in Section 3: it is mainly a collection of known results, or results which are fairly easy to prove, some of their proofs are collected in Appendix A. Less standard (but still elementary) is the analysis of initial data for which the trajectories are not crossing: this is done in Section 3.1, where it is shown that we can perturb an initial data (in case splitting particles) and get that for a fixed small interval the trajectories are not intersecting (Proposition 3.8.

The key part of the paper begins in Section 4. Here the definition of dissipative solution is given, Definition 4.1, and it is shown that it enjoys some properties: it has finite energy (Lemma 4.4), it enjoys a concatenation property (Lemma 4.5), and $v(t)$ belongs to $\mathrm{BV}^{1/2}_t L^2_\eta$ (Lemma 4.6). The latter property gives that the incremental ratio $\frac{\gamma(t+s)-\gamma(t)}{s}$ converges as $s \searrow 0$ to $\dot\gamma$ uniformly in $L^2_{\mathscr{L}^1 \times \eta}$ (Lemma 4.7).
These estimates are necessary to construct a forward piecewise conservative approximation to a given dissipative solution, Proposition 4.12. There are two key estimates here: the choice of the time interval where the dissipation is small (Lemma 4.9), and the comparison between the projection of the conservative solution and the dissipative solution (Lemma 4.10 and Corollary 4.11).
The nice estimate showing that the distance between the dissipative solution and the conservative one is proportional to the square root of the dissipation of energy is in Proposition 4.15, Section 4.2, while the analysis of the case where H is purely quadratic (and then the ODEs (1) are linear) is in Section 4.3, where it is shown that the conservative evolution and the projection $\mathbb{P}_t$ commute. In

the same section it is shown that, in this case, the fact that the unique dissipative solution is the conservative one implies that the conservative trajectories are disjoint, Proposition 4.18.

Section 5 uses standard arguments to deduce that if $\eta_n, v_n(t,\gamma)$ is a family of dissipative solutions converging to $\eta, v(t,\gamma)$ weakly, then the measures $(\gamma(t),\dot\gamma, v_n(t,\gamma))_\sharp\eta_n$ converges weakly to $(\gamma(t),\dot\gamma, v(t,\gamma))_\sharp\eta$, Proposition 5.1. This gives a proof of the compactness of dissipative solution, Theorem 5.2, which is simpler w.r.t. the proof of the analogous result contained in [5].

Section 6 concerns the construction of approximations to a dissipative solution made of finitely many sticky particles. The approach is standard, and follows the ideas of [5]. First, if we give final data (position and speed of the particles) and finite set of times t_i and functions $\Upsilon_i(\gamma)$ with $\mathbb{P}_{t_i}(\Upsilon_i) = 0$, one can construct a backward dissipative solution by alternating the backward conservative flow in $[t_i, t_{i+1})$ and requiring that at t_i the projection $\mathbb{P}_{t_i}$ is acting as

$$v(t_i,\gamma) - v(t_i-,\gamma) = \Upsilon_i.$$

In other words we are specifying the times and the action of the projection. Lemma 6.2 shows that this construction is coherent and the result is a dissipative solution.
In the rest of the section one finds suitable times t_i and functions $\Upsilon_i(\gamma_i)$ to obtain the desired approximation to a dissipative solution: first requiring that $\mathbb{P}_t$ acts only at the time t_i (Proposition 6.4), then requiring that the approximating dissipative is made of finitely many particles (Proposition 6.6), and finally that the backward trajectories of these particles are not intersecting (Proposition 6.8).
The result of this section, together with the weak closure of the family of dissipative solutions, implies that the weak closure of the sticky particle solutions are the dissipative solutions, Theorem 6.9: this shows that it is the natural set for studying this kind of problems.

The last section, Section 7, concerns the fact that the set of initial data for which the only dissipative solution is the conservative one is a residual set, Theorem 7.1. Its proof uses quite standard argument, once it is known that $\mu \mapsto H(\mu)$ is l.s.c. (in this section V, W are assumed to be convex), the set of finite particles solutions such that the trajectories are not intersecting is dense (Proposition 3.6), and that if the trajectories are non intersecting then the unique solution is the dissipative one (Lemma 4.16).

The appendix contains the proof of some elementary facts about the conservative solutions (Appendix A), an example of non uniqueness for the conservative flow if the Hamiltonian does not satisfy the assumptions of Section 2.1 (Appendix B), and an example where the trajectories of the conservative solution are intersecting, but nevertheless the unique solution is the conservative one (giving a counterexample to the converse of Proposition 3.8, Appendix C).

2 Definitions, assumptions and notations

Some general notation.

- We will work in the space $(t,x) \in [0,T] \times \mathbb{R}^d$ or $(t,q,p) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$. The letter d will always denote the space dimension.
- In a metric space (X,d) the ball or radius r about $x \in X$ is written as $B^X_r(x)$. Sometimes $B_r(x)$ if the space is clear from the context.
- The symplectic matrix J is

$$\mathrm{J} = \begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix} \in \mathbb{R}^{2d\times 2d}. \tag{4}$$

- The norm in $\mathbb{R}^d$ is $|\cdot|$, and its scalar product by $(\cdot,\cdot)$. More generally, $\|\cdot\|$ and $(\cdot,\cdot)$ will denote respectively the norm and scalar product on an Hilbert space (here most of the time L^2_ν for some measure ν), and which space is under consideration will be usually clear from the context.

- The letters s, t, τ are reserved for time variable, x, y, z for the space variable in $\mathbb{R}^d$, and for the Hamiltonian variables we will use $(q,p) \in \mathbb{R}^d \times \mathbb{R}^d$. The capital letters X, Y, Z denotes the coordinates of N-particles in $\mathbb{R}^d$, and (Q,P) the Hamiltonian coordinates for N-particles; we will also use Q, P in case of a countable or a continuum family of particles. The time interval in which we consider the solution is $[0,T]$. To differentiate variable we will use $x', \tilde{x}, \hat{x}, \dots$, and the same for $y, z, q, p, X, Q, P, \dots$.

- We write x_i for the i-th component of $x = (x_1, x_2, \dots)$.

- A generic constant is C (if supposedly large), or c (if supposedly small). We use them if their value depends only on some parameters of the problem, if we do not care about this dependence we will use the symbol $\mathcal{O}(1), o(1)$.

- The letter $v = v(t,\gamma)$ is reserved for the Hamiltonian coordinate P on the space of path $t \mapsto \gamma(t) \in \mathbb{R}^d$. The evaluation map $t, \gamma \mapsto \gamma(t)$ will be denoted with $\gamma(t)$ (this slightly differs from the standard notation $e(t)$).

- Since we are in $\mathbb{R}^d$, we will not distinguish between gradient and differential, noting both with ∇_x, ∂_x. For Lipschitz curves $t \mapsto \gamma(t) \in \mathbb{R}^d$ we will use the special notation $d\gamma/dt = \dot{\gamma}$, and more generally
$$\frac{d}{dt}\phi(t,\gamma(t)) = \dot{\phi}(t,\gamma(t)).$$

- We will use the Greek letter μ for a measure on the phase space $\mathbb{R}^d \times \mathbb{R}^d$, η for a measure of the space of path $\Gamma = L^2((0,T),\mathbb{R}^d)$, $\mathscr{L}$ for the Lebesgue measure. Sometimes we will parameterize curves as $t \mapsto x(\alpha,t), Q(\alpha,t), P(\alpha,t), \dots$ with $\alpha \in [0,1]$, and we will use the measure $\varpi(d\alpha)$ on the space of parameters: in general, we can take $\varpi = \mathscr{L}^1 \llcorner_{[0,1]}$. A generic measure will be denoted by ν, π, the product of measures will be denoted by $\cdot \times \cdot$.

- The push-forward of a measure ν according to a map $\mathbb{T} : X \to Y$ is denoted with $\mathbb{T}_\sharp \nu$.

- The disintegration of a measure ν according to a Borel map $\mathbb{T} : A \to B$, A, B Polish space, is written as
$$\nu = \int \nu_y m(db), \quad m = \mathbb{T}_\sharp \nu.$$
We often interpret m as the restriction of ν to the σ-algebra $\mathbb{T}^{-1}(\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra on Y, and we will also write
$$\nu = \int \nu_{\mathbb{T}(a)} \nu(da) = \int \nu_a \nu(da).$$

- We will use the notation $L^2_\nu(X,Y)$, Y Hilbert space, for the space of functions $f : X \to Y$ such that
$$\int \|f(x)\|^2_Y \nu(dx) < \infty.$$

- $\mathscr{P}_2(X)$, X Banach space, is the set of probability measures with finite quadratic moments in X. The topology of $\mathscr{P}_2(X)$ is either the narrow topology or the Wasserstein-2 distance W_2.
- The space $\mathrm{BV}^{1/2}([0,T],X)$, X Banach, is defined as the functions $f:[0,T]\mapsto X$ such that

$$\sup\left\{\sum_{i=1}^N \|f(t_i)-f(t_{i-1})\|_X^2, 0\leq t_0\leq t_1\leq\cdots\leq t_N\leq T\right\}<\infty.$$

  We will often shorten the notation to $\mathrm{BV}^{1/2}_{\mathscr{L}^1}X$ or $\mathrm{BV}^{1/2}_t X$, e.g. $\mathrm{BV}^{1/2}_t L^2_\eta$.
- A projection operator is denoted by $\mathbb{P}$, with some index in case of dependence from a parameter or to denote the target space.

2.1 Hamiltonian

The assumptions in this sections are standard for existence of a solution the well posedness of the conservative solution (see Section 3, and for more general Hamiltonians see [4, 12]).

We consider the family of Hamiltonians $H:\mathscr{P}_2(\mathbb{R}^d\times\mathbb{R}^d)\to\mathbb{R}$ defined as

$$H(\mu)=\int V(q,p)\mu(dqdp)+\frac{1}{2}\int\int W(q,p,q',p')\mu(dq'dp')\times\mu(dqdp),$$

where $V:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, $W:(\mathbb{R}^d\times\mathbb{R}^d)^2\to\mathbb{R}$ are functions such that with the assumptions:

1. $V(0)=W(0)=0$, $\nabla V(0)=0$ and $\nabla W(0)=0$;
2. $W(q,p,q',p')=W(q',p',q,p)$;
3. V,W and functions of class $C^{2.1}$ with first derivatives L-Lipschitz:

$$\big|\nabla V(q,p)-\nabla V(q',p')\big|\leq L|(q,p)-(q',p')|, \tag{5a}$$

$$\big|\nabla W(x,v,x',v')-\nabla W(y,w,y',w')\big|\leq L\big|(x,v,y,w)-(y,w,y',w')\big|; \tag{5b}$$

4. there exists $\lambda\geq 0$ such that $(x,v)\mapsto V(x,v)+\lambda|x|^2/2$ is convex, and by (5a) it has at most quadratic growth;
5. $(x,v)\mapsto W(x,v,x',v')$ is convex;
6. for all (x,v) it holds

$$\Lambda\mathbb{I}\leq\nabla_{pp}V(x,v)\leq L\mathbb{I},\qquad \mathbb{I}\text{ identity matrix in }\mathbb{R}^d,$$

   i.e. it is uniformly convex w.r.t. v.

Here above and in the following we will denote the differentials of V,W w.r.t. the q,p,q',p' components as $\nabla_q,\nabla_p,\nabla_{q'},\nabla_{p'},\nabla_{qq},\nabla_{pp},\dots$.

For the measure $\varpi(d\alpha)$ and the map $(\alpha,t)\mapsto(Q(\alpha,t),P(\alpha,t))$, we will use the notation

$$H(t,\varpi)=H(\mu(t)),\quad \mu(t)=(q(t),p(t))_\sharp\varpi,$$

i.e. more explicitly

$$H(t,\varpi)=\int V(Q(\alpha,t),P(\alpha,t))\varpi(d\alpha)+\frac{1}{2}\int\int W(Q(\alpha,t),P(\alpha,t),q(t,\alpha'),p(t,\alpha'))\varpi(d\alpha)\times\varpi(d\alpha').$$

A similar notation will be used for the gradient of H in $\mathscr{P}_2(\mathbb{R}^d\times\mathbb{R}^d)$ w.r.t. the Wasserstein distance:

$$\nabla H(\mu)=\nabla V(q,p)+\int\nabla W(q,p,q',p')\mu(dq'dp'),$$

and $\nabla H(t,\varpi)=\nabla H(\mu(t))$, $\mu(t)=(q(t),p(t))_\sharp\varpi$: we shorten the notation for $\nabla W(q,p,q',p')=(\nabla_q W(q,p,q',p'),\nabla_p W(q,p,q',p'))$.

2.2 Probability space of curves

To shorten the notation we will write

$$\Gamma:=L^2((0,T),\mathbb{R}^d),\qquad \Gamma_t:=L^2((t,T),\mathbb{R}^d),$$

and denote by $\mathbb{T}_t:\Gamma\to\Gamma_t$ the *restriction map*

$$\mathbb{T}_t(\gamma)=\gamma\llcorner_{[t,T]}.$$

The measures η which we are going to consider will be supported on the set

$$\mathscr{M}(\Gamma)=\bigg\{\eta\in\mathscr{P}(\Gamma):\ \frac{1}{2}\int|\gamma(0)|^2\eta(d\gamma)\le\tilde{C}_1,\ \frac{1}{2}\int\int_0^T|\dot\gamma(t)|^2dt\eta(d\gamma)\le\tilde{C}_2\bigg\},$$

for some constant $\tilde{C}_1$, $\tilde{C}_2$ which will be estimated explicitly for the solutions (conservative or dissipative) we consider below. It is standard to verify that $\mathscr{M}(\Gamma)$ is a compact subset of $\mathscr{P}_2(\Gamma)$ w.r.t. the narrow convergence. Notice that the bound on $\int\int_0^T|\dot\gamma(t)|^2dt\eta(d\gamma)$ implies that γ is η-a.e. absolutely continuous (a.c. in the following), so that the value $\gamma(0)$ is well defined η-a.e. giving sense to the first condition in the definition of $\mathscr{M}(\Gamma)$.

It is immediate to see that if $\eta\in\mathscr{M}(\Gamma)$, $\mathbb{T}_t$ is equivalent to the projection

$$\gamma\mapsto\hat{\mathbb{P}}_t\gamma(\tau)=\gamma(t)\mathbb{1}_{[0,t)}(\tau)+\gamma(\tau)\mathbb{1}_{[t,T]}(\tau),$$

and that $(\hat{\mathbb{P}}_t)_\sharp\eta\in\mathscr{M}(\Gamma)$.

Let Ω_t be the descending Borel fibration generated by $\mathbb{T}_t$,

$$\Omega_t=\mathbb{T}_t^{-1}\big(\mathcal{B}(\Gamma_t)\big),$$

i.e. the smallest σ-algebra such that $\mathbb{T}_t$ is Borel, and let $\mathbb{P}_t:L^2(\eta)\to L^2(\eta)$ be the corresponding projection. The latter functional can be represented by means of the disintegration of $\eta\in\mathscr{P}(\Gamma)$ according to the map $\mathbb{T}_t$: indeed if

$$\eta=\int\omega^t_{\gamma'}(d\gamma)\mathbb{T}_{t\sharp}\eta(d\gamma')=\int\omega_{\mathbb{T}_t(\gamma')}\eta(d\gamma'),$$

where $\Gamma \ni \gamma' \mapsto \omega_{\mathbb{T}_t(\gamma')} \in \mathscr{P}(\Gamma)$ can be taken to be Borel, then [6, Proposition 10.4.18]

$$(\mathbb{P}_t f)(\gamma) = \int f(\gamma') \omega^t_{\mathbb{T}_t(\gamma)}(d\gamma'). \tag{6}$$

The following concatenation property holds: for $s < t$, let $\mathbb{T}_{s\to t} : \Gamma_s \to \Gamma_t$ be the restriction map such that $\mathbb{T}_{s\to t} \circ \mathbb{T}_s = \mathbb{T}_t$. Then by disintegrating according to $\mathbb{T}_{s\to t}$

$$(\mathbb{T}_s)_\sharp \eta(d\gamma') = \int \omega^{s\to t}_{\gamma''}(d\gamma')(\mathbb{T}_t)_\sharp \eta(d\gamma'').$$

Hence

$$\eta = \int \omega^s_{\gamma'}(\mathbb{T}_s)_\sharp \eta = \int \bigg[\int \omega^s_{\gamma'} \omega^{s\to t}_{\gamma''}(d\gamma') \bigg] (T_{s\to t})_\sharp((\mathbb{T}_s)_\sharp \eta)(d\gamma'') = \int \bigg[\int \omega^s_{\gamma'} \omega^{s\to t}_{\gamma''}(d\gamma') \bigg] (\mathbb{T}_t)_\sharp \eta(d\gamma''),$$

and from the uniqueness of disintegration

$$\omega^t_{\gamma''}(d\gamma) = \int \omega^s_{\gamma'}(d\gamma) \omega^{s\to t}_{\gamma''}(d\gamma') \qquad (\mathbb{T}_t)_\sharp \eta\text{-a.e. } \gamma''.$$

The above formula corresponds to the composition property

$$\mathbb{P}_{s\to t} \circ \mathbb{P}_s = \mathbb{P}_t, \qquad (\mathbb{P}_{s\to t} f)(\gamma) = \int f \omega^{s\to t}_\gamma. \tag{7}$$

Lemma 2.1 ([6, Theorem 10.2.1])**.** *It holds*

$$\lim_{t \searrow s} \mathbb{P}_{s\to t} = \mathbb{P}_s$$

strongly in L^2_η.

3 Conservative solutions

Let $H : \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$ be the Hamiltonian considered in Section 2.1. Here we construct the Hamiltonian flow: the results are classical, we adapt them to a suitable form which will be useful for the study of dissipative solutions. Since this flow preserves the energy, we will call it *conservative flow/solution*, opposed to the *dissipative flow/solution* of Section 4.

The results in this section are pretty much standard: we give for simplicity the proofs in the appendix.

Definition 3.1. The measure $\eta \in \mathscr{P}_2(\Gamma)$ is a *conservative solution* if there is an $L^2(\eta)$-function $v : \Gamma \to W^{1,2}((0,T), \mathbb{R}^d)$ such that for η-a.e. γ it holds

$$\gamma(t) = \gamma(0) + \int_0^t \bigg[\nabla_v V(\gamma(s), v(s,\gamma)) + \int \nabla_v W(\gamma(s), v(s,\gamma), \gamma'(s), v(s,\gamma'(s)) \eta(d\gamma') \bigg] ds, \tag{8a}$$

$$v(t,\gamma) = v_0(\gamma) - \int_0^t \bigg[\nabla_q V(\gamma(s), v(s,\gamma)) + \int \nabla_q W(\gamma(s), v(s,\gamma), \gamma'(s), v(s,\gamma'(s)) \eta(d\gamma') \bigg] ds, \tag{8b}$$

where $v_0 \in L^2_\eta$.

Let $\mu(t) \in \mathscr{P}(\mathbb{R}^d \times \mathbb{R}^d)$ be the measure

$$\mu(t) = (\gamma(t), v(t))_\sharp \eta.$$

The conservative solution η can be interpreted as the Lagrangian representation of the measure valued solution $\mu(t)$ to the PDE

$$\partial_t \mu(t) + \mathrm{div}(\mathrm{J}\nabla H(\mu(t))\mu(t)) = 0, \tag{9}$$

where J is the $2d \times 2d$ symplectic matrix (4). We thus are in the setting of [4] for existence and actually uniqueness of a Hamiltonian flow $t \to \mu_t \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ solving (9) and preserving H. The existence and uniqueness of the solution $\mu(t)$ is strictly related to the existence and uniqueness of the Lagrangian representation η: in the linear transport case this is well established [2], while in our case the fact that the measure η is a probability on trajectories $\gamma : [0,T] \to \mathbb{R}^d$ instead of $\gamma : [0,d] \to \mathbb{R}^{2d}$ follows from the fact that the time derivative of (8a),

$$\dot\gamma(t) = \nabla_v V(\gamma(s), v(s,\gamma)) + \int \nabla_v W(\gamma(s), v(s,\gamma), \gamma'(s), v(s,\gamma'(s))\eta(d\gamma'),$$

is a bijection between $\dot\gamma$ and $v(\gamma)$.

We give a self-contained proof of these facts in Appendix A.

Define

$$\begin{aligned} F(t) \ : \ L^2(\eta) \ &\to \ L^2(\eta) \\ v \ &\mapsto \ F(t,v)(\gamma) = \nabla_v V(\gamma(t), v(\gamma)) + \int \nabla_v W(\gamma(t), v(\gamma), \gamma'(t), v(\gamma'))\eta(d\gamma') \end{aligned} \tag{10}$$

Recalling that V is uniformly convex in v and W is convex in v, it is easy to verify that F is well defined for all t, and moreover the next proposition holds.

Proposition 3.2. *The operator* (10) *is uniformly monotone, namely*

$$\Lambda \|v_1 - v_2\|_2^2 \le \big(v_1 - v_2, F(t,v_1) - F(t,v_2)\big) \le 3L\|v_1 - v_2\|_2^2.$$

In particular $F(t)$ is a bi-Lipschitz map of $L^2(\eta)$ into itself. The proof is in Appendix A, page 37.

The next result gives the existence, uniqueness and continuous dependence. Let $(0,1) \ni \alpha \mapsto Q_0(\alpha), P_0(\alpha)$ be given functions in $L^2(0,1)$.

Proposition 3.3. *There exist unique functions* $Q(\alpha,t), P(\alpha,t) \in C_0([0,T], L^2(0,1))$ *satisfying*

$$Q(\alpha,t) = Q_0(\alpha) + \int_0^t \bigg[\nabla_p V(Q(\alpha,s), P(\alpha,s)) + \int \nabla_p W(Q(\alpha,s), P(\alpha,s), Q(\alpha',s), P(\alpha',s) d\alpha'\bigg] ds,$$

$$P(\alpha,t) = P_0(\alpha) - \int_0^t \bigg[\nabla_q V(Q(\alpha,s), P(\alpha,s)) + \int \nabla_q W(Q(\alpha,s), P(\alpha,s), Q(\alpha',s), P(\alpha',s) d\alpha'\bigg] ds.$$

Moreover

$$t \mapsto H(t, \mathcal{L}^1 \llcorner_{(0,1)}) = \int V(Q(\alpha,t), P(\alpha,t)) d\alpha + \int\int W(Q(\alpha,t), P(\alpha,t), Q(\alpha',t), P(\alpha',t)) d\alpha d\alpha'$$

is constant, $(\partial_t Q(t), P(t)) \in L^2((0,1), W^{1,2}(0,T))$ *with*

$$\|\partial_t Q(t)\|_{L^2(0,1)}, \|\partial_t P(t)\|_{L^2(0,1)}, \frac{1}{3L}\|\partial_t^2 Q(t)\|_{L^2(0,1)} \leq 3Le^{3Lt}\|(Q_0,P_0)\|_{L^2(0,1)}.$$

Finally, if $(Q_0,P_0),(Q_0',P_0')$ *are two different initial data and* $(Q(t),P(t)),(Q'(t),P'(t))$ *the corresponding solutions, then it holds*

$$\big\|(Q(t),P(t)) - (Q'(t),P'(t))\big\|_{L^2(0,1)} \leq e^{3Lt}\big\|(Q_0,P_0) - (Q_0',P_0')\big\|_{L^2(0,1)}.$$

Corollary 3.4. *The measure* $\eta = Q(\alpha,t)_\sharp \mathscr{L}^1(d\alpha)$ *is a conservative solution concentrated on* $\mathscr{M}(\Gamma)$ *with* $\tilde{C}_1, \tilde{C}_2 = 3LTe^{3LT}\|(x_0,v_0)\|^2_{L^2(0,1)}$.

The measure

$$\int \phi(x,v)\mu(t;dxdv) = \int_0^1 \phi(Q(\alpha,t),P(\alpha,t))d\alpha$$

is the unique solution to the transport equation (9) *with initial data* $\mu(t=0)$.

Proof. The first statement follows by the definition of conservative solution. The fact that μ is unique and solves the transport equation follows by observing that $t \mapsto (Q(\alpha,t),P(\alpha,t))$ is a characteristic for μ-a.e. α, and the uniqueness result above. □

For the flow at the level of the ODE in the phase space $\mathbb{R}^d \times \mathbb{R}^d$, we will use the following notation: for a given initial data $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ let $(Q(t,q,p;\mu_0), P(t,q,p;\mu_0))$ be the unique flow in $L^\infty((0,T), L^2_{\mu_0}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d))$ such that

$$\begin{aligned} Q(t,q,p;\mu_0) = q &+ \int_0^t \nabla_v V(Q(s,q,p;\mu_0),P(s,q,p;\mu_0))ds \\ &+ \int_0^t \int \nabla_v W\big(Q(s,q,p;\mu_0),P(s,q,p;\mu_0),Q(s,q',p';\mu_0),P(s,q',p';\mu_0)\big)\mu_0(dq'dp')ds, \end{aligned} \tag{11a}$$

$$\begin{aligned} P(t,q,p;\mu_0) = p &- \int_0^t \nabla_q V(Q(s,q,p;\mu_0),P(s,q,p;\mu_0))ds \\ &- \int_0^t \int \nabla_q W\big(Q(s,q,p;\mu_0),P(s,q,p;\mu_0),Q(s,q',p';\mu_0),P(s,q',p';\mu_0)\big)\mu_0(dq'dp')ds. \end{aligned} \tag{11b}$$

These equations correspond to the projection on $\mathbb{R}^d \times \mathbb{R}^d$ of (8), as it can be easily seen because

$$(\gamma(t),v(t))_\sharp \eta = \mu_t = (Q(t),P(t))_\sharp \mu_0.$$

Using the semigroup property and the fact that (11) are time independent, we have also

$$\begin{aligned} Q(t-s,Q(s,q,p;\mu_0),P(s,q,p;\mu_0);\mu_s) &= Q(t,q,p;\mu_0), \\ P(t-s,Q(s,q,p;\mu_0),P(s,q,p;\mu_0);\mu_s) &= P(t,q,p;\mu_0). \end{aligned}$$

Remark 3.5. The conservative flow (Q, P) can be defined to all $\mathbb{R}^d \times \mathbb{R}^d$: indeed the solution μ_t is uniquely defined, and the vector field

$$(q, p) \mapsto \mathrm{J}\nabla H(q, p; \mu_t) = \mathrm{J}\bigg(\nabla V(q, p) + \int \nabla W(q, p, q', p')\mu_t(dq'dp')\bigg)$$

is uniformly Lipschitz. We will use the same notation $(Q, P)(t, q, p; \mu_0)$ as above for the flow extended to the whole $\mathbb{R}^d \times \mathbb{R}^d$.

3.1 Non crossing of trajectories

In the following we will need to study how generic is the crossing of trajectories of N particles satisfying the Hamiltonian ODE, i.e. solving (11) with μ_0 finite sum of Dirac deltas. We will use the notation

$$q_i(t, Q_0, P_0) = Q(t, q_i, p_i; \mu_0), \;\; p_i(t, Q_0, P_0) = P(t, q_i, p_i; \mu_0), \quad \mu_0 = \frac{1}{N}\sum_i \delta_{(q_i, p_i)}.$$

Proposition 3.6. *For conservative solutions made of N particles, the set of initial data such at least two trajectories cross is of codimention $(d-1)$ in $(\mathbb{R}^d \times \mathbb{R}^d)^N$.*

Proof. The condition of intersection of the particles w.l.o.g. labelled $1, 2$ is

$$\big\{(Q_0, P_0) \in \mathbb{R}^d \times \mathbb{R}^d : \exists t \in \mathbb{R}\big(q_1(t, Q_0, P_0) = q_2(t, Q_0, P_0)\big)\big\}.$$

By the implicit function theorem, the condition above defines a $(d-1)$-codimensional surface if

$$\mathrm{rank}\big(\nabla_{Q_0,P_0}(q_1 - q_2)\big) = d.$$

This is implied by the divergence free property of Hamiltonian flows

$$\det\left(\begin{bmatrix} \nabla_{Q_0,P_0}(q_1 - q_2) \\ \vdots \\ \nabla_{Q_0,P_0} q_N \\ \nabla_{Q_0,P_0} p_1 \\ \vdots \\ \nabla_{Q_0,P_0} p_N \end{bmatrix}\right) = \det\left(\begin{bmatrix} \nabla_{Q_0,P_0} q_1 \\ \vdots \\ \nabla_{Q_0,P_0} q_N \\ \nabla_{Q_0,P_0} p_1 \\ \vdots \\ \nabla_{Q_0,P_0} p_N \end{bmatrix}\right) = 1. \qquad \square$$

In the following we will need to perturb a finite particle conservative solution preserving the initial position of the particles and the average speed: more precisely, the initial data are $\{q_{i,j}, p_{i,j}\}_{i,j}$ with

$$q_{i,j} = \bar{q}_i, \quad \sum_j m_{i,j}\bar{p}_{i,j} = \bigg(\sum_j m_{i,j}\bigg)\bar{p}_i, \tag{12}$$

for some constants $m_{i,j} > 0$. In other words, given $\{\bar{q}_i, \bar{p}_i\}_i$, we are allowed to split each particle $\bar{q}_i, \bar{p}_i$ into the particles $\{q_{i,j}, p_{i,j}\}_j$, assigning new initial speeds but preserving the average speed of the particles starting in the same point. The goal of this splitting is again to avoid crossing of trajectories, at least for an interval of time independent on the data $\{\bar{q}_i, \bar{p}_i\}_i, \{q_{i,j}, p_{i,j}\}_{i,j}$.

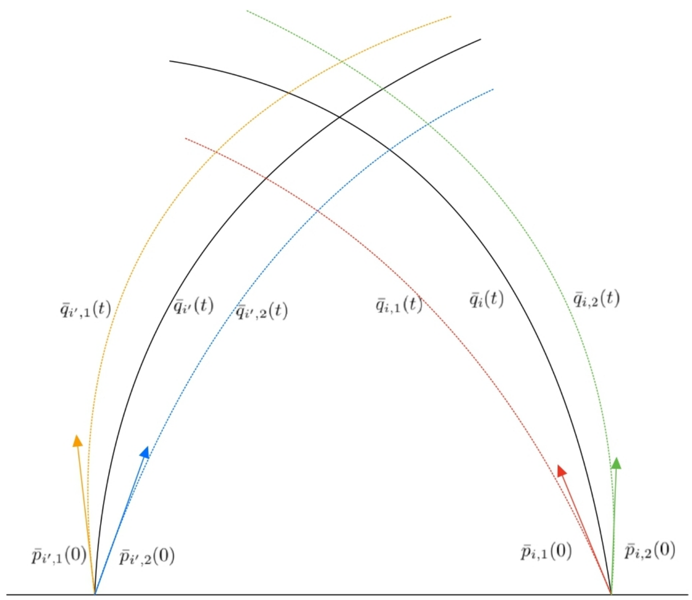

Figure 2: two crossing particles are perturbed according Proposition 3.8 in order to avoid the intersection of the new trajectories.

Remark 3.7. A simple example with two distinct particles shows that at least we have to split one trajectory.

Moreover, it is not possible to perturb the trajectories as in (12) requiring them not to join forever in the future (or in the past): this can be easily seen in the harmonic case $V = p^2/2$, $W = (q-q')^2/2$. We observe that the situation is different in the case of free motion, i.e. $V = v^2/2$, $W = 0$: indeed one can actually require that the particles do not meet for every $t \neq 0$.

Consider now a conservative solution made of finitely many Dirac deltas $\mu_0 = \sum_i^I m_i \delta_{\bar{q}_i,\bar{p}_i}$.

Proposition 3.8. *There exists a time interval* $(0,\bar{t})$*, independent of* μ_0*, such that for all* $\epsilon > 0$ *there is a finite particle solution*

$$\mu_0' = \sum_i \frac{m_i}{2}\big(\delta_{\bar{q}_i,p_{i,1}} + \delta_{\bar{q}_i,p_{i,2}}\big), \qquad \frac{p_{i,1}+p_{i,2}}{2} = \bar{p}_i,$$

such that the trajectories $\{q_{i,j}(t)\}$ *are not intersecting for* $t \in (0,\bar{t})$ *and*

$$\big|p_{i,j} - \bar{p}_i\big| < \epsilon.$$

Hence it is sufficient to split each particle $(\bar{q}_i,\bar{p}_i)$ in half, see Figure 3.1.

Proof. Being the conservative flow differentiable w.r.t. the initial data, we compute the derivative of the flow $\bar{Q}(t), \bar{P}(t)$ w.r.t. a perturbation of the form

$$\delta Q_1(0) = (\delta q_{i,1}(0))_i = 0, \;\; \delta Q_2(0) = (\delta q_{i,2}(0))_i = 0,$$
$$\delta P_1(0) = (\delta p_{i,1}(0))_i = -(\delta p_{i,2}(0))_i = -\delta P_2(0).$$

It is easy to see that the solution satisfies $(\delta Q_1, \delta P_1) = (\delta Q_2, \delta P_2)$ and that the ODE reduces to

$$\frac{d}{dt}\begin{pmatrix} \delta Q_1 \\ \delta P_1 \\ \delta Q_2 \\ \delta P_2 \end{pmatrix} = \mathrm{diag}\big(\nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar q_i(t), \bar p_i(t); \mu_t)\big) \begin{pmatrix} \delta Q_1 \\ \delta P_1 \\ \delta Q_2 \\ \delta P_2 \end{pmatrix},$$

$$\nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar q_i(t), \bar p_i(t); \mu_t) = \begin{bmatrix} \nabla_{q_i p_i} V & \nabla_{p_i p_i} V \\ -\nabla_{q_i q_i} V & -\nabla_{q_i p_i} V \end{bmatrix} (\bar q_i(t), \bar p_i(t))$$
$$+ \int \begin{bmatrix} \nabla_{q_i p_i} W & \nabla_{p_i p_i} W \\ -\nabla_{q_i q_i} W & -\nabla_{q_i p_i} W \end{bmatrix} (\bar q_i(t), \bar p_i(t), \bar q_j, \bar p_j) \mu(t; d\bar q_j d\bar p_j),$$

so that the ODEs are decoupled: $\mu(t) = \sum_i m_i \delta_{\bar q_i(t), \bar p_i(t)}$ is the solution of (9) with initial data $\mu_0 = \sum_i m_i \delta_{\bar q_i, \bar p_i}$.

To prove the same codimension estimate as in the proof of the previous proposition, it is sufficient to study the ODE

$$\dot A_i(t) = \nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar q_i(t), \bar p_i(t); \mu_t) A_i(t), \quad A_i(0) = \mathbb{I}.$$

The assumptions on V, W gives that

$$\big|\nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar q_i(t), \bar p_i(t); \mu_t)\big| \leq 3L,$$

so that one obtains

$$|B_i(t)| = \left| A_i(t) - \mathbb{I} - \int_0^t \nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar q_i(s), \bar p_i(s); \mu_s) ds \right| \leq \big(e^{3Lt} - 1 - 3Lt\big) < \frac{\Lambda}{2} t$$

for $t < \bar t$, with $\bar t$ depending only on L.

Hence

$$\begin{pmatrix} \delta q_{i,j}(t) \\ \delta p_{i,j}(t) \end{pmatrix} = \left[\mathbb{I} + \int_0^t \nabla_{q_i,p_i} \mathrm{J} \nabla_{q_i,p_i} H(\bar q_i(s), \bar p_i(s); \mu_s) ds + B_i(t) \right] \begin{pmatrix} 0 \\ \delta P_{i,j}(0) \end{pmatrix}$$
$$= \begin{pmatrix} \big[\int_0^t \nabla^2_{p_i,p_i} H(\bar q_i(s), \bar p_i(s); \mu_s) ds\big] \delta P_{i,j}(0) \\ \big[\mathbb{I} - \int_0^t \nabla_{q_i,p_i} H(\bar q_i(s), \bar p_i(s); \mu_s) ds\big] \delta P_{i,j}(0) \end{pmatrix} + B_i(t) \begin{pmatrix} 0 \\ \delta P_{i,j}(0) \end{pmatrix}.$$

Since from the uniform convexity of V

$$\int_0^t \nabla^2_{p_i,p_i} H(\bar q_i(s), \bar p_i(s); \mu_s) ds \geq \Lambda t \mathbb{I},$$

and $|B_i| \leq \Lambda t/2$ for $t \in [0, \bar t)$, then

$$\int_0^t \nabla^2_{p_i,p_i} H(\bar q_i(s), \bar p_i(s); \mu_s) ds + (B_i)_{1,2}$$

is invertible for $t \in (0, \bar t)$.

We thus conclude that the crossing condition

$$q_{i,j}(t) = q_{i',j'}(t), \quad \text{for some } t \in (0, \bar t), j \neq k,$$

gives a $(d-1)$-codimensional manifold in a neighborhood of $q_{i,j} = \bar q_i$. Hence there are perturbations $p_{i,j}(0) - \bar p_i(0)$ arbitrarily small so that the trajectories do not cross for $t \in (0, \bar t)$.

□

4 Dissipative solution

In this section we define the dissipative solutions for the Hamiltonian system (8), and we show some basic properties.

Definition 4.1. We say that $\eta \in \mathscr{M}(\Gamma)$ is a *dissipative solution* with initial speed $v_0 \in L^2_\eta(\Gamma, \mathbb{R}^d)$ if there is a function $v \in L^2_{\mathscr{L}^1 \times \eta}((0,T) \times \Gamma, \mathbb{R}^d)$ such that for $\mathscr{L}^1$-a.e. t

$$\dot\gamma(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t), \gamma'(t), v(t,\gamma), v(t,\gamma'))\eta(d\gamma'), \tag{13a}$$

$$v(t,\gamma) = \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s), v(s,\gamma))ds - \int_0^t \int \nabla_q W(\gamma(s), \gamma'(s), v(s,\gamma), v(s,\gamma'))\eta(d\gamma')ds\bigg), \tag{13b}$$

where $\mathbb{P}_t$ is the projection (6).

@ Clearly a particular dissipative solution is the conservative solution. We observe also that differently from the conservative case where the initial data is encoded into η, here it is not: for example, considering two particles starting at the same point but with different speed, we can just merge them at $t = 0$, so that their initial speed is different from the initial one. We will see later (Lemma 4.6) that, since one can take the dissipative solutions to be right continuous, we can specify the initial v_0 as the limit of $v(t)$ as $t \searrow 0$, and that in some sense the solution is characterized by the initial data *and* the family of projections $\mathbb{P}_t$ (by constructing an approximating sequence depending on the initial data and the projections, Section 4.1).

Remark 4.2. Equation (13a) is exactly the same as Equation (8a). The second equation (13b) expresses the requirement that when trajectories merge, the function v of the exiting trajectory will be the average of the function v of the incoming trajectories. On the case $V = v^2/2$, $W = W(x,x')$, then $\dot\gamma = v$, so that v coincides with the speed of the trajectory.

It is interesting to note that Equation (13a) is compatible with the projection $\mathbb{P}_t$ used in (13b): indeed, define the measure $\eta_{\bar t} = (\mathbb{T}_{\bar t})_\sharp \eta$, and by Proposition 3.2 find $\tilde v(t) \in L^2_{\eta_{\bar t}}(\Gamma_{\bar t}, \mathbb{R}^d)$ such that

$$\dot\gamma(t) = \nabla_v V(\gamma(t), \tilde v(t,\gamma)) + \int \nabla_v W(\gamma(t), \tilde v(t,\gamma), \gamma'(t), \tilde v(t,\gamma')))\eta_{\bar t}(d\gamma')$$

for $\mathscr{L}^1$-a.e. $t > \bar t$. This function $\tilde v$ is defined on the σ-algebra $\mathcal{B}(\bar t, T) \times \Omega_{\bar t}$, and writing

$$v(t,\gamma) = \tilde v(t, \mathbb{T}_{\bar t}(\gamma)),$$

one deduces immediately that

$$\dot\gamma(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t), v(t,\gamma), \gamma'(t), v(t,\gamma')))\eta(d\gamma'),$$

and Proposition 3.2 yields that this is the only solution for $t > \bar t$. In particular, by letting $\bar t \nearrow t$, one deduces that $v(t,\gamma(t))$ is measurable in Ω_t for $\mathscr{L}^1$-a.e. t.

Since (13a) gives a one-to-one relation between $\dot\gamma$ and v, we can state alternatively that

Definition 4.3. We say that $\eta \in \mathscr{M}(\Gamma)$ is a *dissipative solution* with initial speed $v_0 \in L^2_\eta(\Gamma, \mathbb{R}^d)$ if the function $v \in L^2_{\mathscr{L}^1 \times \eta}((0,T) \times \Gamma, \mathbb{R}^d)$ given by the relation

$$\dot{\gamma}(t) = \nabla_v V(\gamma(t), v(t,\gamma)) + \int \nabla_v W(\gamma(t), \gamma'(t), v(t,\gamma), v(t,\gamma'))\eta(d\gamma'),$$

satisfies

$$v(t,\gamma) = \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(s), v(s,\gamma))ds - \int_0^t \int \nabla_q W(\gamma(s), \gamma'(s), v(s,\gamma), v(s,\gamma'))\eta(d\gamma')ds\bigg),$$

where $\mathbb{P}_t$ is the projection (6).

We begin with a rough energy estimate.

Lemma 4.4. *For every dissipative solution*

$$\|(\gamma(t), v(t))\|_{L^2_\eta} \leq e^{3Lt}\|(\gamma(0), v_0)\|_{L^2_\eta} \quad \mathscr{L}^1\text{-a.e. } t.$$

The energy $E(t) = H((\gamma(t), v(t)_\sharp\eta)$ is actually decreasing, see Proposition 4.15 below. The above lemma shows that the requirement of $\eta \in \mathscr{M}(\Gamma)$ is compatible with the definition of dissipative solution as in Corollary 3.4, because of the relation between $\dot{\gamma}, v$ given by Proposition 3.2.

Proof. This is a standard Gronwall estimate.

Being $\mathbb{P}_t$ a contraction, we have by the Lipschitz estimates on $\nabla V, \nabla W$, Points (3), (1) of Page 9, applied to (13)

$$\|(\gamma(t), v(t))\|_{L^2_\eta} \leq \|(\gamma(0), v_0)\|_{L^2_\eta} + 3L\int_0^t \|(\gamma(s), v(s))\|_{L^2_\eta}dr,$$

for $\mathscr{L}^1$-a.e. t, which gives the statement. □

This next lemma is a concatenation property for dissipative solutions.

Lemma 4.5. *It holds*

$$v(t,\gamma) = \mathbb{P}_{s\to t}\bigg(v(s,\gamma) - \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg),$$

where $\mathbb{P}_{s\to t}$ is the projection (7)*.*

Proof. Using Remark 4.2, for $\mathscr{L}^1$-a.e. $r \geq s$

$$\begin{aligned}&\mathbb{P}_s\bigg(\nabla_q V(\gamma(r), v(r,\gamma))dr + \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')\bigg)\\ &= \nabla_q V(\gamma(r), v(r,\gamma))dr + \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma'),\end{aligned}$$

so that

$$\begin{aligned}&\int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr + \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\\ &= \mathbb{P}_s\bigg(\int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr + \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg).\end{aligned} \tag{14}$$

Hence, directly from the definition of $v(t,\gamma)$ and $\mathbb{P}_t, \mathbb{P}_{s\to t}$

$$
\begin{aligned}
&\mathbb{P}_{s\to t}\bigg(v(s,\gamma) - \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg) \\
&= \mathbb{P}_{s\to t}\bigg(\mathbb{P}_s\bigg(v_0(\gamma) - \int_0^s \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_0^s \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg)\bigg) \\
&\quad + \mathbb{P}_{s\to t}\bigg(- \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg) \\
&= \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^s \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_0^s \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg) \\
&\quad + \mathbb{P}_{s\to t}\bigg(\mathbb{P}_s\bigg(- \int_s^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_s^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg) \\
&= \mathbb{P}_t\bigg(v_0(\gamma) - \int_0^t \nabla_q V(\gamma(r), v(r,\gamma))dr - \int_0^t \int \nabla_q W(\gamma(r), \gamma'(r), v(r,\gamma), v(r,\gamma'))\eta(d\gamma')dr\bigg),
\end{aligned}
$$

where we have used (14) in the third inequality. □

The next estimate plays a key role in the following.

Lemma 4.6. *Let $\eta \in \mathscr{M}(\Gamma)$ be a dissipative solution. Then the map $t \to v(t,\gamma)$ is right continuous and belongs to $\mathrm{BV}^{\frac{1}{2}}([0,T], L^2_\eta(\Gamma, \mathbb{R}^d)))$ with norm*

$$
\|v\|_{\mathrm{BV}_t^{1/2} L^2_\eta} \leq (1 + 6LT)e^{3LT} \|(\gamma(0), v_0)\|^2_{L^2_\eta}. \tag{15}
$$

Let us denote

$$
\nabla_q H(\gamma(t), v(t); \eta) = -\nabla_q V(\gamma(t), v(\gamma)) - \int \nabla_q W(\gamma(t), \gamma'(t), v(\gamma), v(\gamma'))\eta(d\gamma').
$$

The proof stems from the fact that $v(t,\gamma)$ is the integral of the L^2_η-function $\nabla_q H(\gamma(t), v(t); \eta)$ w.r.t. time (which would give the a.c. continuity), composed with projection $\mathbb{P}_t$ (which is responsible for energy dissipation and hence for the $\mathrm{BV}^{1/2}$-norm).

Proof. We compute by Lemma 4.4

$$
\begin{aligned}
\sum_i \|v(t_i)-v(t_{i-1})\|_2^2 &= \sum_i \Big[\|v(t_i)\|_2^2+\|v(t_{i-1})\|_{L^2_\eta}^2 - 2\int (v(t_i),v(t_{i-1}))\eta\Big]\\
&= \sum_i \Big[\|v(t_i)\|_{L^2_\eta}^2+\|v(t_{i-1})\|_{L^2_\eta}^2 - 2\int (v(t_i),\mathbb{P}_{t_{i-1}\to t_i} v(t_{i-1}))\eta\Big]\\
&= \sum_i \Big[\|v(t_i)\|_{L^2_\eta}^2+\|v(t_{i-1})\|_{L^2_\eta}^2\\
&\qquad - 2\int \Big(v(t_i),\mathbb{P}_{t_{i-1}\to t_i}\Big(v(t_{i-1})+\int_{t_{i-1}}^{t_i} G(r,v(r))dr\Big)\Big)\eta\\
&\qquad + 2\int \Big(v(t_i),\mathbb{P}_{t_{i-1}\to t_i}\Big(\int_{t_{i-1}}^{t_i} G(r,v(r))dr\Big)\Big)\eta\Big]\\
&= \sum_i \Big[\|v(t_{i-1})\|_{L^2_\eta}^2-\|v(t_i)\|_{L^2_\eta}^2 + 2\int \Big(v(t_i),\mathbb{P}_{t_{i-1}\to t_i}\Big(\int_{t_{i-1}}^{t_i} G(r,v(r))dr\Big)\Big)\eta\Big]\\
&= \|v(0)\|_{L^2_\eta}^2-\|v(T)\|_{L^2_\eta}^2 + 2\sum_i\int \Big(v(t_i),\mathbb{P}_{t_{i-1}\to t_i}\Big(\int_{t_{i-1}}^{t_i} G(r,v(r))dr\Big)\Big)\eta\\
&\le \|v(0)\|_{L^2_\eta}^2-\|v(T)\|_{L^2_\eta}^2 + 6LT\sup_t \|v(t)\|_{L^2_\eta}^2\\
&\le (6LT+1)\sup_t \|v(t)\|_{L^2_\eta}^2 \le (1+6LT)e^{3LT}\|(\gamma(0),v_0)\|_{L^2_\eta}^2,
\end{aligned}
$$

where in the first inequality we have used the Lipschitz estimate

$$
\big\|\nabla_q H(\gamma(t),v_1(t);\eta)-\nabla_q H(\gamma(t),v_2(t);\eta)\big\|_{L^2_\eta} \le 3L\|v_1-v_2\|_{L^2_\eta}
$$

analogous to the second inequality of Proposition 3.2.

The $\mathrm{BV}_t^{1/2}L^2_\eta$ regularity gives immediately that the function $t\mapsto v(t)$ is strongly continuous in L^2_η outside countably many times. Moreover, as $t\searrow s$, Lemma 2.1 gives

$$
\begin{aligned}
&\lim_{t\searrow s}\mathbb{P}_t\Big(v(s,\gamma)-\int_s^t \nabla_q V(\gamma(r),v(r,\gamma))dr - \int_s^t\int \nabla_q W(\gamma(r),\gamma'(r),v(r,\gamma),v(r,\gamma'))\eta(d\gamma')dr\Big)\\
&= v(s,\gamma)
\end{aligned}
$$

in L^2_η, which is the right continuity property. □

In particular, by Proposition 3.2 we obtain $\dot\gamma\in \mathrm{BV}_t^{1/2}L^2_\eta$ and one can take as initial data

$$
v_0(\gamma)=\lim_{t\searrow 0} v(t,\gamma).
$$

This will be our choice in the following.

The next results use the $\mathrm{BV}_t^{1/2}L^2_\eta$ estimate on $\dot\gamma(t),v(t)$ to deduce some useful approximation properties: these results are actually valid for generic $\mathrm{BV}^{1/2}X$ functions, we state them in the particular form we will use.

Lemma 4.7. *For all η dissipative, it holds*

$$\int_0^{T-s}\int\left|\frac{\gamma(t+s)-\gamma(t)}{s}-\dot\gamma(t)\right|^2\eta(d\gamma)dt\le\Big(2(1+6LT)e^{3LT}\|(\gamma(0),v_0)\|^2_{L^2_\eta}\Big)s=\mathcal{O}(1)s.$$

Proof. Write

$$\begin{aligned}\int_0^{T-s}\int\left|\frac{\gamma(t+s)-\gamma(t)}{s}-\dot\gamma(t)\right|^2\eta(d\gamma)dt&=\int_0^{T-s}\int\left|\frac{1}{s}\int_0^s(\dot\gamma(t+\sigma)-\dot\gamma(t))d\sigma\right|^2\eta(d\gamma)dt\\
&\le\frac{1}{s}\int_0^s\int_0^{T-s}\int|\dot\gamma(t+\sigma)-\dot\gamma(t)|^2\eta(d\gamma)dtd\sigma\\
&=\frac{1}{s}\int_0^s\left\{\sum_{k=0}^{[T/s]-2}\int_{ks}^{(k+1)s}+\int_{([T/s]-1)s}^{T-s}\right\}\left[\int|\dot\gamma(t+\sigma)-\dot\gamma(t)|^2\eta(d\gamma)\right]dtd\sigma\\
&=\frac{1}{s}\int_0^s\int_0^s\left[\sum_{k=0}^{[T/s]-2}\int|\dot\gamma(ks+\tau+\sigma)-\dot\gamma(ks+\tau)|^2\eta(d\gamma)\right]d\tau d\sigma\\
&\quad+\frac{1}{s}\int_0^s\int_{([T/s]-1)s}^{T-s}\left[\int|\dot\gamma(t+\sigma)-\dot\gamma(t)|^2\eta(d\gamma)\right]dtd\sigma.\end{aligned}$$

The first integral is estimated as

$$\begin{aligned}\frac{1}{s}\int_0^s\int_0^s\left[\sum_{k=0}^{[T/s]-2}\int|\dot\gamma(ks+\tau+\sigma)-\dot\gamma(ks+\tau)|^2\eta(d\gamma)\right]d\tau d\sigma&\le\frac{1}{s}\int_0^s\int_0^s\|\dot\gamma\|_{\mathrm{BV}^{1/2}_t(L^2_\eta)}d\tau d\sigma\\
&\le(1+6LT)e^{3LT}\|(\gamma(0),v_0)\|^2_{L^2_\eta}s,\end{aligned}$$

where we have used the estimate (15). Similarly for the last integral

$$\begin{aligned}\frac{1}{s}\int_0^s\int_{([T/s]-1)s}^{T-s}\left[\int|\dot\gamma(t+\sigma)-\dot\gamma(t)|^2\eta(d\gamma)\right]dtd\sigma&\le\frac{1}{s}\int_0^s\int_{([T/s]-1)s}^{T-s}\|\dot\gamma\|_{\mathrm{BV}^{1/2}_t(L^2_\eta)}dtd\sigma\\
&\le(1+6LT)e^{3LT}\|(\gamma(0),v_0)\|^2_{L^2_\eta}s.\end{aligned}$$

Adding the two estimates we obtain the statement. □

Lemma 4.8. *For every $\epsilon>0$ we can find finitely many times $0=t_0<t_1<\cdots<t_N=T$ such that*

$$\sup_{i,t_{i-1}\le s<t_i}\|v(s)-v(t_{i-1})\|_{L^2_\eta}\le\epsilon.$$

Proof. Since $t\mapsto v(t)$ is right continuous, for every t there is δ_t such that

$$\sup_{0\le\tau<\delta_t}\|v(t+\tau)-v(t)\|_{L^2_\eta}<\epsilon,\quad\|v(t+\delta_t)-v(t)\|_{L^2_\eta}\ge\epsilon.$$

Starting from $t=0$, define the sequence of times

$$t_{i+1}=t_{i-1}+\delta_{t_i},\quad t_0=0.$$

The $\mathrm{BV}^{1/2}_t L^2_\eta$-regularity gives that

$$\epsilon\sharp\{t_i\} \leq \sum_i \|v(t_i) - v(t_{i-1})\|^2_{L^2_\eta} \leq (1 + 6LT)e^{3LT}\|(\gamma(0), v_0)\|^2_{L^2_\eta},$$

so that there are at most $\mathcal{O}(\epsilon^{-1})$-many times t_i. ☐

The source of the discontinuities of the map $t \mapsto v(t)$ is due to the projection $\mathbb{P}_t$: in order to get rid of it, define the function

$$\begin{aligned}
\tilde{v}(t, s, \gamma) &= v(s, \gamma) - \int_s^t \nabla_q H(\gamma(\tau), v(\tau, \gamma); \eta) d\tau \\
&= v(s, \gamma) - \int_s^t \bigg[\nabla_q V(\gamma(\tau), v(\tau, \gamma)) ds + \int \nabla_q W(\gamma(\tau), v(\tau, \gamma), \gamma'(\tau), v(\tau, \gamma'))\eta(d\gamma')\bigg] d\tau,
\end{aligned}$$

so that it holds for all $s \in [0, t]$

$$v(t, \gamma) = \mathbb{P}_t(\tilde{v}(t, s))(\gamma).$$

Lemma 4.9. *It holds for $s \leq t$*

$$\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t, s)\|_{L^2_\eta} \leq \|v(t) - v(s)\|_{L^2_\eta} + C(t - s).$$

Proof. Indeed

$$\begin{aligned}
\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t, s)\|_{L^2_\eta} &\leq \|v(t) - v(s)\|_{L^2_\eta} + \int_s^t \|\nabla_q H(\gamma(\tau), v(\tau))\|_{L^2_\eta} d\tau \\
&\leq \|v(t) - v(s)\|_{L^2_\eta} + C(t - s)\|(\gamma(s), v(s))\|_{L^2_\eta}.
\end{aligned}$$ ☐

4.1 Piecewise conservative approximations to dissipative solutions

Aim of the next statements is to compare a dissipative solution and the conservative solution with the same initial data, and to construct a piecewise conservative approximation. To shorten the notation, we will write

$$\begin{aligned}
Q(t, \gamma; s, \eta) &= Q\big(t - s, \gamma(s), v(s, \gamma), (\gamma(s), v(s))_\sharp\eta\big) \\
P(t, \gamma; s, \eta) &= P\big(t - s, \gamma(s), v(s, \gamma), (\gamma(s), v(s))_\sharp\eta\big)
\end{aligned}$$

for the lifting of the conservative solution starting at time s with initial measure $(\gamma(s), v(s))_\sharp\eta$.

The first result is that $(Q(t, \gamma; s, \eta), P(t, \gamma; s, \eta)$ well approximates $(\gamma(t), \tilde{v}(t, s))$.

Lemma 4.10. *It holds*

$$\begin{aligned}
&\big\|\mathbb{P}_t(Q(t, \gamma; s, \eta), P(t, \gamma; s, \eta)) - (\gamma(t), v(t))\big\|_{L^2_\eta} \\
&\quad \leq \big\|(Q(t, \gamma; s, \eta), P(t, \gamma; s, \eta)) - (\gamma(t), \tilde{v}(t, s))\big\|_{L^2_\eta} \leq C \int_s^t \|v(\tau) - \tilde{v}(\tau)\|_{L^2_\eta} d\tau.
\end{aligned}$$

Note that instead

$$\big\|(Q(t, \gamma; s, \eta), P(t, \gamma, s, \eta)) - (\gamma(t), v(t))\big\|_{L^2_\eta} = \mathcal{O}(1)\big(\|v(t) - v(s)\|_{L^2_\eta} + t\big), \tag{16}$$

hence by comparing the projection $\mathbb{P}_t(Q, P)$ we gain an estimate which is regular in time.

Proof. It is enough to set $s = 0$. We can now compute for $0 \le t \le \bar{t}$

$$
\begin{aligned}
\big\|\mathbb{P}_t(Q(t,\gamma;0,\eta),P(t,\gamma;0,\eta)) - (\gamma(t),v(t))\big\|_{L^2_\eta} &\le \big\|(Q(t,\gamma;0,\eta),P(t,\gamma;0,\eta)) - (\gamma(t),\tilde v(t))\big\|_{L^2_\eta} \\
&\le \int_0^t \big\|\nabla H(Q(s,0),P(s,0)) - \nabla H(\gamma(s),v(s))\big\|_{L^2_\eta} ds \\
&\le 3L \int_0^t \big\|(Q(s,0),P(s,0)) - (\gamma(s),v(s))\big\|_{L^2_\eta} ds \\
&\le 3L \int_0^t \big\|(Q(s,0),P(s,0)) - (\gamma(s),\tilde v(s))\big\|_{L^2_\eta} ds \\
&\quad + 3L \int_0^t \|v(s) - \tilde v(s)\|_{L^2_\eta} ds.
\end{aligned}
$$

Hence by Gronwall's estimate we obtain

$$
\big\|(Q(t,\gamma;0,\eta),P(t,\gamma;0,\eta)) - (\gamma(t),\tilde v(t))\big\|_{L^2_\eta} \le \int_0^t 3Le^{3L(t-s)}\|v(s)-\tilde v(s)\|_{L^2_\eta} ds,
$$

which is the second inequality in the statement. The first inequality is deduce from the fact that $\mathbb{P}_t$ is a contraction. □

In Section 6 we need a similar estimate as in the above lemma, but for the backward flow: the trivial estimate one obtains from Lemmas 4.9, 4.10 would give

$$
\big\|(Q(s,\gamma;t,\eta),P(s,\gamma;t,\eta)) - (\gamma(s),v(s))\big\|_{L^2_\eta} \le C\big((t-s) + \|v(t) - \tilde v(t,s)\|_{L^2_\eta}\big).
$$

The next corollary instead shows that we can get rid of the second term in the r.h.s. above by considering the conservative solution starting from $(\gamma(t),\tilde v(t,\gamma))$ instead of $(\gamma(t),v(t,\gamma))$.

Corollary 4.11. *It holds*

$$
\begin{aligned}
&\Big\|\Big(Q\big(s-t,\gamma(t),\tilde v(t,\gamma);(\gamma(t),\tilde v(t))_\sharp\eta\big),P\big(s-t,\gamma(t),\tilde v(t,\gamma);(\gamma(t),\tilde v(t))_\sharp\eta\big)\Big) - \big(\gamma(s),v(s)\big)\Big\|_{L^2_\eta} \\
&\qquad\qquad \le C\int_s^t \|v(\tau)-\tilde v(\tau)\|_{L^2_\eta} d\tau.
\end{aligned}
$$

Recall that

$$
s \mapsto Q\big(s,\gamma(t),\tilde v(t,\gamma);(\gamma(t),\tilde v(t))_\sharp\eta\big),P\big(s,\gamma(t),\tilde v(t,\gamma);(\gamma(t),\tilde v(t))_\sharp\eta\big)
$$

is the solution to the conservative flow starting at time t with measure $(\gamma(t),\tilde v(t))_\sharp\eta$.

Proof. The statement is a consequence of the backward stability estimate for the conservative flow and Lemma 4.10:

$$
\begin{aligned}
&\Big\|\big(Q(s,\gamma(t),\tilde v(t,\gamma);(\gamma(t),v(t))_\sharp\eta),P(s,\gamma(t),\tilde v(t,\gamma);(\gamma(t),v(t))_\sharp\eta)\big) - (\gamma(s),v(s))\Big\|_{L^2_\eta} \\
&\le C\Big\|(\gamma(t),\tilde v(t)) - \big(Q(t,\gamma;s,\eta),P(t,\gamma;s,\eta)\big)\Big\|_{L^2_\eta} \\
&\le C\int_s^t \|v(\tau)-\tilde v(\tau)\|_{L^2_\eta} d\tau.
\end{aligned}
$$

□

We now define the piecewise conservative solutions $\tilde{Q}(t,\gamma;\eta), \tilde{P}(t,\gamma;\eta)$ by alternating the conservative flow $Q(t,q,p;\mu_0), P(t,q,p;\mu_0)$ with the projection operator $\mathbb{P}_t$: if $0 = t_0 < t_1 < \dots < t_N = T$, define for $t \in [0, t_1)$

$$\tilde{Q}(t,\gamma;\eta) = Q(t,\gamma(0),v_0(\gamma);\mu_0),\ \ \tilde{P}(t,\gamma;\eta) = P(t,\gamma(0),v_0(\gamma);\mu_0), \quad \mu_0 = (\gamma(0),v_0(\gamma))_\sharp \eta,$$

$$q_{t_1}(\gamma) = \mathbb{P}_{t_1}(\tilde{Q}(t_1,\gamma;\eta)), \quad p_{t_1}(\gamma) = \mathbb{P}_{t_1}(\tilde{P}(t_1,\gamma;\eta)),$$

and if $q_{t_i}(\gamma), p_{t_i}(\gamma)$ have been constructed, set for $t \in [t_i, t_{i+1})$

$$\mu_{t_i} = (q_{t_i}(\gamma), p_{t_i}(\gamma))_\sharp \eta,$$

$$\tilde{Q}(t,\gamma;\eta) = Q(t-t_i, q_{t_i}(\gamma), p_{t_i}(\gamma); \mu_{t_i}),\ \ \tilde{P}(t,\gamma;\eta) = P(t-t_i, q_{t_i}(\gamma), p_{t_i}(\gamma); \mu_{t_i}),$$
$$q_{t_{i+1}}(\gamma) = \mathbb{P}_{t_{i+1}}(\tilde{Q}(t_{i+1},\gamma;\eta)), \quad p_{t_{i+1}}(\gamma) = \mathbb{P}_{t_{i+1}}(\tilde{P}(t_{i+1},\gamma;\eta)),$$

Proposition 4.12. *For every $\epsilon > 0$ there exists a piecewise conservative approximation $(\tilde{Q},\tilde{P})$ of the dissipative solution η such that*

$$\big\|(\tilde{Q}(t,\gamma;\eta), \tilde{P}(t,\gamma;\eta)) - (\gamma(t), v(t))\big\|_{L^2_\eta} \leq C\epsilon T.$$

Proof. Let $\{t_i\}_i$ be the partition of Lemma 4.8: w.l.o.g. we can assume that

$$|t_{i+1} - t_i| \leq \epsilon, \quad i = 0, \dots, N-1. \tag{17}$$

Lemma 4.9 applied to Lemma 4.10 gives that

$$\big\|\mathbb{P}_{t_{i+1}}(Q(t_{i+1},\gamma;t_i,\eta), P(t_{i+1},\gamma;t_i,\eta)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \leq C\epsilon(t_{i+1} - t_i).$$

In each interval $[t_i, t_{i+1})$ we thus estimate by the continuous dependence of the conservative flow

$$\begin{aligned}
&\big\|(\tilde{Q}(t,\gamma;\eta), \tilde{P}(t,\gamma;\eta)) - (Q(t,\gamma;t_i,\eta), P(t,\gamma;t_i,\eta))\big\|_{L^2_\eta} \\
&\leq e^{3L(t-t_i)}\big\|(\tilde{Q}(t_i,\gamma;\eta), \tilde{P}(t_i,\gamma;\eta)) - (Q(t_i,\gamma;t_i,\eta), P(t_i,\gamma;t_i,\eta))\big\|_{L^2_\eta} \\
&= e^{3L(t-t_i)}\big\|(q_{t_i}(\gamma), p_{t_i}(\gamma)) - (\gamma(t_i), v(t_i))\big\|_{L^2_\eta}.
\end{aligned} \tag{18}$$

In particular we obtain

$$\begin{aligned}
&\big\|(q_{t_{i+1}}(\gamma), p_{t_{i+1}}(\gamma)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \\
&\leq \big\|(q_{t_{i+1}}(\gamma), p_{t_{i+1}}(\gamma)) - \mathbb{P}_{t_{i+1}}((Q(t_{i+1},\gamma;t_i,\eta), P(t_{i+1},\gamma;t_i,\eta))\big\|_{L^2_\eta} \\
&\quad + \big\|\mathbb{P}_{t_{i+1}}(Q(t,\gamma;t_i,\eta), P(t,\gamma;t_i,\eta)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \\
&\leq \big\|(\tilde{Q}(t_{i+1},\gamma;\eta), \tilde{P}(t_{i+1},\gamma;\eta)) - (Q(t_{i+1},\gamma;t_i,\eta), P(t_{i+1},\gamma;t_i,\eta))\big\|_{L^2_\eta} \\
&\quad + \big\|\mathbb{P}_{t_{i+1}}(Q(t,\gamma;t_i,\eta), P(t,\gamma;t_i,\eta)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \\
&\leq e^{3L(t_{i+1}-t_i)}\big\|(q_{t_i}(\gamma), p_{t_i}(\gamma) - (\gamma(t_i), v(t_i))\big\|_{L^2_\eta} + C\epsilon(t_{i+1} - t_i).
\end{aligned}$$

From the explicit solution to the difference equation

$$a_i = \lambda_i a_{i-1} + b_i, \quad a_i = \bigg(\prod_{j=1}^{i} \lambda_j\bigg) a_0 + \sum_{j=1}^{i} \bigg(\prod_{k=j+1}^{i} \lambda_k\bigg) b_j, \tag{19}$$

with the convention $\prod_\emptyset \lambda = 1$, we conclude that

$$\begin{aligned} &\big\|(q_{t_{i+1}}(\gamma), p_{t_{i+1}}(\gamma)) - (\gamma(t_{i+1}), v(t_{i+1}))\big\|_{L^2_\eta} \\ &\leq \sum_{j=1}^{i} \bigg(\prod_{k=j+1}^{i} e^{3L(t_{i+1}-t_i)}\bigg) C\epsilon(t_{i+1} - t_i) \leq CT\epsilon. \end{aligned} \tag{20}$$

By (18), (16) and the choice of the intervals as in Lemma 4.8 and (17), we obtain the statement. □

Remark 4.13. The proof above shows that a dissipative solution is uniquely characterized by the initial data and the family of projections $\{\mathbb{P}_t\}_t$. However, these projections are such that $\mathbb{P}_t\gamma(t) = \gamma(t)$: one must check that in the limit the trajectories $t \mapsto \gamma(t)$ are a.c., which is not the case for $t \mapsto \tilde{Q}(t, \gamma)$.

4.2 Some useful estimates for dissipative solutions

We now show that the energy

$$\begin{aligned} t \mapsto E(t) &= H((\gamma(t), v(t))_\sharp \eta) \\ &= \int V(\gamma(t), v(t,\gamma))\eta(d\gamma) + \int\int W(\gamma(t), v(t,\gamma), \gamma'(t), v(t,\gamma'))\eta(d\gamma)\eta(d\gamma') \end{aligned}$$

is decreasing, and relate its decrease with the distance of the dissipative solution to the conservative one. The first step is the following estimate.

Lemma 4.14. *It holds for $s \leq t$*

$$\Lambda \|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t,s)\|_2^2 \leq C \int_s^t \|(\mathbb{I} - \mathbb{P}_\tau)\tilde{v}(\tau,s)\|_2^2 ds + E(s) - E(t). \tag{21}$$

Proof. We compute

$$\begin{aligned} E(t) - E(s) &= H((\gamma(t), \mathbb{P}_t\tilde{v}(t,s))_\sharp\eta) - H((\gamma(s), \tilde{v}(s,s))_\sharp\eta) \\ &\leq -\Lambda\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t,s)\|_2^2 + H((\gamma(t), \tilde{v}(t,s))_\sharp\eta) - H((\gamma(s), \tilde{v}(s,s))_\sharp\eta) \\ &= -\Lambda\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t,s)\|_2^2 + \int_s^t \int \nabla H(\gamma(\tau), \tilde{v}(\tau,s)) \mathrm{J} \nabla H(\gamma(\tau), v(\tau))\eta(d\gamma)d\tau. \end{aligned}$$

The latter integrand is for $V, W \in C^{2,1}$

$$\begin{aligned} &\int \nabla H(\gamma(\tau), \tilde{v}(\tau,s)) \mathrm{J} \nabla H(\gamma(\tau), v(\tau))\eta(d\gamma) \\ &= \int \Big[\nabla H(\gamma(\tau), \mathbb{P}_\tau\tilde{v}(\tau,s)) + \nabla_v \nabla H(\gamma(\tau), \mathbb{P}_\tau\tilde{v}(\tau,s))(\tilde{v}(\tau,s) - \mathbb{P}_\tau\tilde{v}(\tau,s)) \\ &\qquad + \mathcal{O}((\mathbb{I} - \mathbb{P}_\tau)\tilde{v}(\tau,s))^2\Big] \mathrm{J}\nabla H(\gamma(\tau), v(\tau))\eta(d\gamma) \\ &\leq C\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t)\|_2^2, \end{aligned}$$

where we used that for every $g \in L^2_\eta$ by the very definition of projection

$$\int \big(\tilde{v}(\tau, s) - \mathbb{P}_\tau \tilde{v}(\tau, s)\big)\big)\mathbb{P}_\tau g\eta = 0.$$

This is the desired estimate. □

Proposition 4.15. *The energy $E(t)$ is decreasing in time and it holds*

$$\big\|(Q(t,\gamma;s,\eta), P(t,\gamma;s,\gamma)) - (\gamma(t), \tilde{v}(t,\gamma))\big\|_{L^2_\eta} \le C(t-s)\sqrt{E(s) - E(t)},$$

$$\big\|(Q(t,\gamma;s,\eta), P(t,\gamma;s,\gamma)) - (\gamma(t), v(t,\gamma))\big\|_{L^2_\eta} \le C\sqrt{E(s) - E(t)},$$

Proof. From Lemma 4.14 and the right continuity of $t \mapsto v(t)$ we deduce that

$$\begin{aligned}\limsup_{t \searrow s} \frac{E(t) - E(s)}{t - s} &\le \limsup_{t \searrow s} \frac{1}{t-s} \int_s^t \|(\mathbb{I} - \mathbb{P}_\tau)\tilde{v}(\tau, s)\|_2^2 d\tau \\ &\le \limsup_{s \searrow 0} \frac{1}{2}\|v(t) - v(s)\|_2^2 = 0.\end{aligned}$$

Hence $t \mapsto E(t)$ is decreasing.

A Gronwall estimate for (21) gives

$$\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t)\|_2^2 \le -\int_0^t e^{C(t-s)} DE(ds) \le C\big(E(0) - E(t)\big),$$

where DE is the measure derivative of the decreasing function $E(t)$, and then by Lemma 4.10

$$\begin{aligned}\big\|(Q(t,\gamma;s,\eta), P(t,\gamma;s,\gamma)) - (\gamma(t), \tilde{v}(t,\gamma))\big\|_{L^2_\eta} &\le C\int_s^t \big\|(\mathbb{I} - \mathbb{P}_\tau)\tilde{v}(\tau)\big\|_{L^2_\eta} d\tau \\ &\le C(t-s)\sqrt{E(s) - E(t)},\end{aligned}$$

$$\begin{aligned}&\big\|(Q(t,\gamma;0,\eta), P(t,\gamma;0,\gamma)) - (\gamma(t), v(t,\gamma))\big\|_{L^2_\eta} \\ &\le \big\|(Q(t,\gamma;0,\eta), P(t,\gamma;0,\gamma)) - (\gamma(t), \tilde{v}(t,\gamma))\big\|_{L^2_\eta} + \big\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t)\big\|_{L^2_\eta} \\ &\le C(t-s)\sqrt{E(s) - E(t)} + \big\|(\mathbb{I} - \mathbb{P}_t)\tilde{v}(t)\big\|_{L^2_\eta} \\ &\le C\sqrt{E(s) - E(t)}.\end{aligned}$$

This concludes the proof. □

4.3 Some special cases for dissipative solutions

We conclude this section with some special cases, namely when the data are a finite number of Dirac deltas and when the Hamiltonian is purely quadratic.

Lemma 4.16. *Assume that*

$$\mu_0 = \sum_{n=1}^{N} m_n \delta_{(q_n,q_n)}$$

and let $Q(t,q_n,p_n;\mu_0), P(t,q_n,p_n;\mu_0)$ be the conservative solution with initial condition μ_0. If the trajectories $\{Q(t,q_n,p_n;\mu_0)\}_n$ do not intersect, then there is a unique dissipative solution with initial data μ_0: in particular it coincides with the conservative one.

Proof. The proof is immediate by observing that since the trajectories never meet then $\mathbb{P}_t = \mathbb{I}$ for all $t \geq 0$. $\square$

In the case of

$$V(x,v) = \frac{1}{2}(q,p)^T A(q,p), \quad W(x,v,x',v') = \frac{1}{2}(x-x',v-v')^T B(q-q',p-p'), \tag{22}$$

i.e. V, W are quadratic, with

$$A^T = A, \ A_{22} \geq \Lambda\mathbb{I}, \quad B^T = B \geq 0,$$

then the trajectories of the dissipative solution can be computed by projecting the solution to the conservative one, as in the standard pressurless dynamics. Indeed, the ODEs for the trajectories are

$$\frac{d}{dt}\begin{pmatrix} Q(t,q,p;\mu_0) \\ P(t,q,p;\mu_0) \end{pmatrix} = \mathrm{J}(A+B)\begin{pmatrix} Q(t,q,p;\mu_0) \\ P(t,q,p;\mu_0) \end{pmatrix} + \int \mathrm{J}B \begin{pmatrix} Q(t,q',p';\mu_0) \\ P(t,q',p';\mu_0) \end{pmatrix} \mu_0(dq'dp')$$

Hence assuming

$$\int (q,p)\mu_0(dqdp) = 0 \quad \Rightarrow \quad \int \begin{pmatrix} Q(t,q,p;\mu_0) \\ P(t,q,p;\mu_0) \end{pmatrix} \mu_0(dqdp) = 0$$

(i.e. it is preserved in time), we obtain

$$\begin{pmatrix} Q(t,q,p;\mu_0) \\ P(t,q,p:\mu_0) \end{pmatrix} = e^{\mathrm{J}(A+B)t}\begin{pmatrix} q \\ p \end{pmatrix},$$

i.e. the conservative flow is independent from μ_0.

Next, consider the piecewise conservative solution constructed in Proposition 4.12: being the projection operator linear, it follows that

$$(\tilde{Q}(t,\gamma;0,\eta), \tilde{P}(t,\gamma;0,\eta)) = e^{\mathrm{J}(A+B)t}\mathbb{P}_t(\gamma(0), v_0(\gamma)).$$

Being the above formula independent of the approximation parameter ϵ, we conclude that

Proposition 4.17. *If V, W are quadratic, and η is a dissipative solution with associated descending fibration $\{\Omega_t\}_t$ and projections $\mathbb{P}_t$, then*

$$(\gamma(t), v(t,\gamma)) = e^{\mathrm{J}(A+B)t}\mathbb{P}_t(\gamma(0), v_0).$$

For the quadratic case (22), a converse of Lemma 4.16 holds: if there is only the conservative solution, then the particle trajectories do not intersect. Note that the examples in [1] show that the existence of a conservative solution with non intersecting trajectories does not implies that all solutions are conservative (hence there is only one).

Proposition 4.18. *If V, W are quadratic and the only solution is the conservative one, then η is concentrated on a family of non-intersecting curves.*

Proof. We need the following duality result [13]: if $\nu \in \mathscr{P}(X)$, $\nu' \in \mathscr{P}(X')$, X, X' Polish, $\mathcal{Z} \subset X \times X'$ Borel, and

$$\Pi^{\leq}(\nu,\nu') = \bigg\{ \pi \text{ Borel measure}, \int \phi(x_1)\pi(dx_1dx_2) \leq \int \phi\nu, \int \phi(x_2)\pi(dx_1dx_2) \leq \int \phi\nu' \bigg\},$$

then

$$\sup\big\{\pi(\mathcal{Z}), \pi \in \Pi^{\leq}(\nu,\nu')\big\} = \min\big\{\nu(A) + \nu'(A'), \mathcal{Z} \subset A \times X' \cup X \times A', A, A' \text{ Borel}\big\}. \tag{23}$$

Let η be a dissipative solution such that $(\gamma(0), v_0)_\sharp \eta = \mu_0$. Consider the set

$$\mathcal{Z} = \Big\{(\gamma,\gamma') : \Gamma \times \Gamma : \gamma \neq \gamma' \text{ and } \exists t \in [0,T] \text{ such that } \gamma(t) = \gamma'(t)\Big\},$$

and assume that there is $\pi \in \Pi^{\leq}(\eta,\eta)$ such that $\pi(\mathcal{Z}) > 0$: we can require π to be symmetric, i.e.

$$\int \phi(\gamma,\gamma')\pi(d\gamma d\gamma') = \int \phi(\gamma',\gamma)\pi(d\gamma d\gamma'),$$

because $\mathcal{Z}$ is symmetric. Let

$$\pi = \int \pi_\gamma \eta(d\gamma)$$

be the (not normalized) disintegration.

Define the map

$$(\gamma,\gamma') \mapsto \tau_{\gamma,\gamma'} = \arg\min\big\{\gamma(t) = \gamma'(t)\big\}.$$

and define

$$\varpi(d\gamma d\gamma') = \pi + (\mathbb{I},\mathbb{I})_\sharp(\eta - (P_1)_\sharp \pi).$$

This does not correspond to a measure on Γ, and indeed the same curve γ intersects many others γ': however $(\mathbb{P}_1)_\sharp \varpi$ is. Let

$$E_t = \tau^{-1}_{\gamma,\gamma'}([0,t]),$$

i.e. the couples of curves which cross before t, let $\tilde{\mathbb{P}}_t$ be the corresponding projection, and set

$$t \mapsto (\tilde\gamma(t), \tilde\gamma'(t)) = \tilde{\mathbb{P}}_t e^{\mathrm{J}(A+B)t}(\gamma(0),\gamma'(0)).$$

It is fairly easy to see that $t \mapsto \tilde\gamma(t)$ is continuous, and then that $(\tilde\gamma)_\sharp\eta$ is a dissipative solution verifying Proposition 4.17. Hence from the assumption that there are no dissipative solutions we deduce that

$$\sup\big\{\pi(\mathcal{Z}), \pi \in \Pi^{\leq}(\eta,\eta)\big\} = 0$$

The duality (23) implies that there is an η-negligible set $N = A \cup A'$ such that $\mathcal{Z} \subset N \times N$, and then the proposition is proved. $\square$

In general, when the Hamiltonian is not purely quadratic, it may happen that

$$\sup\big\{\pi(\mathcal{Z}), \pi \in \Pi^{\leq}(\eta,\eta)\big\} > 0$$

even if the unique solution is the conservative one: we will give an explicit example in Appendix C.

5 Compactness of Dissipative solutions

It is well known that by Prokhorov's theorem $\mathscr{M}(\Gamma)$ is compact w.r.t. the Wasserstein distance W_p, $p < 2$, being η concentrated on curves in $W^{1,2}$ with uniformly bounded energy. Since the set $\mathscr{M}(\Gamma)$ is tight w.r.t. the cost $\|\cdot\|^2_{L^2}$, the Wasserstein distance W_p, $p < 2$, is equivalent to the narrow convergence.

Proposition 5.1. *Let $\{\eta_n\}_{n\in\mathbb{N}},\eta$ dissipative solutions in $\mathscr{M}(\Gamma)$ and suppose $W_p(\eta_n,\eta) \to 0$, $p > 1$. For every continuous bounded function $\phi : (L^2(0,T))^3 \to \mathbb{R}$, it holds*

$$\int \phi(\gamma,\dot\gamma,v_n(\gamma))\eta_n(d\gamma) \to \int \phi(\gamma,\dot\gamma,v(\gamma))\eta(d\gamma).$$

Recall that $v(\gamma)$ is computed by (13a).

Proof. The proof is divided into two steps.

Step 1. First of all, we show that there are a family of maps $R_n, R : [0,1] \to \Gamma$ such that

$$\eta_n = (R_n)_\sharp \mathscr{L}^1,\ \eta = R_\sharp \mathscr{L}^1, \quad \lim_n \|R_n - R\|_{L^2(0,1)} = 0.$$

The construction is standard, we repeat it for reader's convenience.

Let $B_i = B_{r_i}(\gamma_i)$, $i \in \mathbb{N}$, be a family of open balls generating the topology of Γ, and such that

$$\eta(\partial B_i) = 0. \tag{24}$$

Define the map $S : \Gamma \to [0,1]$ such that

$$\gamma \mapsto \alpha = S(\gamma) = \sum_i 3^{-i}\chi_{B_i}(\gamma) \in [0,1].$$

The map S is clearly injective.

Define the measures $\mu_n = S_\sharp\eta_n$, $\mu = S_\sharp\eta$, and let $\phi \in C([0,1])$. Then the function

$$\gamma \mapsto \phi\bigg(\sum_i 3^{-i}\chi_{B_i}(\gamma)\bigg)$$

is bounded and continuous outside the set $\cup_i\partial B_i$, so that by [3, Prop.1.62 b] and (24) it follows

$$\lim_n \int \phi(\alpha)S_\sharp\eta_n(d\alpha) = \lim_n \int \phi(S(\gamma))\eta_n(d\gamma) = \int \phi(S(\gamma))\eta(d\gamma) = \int \phi(\alpha)S_\sharp\eta(d\alpha),$$

so that

$$\mu_n = S_\sharp\eta_n \rightharpoonup S_\sharp\eta = \mu,$$

i.e. the measures μ_n converges weakly to μ.

Next, consider the unique monotone transport maps $G^n, G : [0,1] \to [0,1]$ such that

$$\mu_n = (G_n)_\sharp\mathscr{L}^1, \quad \mu = (G)_\sharp\mathscr{L}^1.$$

It is elementary to see that

$$\lim_n \|G_n - G\|_{L^p(0,1)} = 0.$$

Finally, if $S^{-1}:[0,1]\to\Gamma$ is a left inverse of S, define the maps

$$R_n = S^{-1}\circ G_n, \quad R = S^{-1}\circ G. \tag{25}$$

If $x_n = S(\gamma_n)$, $x = S(\gamma)$ and $x_n \to x$, then every $B_{r_i}(\gamma_i)\ni\gamma$ contains definitely γ_n, hence $\gamma_n\to\gamma$. This shows that $S^{-1}\llcorner_{S(\Gamma)}$ is continuous. Observing now that $G_n(\alpha)\to G(\alpha)$ for $\mathscr{L}^1$-a.e. $\alpha\in[0,1]$, we obtain that $R_n = S^{-1}\circ G_n:[0,1]\to\Gamma$ converges $\mathcal{L}^1$-almost everywhere to $R = S^{-1}\circ G$. Using the estimates

$$\int_0^1 \|R_n(\alpha)\|^2_{L^2}\mathscr{L}^1(d\alpha) = \int_{[0,1]}\|S^{-1}(\alpha)\|^2_{L^2}\mu_n(d\alpha) = \int_{L^2(0,T)}\|\gamma\|^2_{L^2}\eta_n(d\gamma),$$

$$\int_0^1 \|S^{-1}\circ G(\alpha)\|^2_{L^2}\mathcal{L}^1(d\alpha) = \int_{L^2(0,T)}\|\gamma\|^2_{L^2}\eta(d\gamma),$$

we deduce that $\|R_n\|_{L^p((0,1),\Gamma)}$ converges to $\|R\|_{L^p((0,1),\Gamma)}$: this together with the $\mathscr{L}^1$-a.e. pointwise convergence implies that $R_n\to R$ in $L^p((0,1),\Gamma)$.

Step 2. The statement thus reduces to

$$\lim_n \int_0^1 \phi\big(R_n(\alpha),\dot R_n(\alpha),v_n(R_n(\alpha))\big)d\alpha = \int_0^1 \phi\big(R(\alpha),\dot R(\alpha),v(R(\alpha))\big)d\alpha.$$

We claim that

$$\dot R_n(\alpha) = \frac{\partial}{\partial t}(R_n(\alpha)) \underset{L^2((0,1),\Gamma)}{\longrightarrow} \frac{\partial}{\partial t}(R(\alpha)) = \dot R(\alpha).$$

By Lemma 4.7 we have

$$\int_0^{T-s}\left|\frac{R_n(t+s,\alpha)-R_n(\alpha,t)}{s} - \dot R_n(\alpha,t)\right|^2 dtd\alpha \le C(T)s\|(R_n(0),v_n(0))\|^2_{L^2(0,1)},$$

$$\int_0^{T-s}\left|\frac{R(t+s,\alpha)-R(\alpha,t)}{s} - \dot R(\alpha,t)\right|^2 dtd\alpha \le C(T)s\|(R(0),v(0))\|^2_{L^2(0,1)},$$

Hence by triangle inequality and the convergence $R_n\to R$

$$\begin{aligned}
&\limsup_n \int\bigg(\int_0^{T-s}\big|\dot R(\alpha,t)-\dot R_n(\alpha,t)\big|^2dt\bigg)^{p/2}d\alpha \\
&\quad\le C(p)\limsup_n\int\bigg(\int_0^{T-s}\left|\frac{R(t+s,\alpha)-R(\alpha,t)}{s}-\frac{R_n(t+s,\alpha)-R_n(\alpha,t)}{s}\right|^2ds\bigg)^{p/2}d\alpha \\
&\qquad + C(T,p)s\|(R(0),v(0))\|^2_{L^2(0,1)} \\
&\quad= 6C(T,p)s\|(R(0),v(0))\|^2_{L^2(0,1)}.
\end{aligned}$$

Letting $s\searrow 0$ we obtain the desired convergence $\dot R_n\to\dot R$. Using again Proposition 3.2 we deduce that the function $v_n(t,R_n(\alpha))$ converges to $v(t,R(\alpha))$ in $L^2(0,1)$ for $\mathscr{L}^1$-a.e. α.

Finally for a continuous bounded function $\phi:(L^2(0,1))^3\to\mathbb{R}$

$$\begin{aligned}
\lim_n\int\phi(\gamma,\dot\gamma,v_n)\eta_n(d\gamma) &= \lim_n\int\phi\bigg(R_n(\alpha),\frac{\partial R_n(\alpha)}{\partial t},v_n(R_n(\alpha))\bigg)d\alpha \\
&= \int\phi\bigg(R(\alpha),\frac{\partial R(\alpha)}{\partial t},v(R(\alpha))\bigg)d\alpha = \int\phi(\gamma,\dot\gamma,v(\gamma))\eta(d\gamma). \qquad\square
\end{aligned}$$

Theorem 5.2. *Let $\{\eta_n\}_{n\in\mathbb{N}}$ be a sequence of dissipative solutions supported on $\mathscr{M}$ such that $W_p(\eta_n, \eta) \searrow 0$, $p > 1$. Then η is dissipative solution.*

Proof. Since (13a) is satisfied because of Proposition 5.1, we have to prove that equation (13b) passes to the limit: if $R_n, R : [0,1] \to \Gamma$ are the functions (25) in the proof of Proposition 5.1, then (13b) can be rewritten as

$$\begin{aligned}\dot{v}_n(t, R_n(\alpha, t)) &= \mathbb{P}_{t,n}\big(F(R_n, v_n(R_n))\big) \\ &= \mathbb{P}_{t,n}\bigg(v_{0,n}(R_n(\alpha)) - \int_0^t \nabla_q V\big(R_n(\alpha, s), v_n(s, R_n(\alpha))\big) \\ &\qquad - \int_0^t \nabla_q W\big(R_n(\alpha, s), v_n(s, R_n(\alpha)), R_n(s, \alpha'), v_n(s, R_n(\alpha'))\big) d\alpha'\bigg),\end{aligned}$$

where $\mathbb{P}_{t,n}$ is the projection in $L^2(0,1)$ corresponding to the descending fibration in $(0,1)$ obtained through the map $\mathbb{T}_t \circ R_n$,

$$(0,1) \ni \alpha \mapsto \mathbb{T}_t(R_n(\alpha))(\tau) = R_n(\alpha, t)\mathbf{1}_{[0,t)} + R_n(\alpha, \tau)\mathbf{1}_{[t,T]} \in \Gamma.$$

If $\psi : \Gamma \to \mathbb{R}$ is continuous, then

$$\begin{aligned}\int \psi(\mathbb{T}_t \circ R_n(\alpha)) F(R_n(\alpha), v_n(R_n(\alpha)))(t) d\alpha &= \int \psi(\mathbb{T}_t \circ R_n(\alpha)) \mathbb{P}_{t,n}\big(F(R_n(\alpha), v_n(R_n(\alpha)))(t)\big) d\alpha, \\ &= \int \psi(\mathbb{T}_t \circ R_n(\alpha)) v_n(t, R_n(\alpha)) d\alpha,\end{aligned}$$

so that, passing to the limit and by the pointwise convergence of R_n, v_n we obtain

$$\int \psi(\mathbb{T}_t \circ R(\alpha)) F(R(\alpha), v(R(\alpha)))(t) d\alpha = \int \psi(\mathbb{T}_t \circ R(\alpha)) v(t, R(\alpha)) d\alpha,$$

which, due to the arbitrariness of ψ, reads as

$$\mathbb{P}_t(F(R(\alpha), v(R(\alpha)))(t)) = \mathbb{P}_t(v(t, R(\alpha))),$$

or in the original coordinates

$$\mathbb{P}_t(F(\gamma(t), v(t, \gamma))) = \mathbb{P}_t(v(t, \gamma)).$$

By (13a) the functions $v(t, \gamma)$ depends only on $(t, \mathbb{T}_t(\gamma))$: this together with the right continuity gives that $\mathbb{P}_t(v(t, \gamma)) = v(t, \gamma)$, and therefore

$$\mathbb{P}_t(F(\gamma(t), v(t, \gamma))) = \mathbb{P}_t(v(t, \gamma)) = v(t, \gamma).$$

which is the requirement to be a dissipative solution. □

The following statement is elementary, because of the quadratic growth of V, W.

Lemma 5.3. *The energy $\eta \mapsto H((\gamma(t), v(t)_\sharp \eta)$ is continuous w.r.t. the Wasserstein-2 convergence.*

Remark 5.4. Note that for the Hamiltonian

$$H(\mu) = \int \left(\frac{p^2}{2} - \frac{q^2}{2} \right) \mu(dqdp),$$

with the initial data

$$\mu_n = \frac{1}{n^2} \big(\delta_{(n,0)} + \delta_{(-n,0)} \big) + \left(1 - \frac{2}{n^2} \right) \delta_{0,0},$$

the energy is not l.s.c., being

$$H(\mu_n) = -2 < H(\mu_\infty) = 0.$$

Clearly μ_n is not converging to μ w.r.t. Wasserstein-2, but it converges for all $p < 2$.

6 Discretization

Aim of this section is to prove that the set of dissipative solutions is the closure of the set of finite particle dissipative solutions. We will first approximate a given dissipative solution with a dissipative solution with dissipation only at finitely many times, then with a dissipative solution with finitely many particles, and finally with a sticky particle solution.

Definition 6.1. A dissipative solution is a *discrete in time dissipative solution* if there exists a partition $0 = t_0 < t_1 < \cdots < t_N = T$ such that in every interval $[t_i, t_{i+1})$ the solution $v(t)$ coincides with the conservative solution with initial measure $(\gamma(t_i), v(t_i))_\sharp \eta$.

Recall that $(Q(t,q,p;\mu_0), P(t,q,p;\mu_0))$ is the conservative trajectory starting from q, p with initial measure μ_0. Thus the above definition can be rewritten as

$$\gamma(t) = Q\big(t - t_i, \gamma(t_i), v(t_i,\gamma); \mu_{t_i}\big), \quad v(t,\gamma) = P\big(t - t_i, \gamma(t_i), v(t_i,\gamma); \mu_{t_i}\big),$$

for $t \in [t_i, t_{i+1})$, with

$$\mu_{t_i} = (\gamma(t_i), v(t_i))_\sharp \eta.$$

We begin by introducing a general method of constructing discrete in time dissipative solutions. Let $\eta \in \mathscr{P}(\Gamma)$, and consider L^2_η functions $(y(\gamma), w(\gamma))$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partition of $[0,T]$ and for every $i = 1, \dots, N$ let $\Upsilon_i(\gamma) \in L^2_\eta$ be given functions such that

$$\mathbb{P}_{t_i}(\Upsilon_i) = 0.$$

Define the functions $X(t,\gamma), Y(t,\gamma)$ recursively as follows: for $t \in [t_{N-1}, t_N]$ set

$$\begin{cases} X(t,\gamma) = Q\big(t - t_N, y(\gamma), w(\gamma); \mu_T\big), \\ Y(t,\gamma) = P\big(t - t_N, y(\gamma), w(\gamma); \mu_T\big), \end{cases} \qquad \mu_N = (y, w)_\sharp \eta,$$

and for $t \in [t_{i-1}, t_i)$, $i = 1, \dots, N-1$,

$$\begin{cases} X(t,\gamma) = Q\big(t - t_i, X(t_i,\gamma), Y(t_i,\gamma) + \Upsilon_i(\gamma); \mu_{t_i}\big), \\ Y(t,\gamma) = P\big(t - t_i, X(t_i,\gamma), Y(t_i,\gamma) + \Upsilon_i(\gamma); \mu_{t_i}\big), \end{cases} \qquad \mu_i = (X(t_i), Y(t_i))_\sharp \eta. \tag{26}$$

In other words, the trajectories X, Y are constructed by alternating the conservative flow $(Q,P)(t,q,p;\mu)$ with the projection $\mathbb{P}_{t_i}$, $i = 1, \dots, N-1$ (Figure 6): instead of assigning the initial data, we assign the projections

$$\Upsilon_i = Y(t_i-) - Y(t_i) = (\mathbb{I} - \mathbb{P}_{t_i}) Y(t_i-).$$

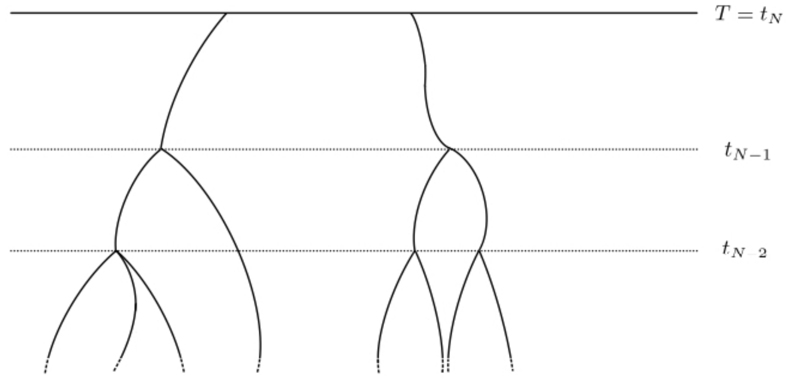

Figure 3: the discrete in time dissipative solution of Lemma 6.2.

Lemma 6.2. *The measure $\tilde{\eta} = X_\sharp \eta$ is a dissipative solution with initial velocity $v_0(\gamma) = Y(0,\gamma)$. Moreover*

$$E(0) \leq E(T) + C \sum_i \|\Upsilon_i\|_\eta^2.$$

Proof. The function $\dot{X}(t,\gamma)$ and $Y(t,\gamma)$ satisfies Equation (13a) by construction, and Equation (8b) holds in each internal $[t_i, t_{i+1})$ with initial data $v(t_i,\gamma)$. We have thus only to verify that

$$v(t_i,\gamma) = \mathbb{P}_{t_i}(v(t_i-))(\gamma), \quad i = 1,\dots,N-1.$$

By construction, $v(t,\gamma)$ depends only on $\gamma \llcorner_{[t,T]}$, so that

$$v(t_i,\gamma) - \mathbb{P}_{t_i}(v(t_i-))(\gamma) = \mathbb{P}_{t_i}\big(v(t_i) - v(t_i-)\big)(\gamma) = \mathbb{P}_{t_i}(-\Upsilon_i)(\gamma) = 0.$$

The energy is jumping only at times t_i of the amount

$$E(t_i-) - E(t_i) = \mathcal{O}(1)\|\Upsilon_i\|_\eta^2,$$

so that the energy estimate holds. □

We next study the stability w.r.t. the data $(y,w), \{\Upsilon_i\}_i$. Let (X,Y), $(X',\tilde{Y}')$ be discrete in time dissipative solution with the same time partition and constructed with initial data $(y,w),(y',w')$ and Υ_i, Υ'_i, $i = 1,\dots,N-1$.

Lemma 6.3. *It holds*

$$\|(X(t),Y(t)) - (X'(t),Y'(t))\|_{L^2_\eta} \leq C\bigg(\|(y,w)-(y',w')\|_{L^2_\eta} + \sum_{i=1}^{N-1} \|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta}\bigg).$$

In particular, by taking $(y',w') = 0, \Upsilon'_i = 0$ we obtain that the solution belongs to $\mathscr{M}(\Gamma)$, with

$$\tilde{C}_1, \tilde{C}_2 = \mathcal{O}(1)\bigg(\|(y,w)\|_{L^2_\eta} + \sum_{i=1}^{N-1} \|\Upsilon_i\|_{L^2_\eta}\bigg).$$

Proof. By stability, for $t \in [t_i, t_{i+1})$ it holds

$$\begin{aligned}
&\|(X(t), Y(t)) - (X'(t), Y'(t))\|_{L^2_\eta} \\
&\quad \le e^{3L(t_{i+1}-t)} \|(X(t_{i+1}-), Y(t_{i+1}-)) - (X'(t_{i+1}-), Y'(t_{i+1}-))\|_{L^2_\eta} \\
&\quad \le e^{3L(t_{i+1}-t)} \big(\|\Upsilon_{i+1} - \Upsilon'_{i+1}\|_{L^2_\eta} + \|(X(t_{i+1}), Y(t_{i+1})) - (X'(t_{i+1}), Y'(t_{i+1}))\|_{L^2_\eta} \big).
\end{aligned}$$

The statement is thus a direct application of (19) as in the proof of Proposition 4.12. □

Proposition 6.4. *If η is a dissipative solution, then for every $\epsilon > 0$ there is a discrete in time dissipative solution $\eta' = (X, Y)_\sharp \eta$ such that for all $t \in [0, T]$*

$$\|(X(t), Y(t)) - (\gamma(t), v(t))\|_{L^2_\eta} < \epsilon.$$

Proof. The proof is analogous to the proof of Proposition 4.12, the only difference being that we will follow the backward solution of Lemma 6.2 above, so that we do not need to apply the projection to the variable Q. In particular the constructed function is a dissipative solution, as stated in Lemma 6.2.

Consider the partition $0 = t_0 < t_1 < \dots < t_N = T$ of Lemma 4.8, and define

$$\Upsilon_i = \tilde{v}(t_i, t_{i-1}, \gamma) - v(t_i, \gamma).$$

Let X, Y be the discrete in time dissipative solutions constructed in Lemma 6.2: at each time step $[t_i, t_{i+1})$ we obtain from Corollary 4.11

$$\begin{aligned}
&\big\|(X(t,\gamma), Y(t,\gamma)) - (\gamma(t), v(t))\big\|_{L^2_\eta} \\
&\le \Big\|(X(t,\gamma), Y(t,\gamma)) \\
&\qquad - \Big(Q\big(t - t_{i+1}, \gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma); (\gamma(t_{i+1}), \tilde{v}(t_{i+1}))_\sharp \eta\big), \\
&\qquad\qquad P(t - t_{i+1}, \gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma); (\gamma(t_{i+1}), \tilde{v}(t_{i+1}))_\sharp \eta)\Big)\Big\|_{L^2_\eta} \\
&\quad + \Big\|\Big(Q\big(t - t_{i+1}, \gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma); (\gamma(t_{i+1}), \tilde{v}(t_{i+1}))_\sharp \eta\big), \\
&\qquad\qquad P\big(t - t_{i+1}, \gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma); (\gamma(t_{i+1}), \tilde{v}(t_{i+1}))_\sharp \eta\big)\Big) \\
&\qquad\qquad\quad - (\gamma(t), v(t))\Big\|_{L^2_\eta} \\
&\le e^{3L(t_i - t)} \big\|(X(t_{i+1}-, \gamma), Y(t_{i+1}-, \gamma)) - (\gamma(t_{i+1}), \tilde{v}(t_{i+1}, t_i, \gamma))\big\|_{L^2_\eta} \\
&\quad + C \int_{t_i}^{t_{i+1}} \|v(s) - \tilde{v}(s)\|_{L^2_\eta} ds \\
&\le e^{3L(t_i - t)} \big\|(X(t_{i+1}, \gamma), Y(t_{i+1}, \gamma)) - (\gamma(t_{i+1}), v(t_{i+1}, \gamma))\big\|_{L^2_\eta} \\
&\quad + C\epsilon(t_{i+1} - t_i).
\end{aligned}$$

Hence, applying the solution formula (19) to the above formula when $t = t_i$ as in (20),

$$\big\|(X(t,\gamma), Y(t,\gamma)) - (\gamma(t), v(t))\big\|_{L^2_\eta} \le C\epsilon.$$

The measure $(X, Y)_\sharp \eta$ satisfies the statement. □

The next step is to discretize the number of particles.

Definition 6.5. A discrete in time dissipative solution is a *discrete in time dissipative particle solution* if η is made of Dirac masses. If the number of deltas is finite, it is a *dissipative finite particle solution.*

Proposition 6.6. *If η is a discrete in time dissipative solution and $\epsilon > 0$, then there exists a finite particle solution $\eta' = (X, Y)_\sharp \eta$ such that for all $t \in [0, T]$*

$$\|(X(t), Y(t)) - (\gamma(t), v(t))\|_{L^2_\eta} \le \epsilon.$$

Proof. The proof follows immediately from Lemma 6.3, if we can find simple functions $(y', w', \{\Upsilon'_i\}_i)$ approximating

$$y(\gamma) = \gamma(T), \quad w(\gamma) = v(T, \gamma), \quad \Upsilon_i(\gamma) = v(t_i-, \gamma) - v(t_i, \gamma)$$

(in the last formula we have used that it is a discrete in time solution where the projection is applied at times t_i), with the property that

$$\|y - y'\|_{L^2_\eta} + \|w - w'\|_{L^2_\eta} + \sum_i \|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta} < \epsilon.$$

The existence of such approximations is elementary.

It remains only to prove that the solution (X', Y') of constructed in Lemma 6.2 by using $y', w', \{\Upsilon'\}_i$ simple functions is a finite particle solution: this is immediate, since from the explicit form of the solution (26) the functions (X', Y') are measureable in the finite algebra generated by $(y', w', \{\Upsilon\}_i)$. □

The last step is to prove that we can construct a sticky particle solution made of finitely many particles.

Definition 6.7. A discrete finite particle solution is a *finite sticky particle solution* if for every $t \in [0, T]$ the maps $\mathbb{T}_t$ and e_t induce the same equivalence relation.

Proposition 6.8. *If η is a finite dissipative solution and $\epsilon > 0$, there exists $\eta' = (X, Y)_\sharp \eta$ finite sticky particle solution such that*

$$\|(X(t), Y(t)) - (\gamma(t), v(t))\|_{L^2_\eta} \le \epsilon.$$

Proof. As in the previous proof, it is enough to find simple functions $y', w', \{\Upsilon'_i\}_i$ approximating

$$y(\gamma) = \gamma(T), \quad w(\gamma) = v(T, \gamma), \quad \Upsilon_i(\gamma) = v(t_i-, \gamma) - v(t_i, \gamma),$$

with the property that

$$\|y - y'\|_{L^2_\eta} + \|w - w'\|_{L^2_\eta} + \sum_i \|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta} < \epsilon,$$

and moreover such that the dissipative solution constructed by Lemma 6.2 is actually a sticky particle solution. W.l.o.g. we can assume that the time steps t_i satisfies

$$t_i - t_{i-1} < \delta_t,$$

being δ_t the time step for which Proposition 3.8 holds.

We begin by considering the final data y, w, and let M be the number of particles. If the backward trajectories are not intersecting, then no perturbation is needed. Otherwise, by Proposition 3.8 there are arbitrarily small perturbations $y' - y, w' - y$ such that the trajectories are not intersecting in $[t_{N-1}, t_N)$. In particular we can assume that

$$\|y - y'\|_{L^2_\eta} + \|w - w'\|_{L^2_\eta} < \epsilon 2^{-T/\delta_t}.$$

The number of particles is increased of at most $2M$.

Assume to have found perturbations up to time t_{i+1} such that in each time interval $[t_j, t_{j+1})$, $j \geq i+1$, the trajectories are not intersecting, and the number of particles is $2^{N-i-1}M$. The initial data for the backward solution are

$$\gamma(t_{i+1}), \quad v(t_{i+1}, \gamma) + \Upsilon_{i+1}(\gamma).$$

We can then again find perturbations $\Upsilon'_{i+1}(\gamma)$ such that the number of particles is at most $2^{N-i}M$ and

$$\|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta} < \epsilon 2^{-T/\delta_i}.$$

After a finite number of steps we arrive to $t = 0$: the total perturbation is

$$\|y - y'\|_{L^2_\eta} + \|w - w'\|_{L^2_\eta} + \sum_i \|\Upsilon_i - \Upsilon'_i\|_{L^2_\eta} < \frac{T}{\delta_t}\epsilon 2^{-T/\delta_t} < \epsilon,$$

and the number of particles is at most $2^{T/\delta_t}M$. □

The above result implies directly the following.

Theorem 6.9. *The weak closure of the set of finite sticky particle solutions with bounded second order moments for $\gamma(0), v(0)$ is the set of dissipative solutions.*

7 A G_δ dense set of initial data

We have proved that for dissipative solutions the energy $E(t) = H((\gamma(t), v(t))_\sharp \eta)$ is decreasing in time, and actually that the energy dissipation controls the distance from the conservative flow. In this section we want to prove that the set of initial data for which there is only one dissipative solution (which is then the conservative one) is of second category in the set of initial data.

In this section we assume that H is convex, so that $\mu \mapsto H(\mu)$ is l.s.c. w.r.t. the narrow convergence. Define for $\mu \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ the functional

$$D(\mu) = \max\big\{H(\mu) - H((\gamma(T), v(T)_\sharp \eta), (\gamma(0), v(0))_\sharp \eta = \mu\big\}.$$

By compactness of $\mathscr{M}(\Gamma)$ and l.s.c. of $\eta \mapsto H((\gamma(t), v(t)_\sharp \eta)$, the maximum is assumed. Being the supremum of u.s.c. functionals, $D(\mu)$ is u.s.c., and when $D(\mu) = 0$ then every dissipative solution with the initial data μ has 0 dissipation, i.e. it coincides with the conservative one.

Using Proposition 3.6, we deduce the following

Theorem 7.1. *The set $D_0 = \{\mu : D(\mu) = 0\} \subset \mathscr{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ is a dense G_δ set w.r.t. narrow convergence.*

Proof. First of all, D_0 is a G_δ-set, being

$$D_0 = \bigcap_n \big\{\mu : D(\mu) < 2^{-n}\big\}, \quad D(\mu) \text{ u.s.c..}$$

Next, by Proposition 6.8, the finite sticky particle solutions are dense and by Proposition 3.6 the set of initial data so that the trajectories are not intersecting is dense in the set of finite sticky particle solutions. Finally, for the non intersecting trajectories, the unique solution is the conservative one by Lemma 4.16. □

A Proofs of Section 3

Proposition 3.2, page 12. *The operator* (10) *is uniformly monotone, namely*

$$\Lambda \|v_1 - v_2\|_2^2 \leq \big(v_1 - v_2, F(t, v_1) - F(t, v_2)\big) \leq 3L\|v_1 - v_2\|_2^2. \tag{27}$$

Proof of Proposition 3.2, page 12. The bound from above follows immediately by the Lipschitz bounds on $\nabla V, \nabla W$, which gives that $F(t)$ is Lipschitz:

$$\|F(t, v_1) - F(t, v_2)\|_2 \leq 3L\|v_1 - v_2\|_{L^2_\eta}.$$

For the estimate from below, we observe that by symmetry

$$W(x, v, x', v') = W(x', v', x, v) \quad \Rightarrow \quad \nabla_p W(x, v, x', v') = \nabla_{v'} W(x', v', x, v),$$

so that

$$\begin{aligned}
&\int \Big(v_1(\gamma) - v_2(\gamma), F(t, v_1)(\gamma) - F(t, v_2)(\gamma)\Big)\eta(d\gamma) \\
&= \int \Big(v_1(\gamma) - v_2(\gamma), \nabla_v V(\gamma(t), v_1(\gamma)) - \nabla_v V(\gamma(t), v_2(\gamma))\Big)\eta(d\gamma) \\
&\quad + \int\int \Big(v_1(\gamma) - v_2(\gamma), \nabla_v W(\gamma(t), v_1(\gamma), \gamma'(t), v_1(\gamma')) \\
&\qquad\qquad - \nabla_v W(\gamma(t), v_2(\gamma), \gamma'(t), v_2(\gamma'))\Big)\eta(d\gamma')\eta(d\gamma) \\
&\geq \Lambda\|v_1 - v_2\|^2_{L^2_\eta} \\
&\quad + \frac{1}{2}\int\int \Big(v_1(\gamma) - v_2(\gamma), \nabla_v W(\gamma(t), v_1(\gamma), \gamma'(t), v_1(\gamma')) \\
&\qquad\qquad - \nabla_v W(\gamma(t), v_2(\gamma), \gamma'(t), v_2(\gamma'))\Big)\eta(d\gamma')\eta(d\gamma) \\
&\quad + \frac{1}{2}\int\int \Big(v_1(\gamma') - v_2(\gamma'), \nabla_{v'} W(\gamma(t), v_1(\gamma), \gamma'(t), v_1(\gamma')) \\
&\qquad\qquad - \nabla_{v'} W(\gamma(t), \tilde{v}_2(t, \gamma), \gamma'(t), v_2(\gamma'))\Big)\eta(d\gamma')\eta(d\gamma),
\end{aligned}$$

and then by uniform convexity

$$\left(\begin{pmatrix} v_1(\gamma) - v_2(\gamma) \\ v_1(\gamma') - v_2(\gamma') \end{pmatrix}, \begin{pmatrix} \nabla_p W(\gamma(t), v_1(\gamma), \gamma'(t), v_1(\gamma')) - \nabla_p W(\gamma(t), v_2(\gamma), \gamma'(t), v_2(\gamma')) \\ \nabla_{v'} W(\gamma(t), v_1(\gamma), \gamma'(t), v_1(\gamma')) - \nabla_{v'} W(\gamma(t), v_2(\gamma), \gamma'(t), v_2(\gamma')) \end{pmatrix}\right) \geq 0,$$

and then we conclude

$$\int \Big(v_1(\gamma) - v_2(\gamma), F(t, v_1)(\gamma) - F(t, v_2)(\gamma)\Big)\eta(d\gamma) \geq \Lambda \|v_1 - v_2\|^2_{L^2_\eta}.$$

This is the lower bound of (27). □

Proposition 3.3, page 12. *There exist unique functions* $Q(\alpha, t), P(\alpha, t) \in C_0([0, T], L^2(0, 1))$ *satisfying*

$$Q(\alpha, t) = Q_0(\alpha) + \int_0^t \left[\nabla_p V(Q(\alpha, s), P(\alpha, s)) + \int \nabla_p W(Q(\alpha, s), P(\alpha, s), Q(\alpha', s), P(\alpha', s)d\alpha'\right]ds,$$

$$P(\alpha, t) = P_0(\alpha) - \int_0^t \left[\nabla_q V(Q(\alpha, s), P(\alpha, s)) + \int \nabla_q W(Q(\alpha, s), P(\alpha, s), Q(\alpha', s), P(\alpha', s)d\alpha'\right]ds.$$

Moreover

$$t \mapsto H(t, \mathcal{L}^1 \llcorner_{(0,1)}) = \int V(Q(\alpha, t), P(\alpha, t))d\alpha + \int\int W(Q(\alpha, t), P(\alpha, t), Q(\alpha', t), P(\alpha', t))d\alpha d\alpha'$$

is constant, $(\partial_t Q(t), P(t)) \in L^2((0, 1), W^{1,2}(0, T))$ *with*

$$\|\partial_t Q(t)\|_{L^2(0,1)}, \|\partial_t P(t)\|_{L^2(0,1)}, \frac{1}{3L}\|\partial_t^2 Q(t)\|_{L^2(0,1)} \leq 3Le^{3Lt}\|(Q_0, P_0)\|_{L^2(0,1)}.$$

Finally, if $(Q_0, P_0), (Q_0', P_0')$ *are two different initial data and* $(Q(t), P(t)), (Q'(t), P'(t))$ *the corresponding solutions, then it holds*

$$\big\|(Q(t), P(t)) - (Q'(t), P'(t))\big\|_{L^2(0,1)} \leq e^{3Lt}\big\|(Q_0, P_0) - (Q_0', P_0')\big\|_{L^2(0,1)}.$$

Proof of Proposition 3.3, page 12. The existence, uniqueness and continuous dependence estimates boil down to the same computation: study the Lipschitz constant of the map

$$(Q(\alpha, t), P(\alpha, t)) \mapsto \left(x_0(\alpha) + \int_0^t \nabla_v H(Q(\alpha, s), P(\alpha, s))ds, v_0(\alpha) - \int_0^t \nabla_q H(Q(\alpha, s), P(\alpha, s))ds\right).$$

We show the continuous dependence: using the Lipschitz estimates for V, W,

$$\begin{aligned}\|Q(t) - Q'(t)\|_2 &= \left\|Q_0 - Q_0' + \int_0^t \big(\nabla_v H(Q(s), P(s))ds - \nabla_v H(Q'(s), P'(s))\big)ds\right\|_2 \\ &\leq \|Q_0 - Q_0'\| + 3L\int_0^t \big\|(Q(s), P'(s)) - (Q'(s), P'(s))\big\|_2 ds,\end{aligned}$$

$$\begin{aligned}\|P(t) - P'(t)\|_2 &= \left\|P_0 - P_0' - \int_0^t \big(\nabla_q H(Q(s), P(s))ds - \nabla_q H(Q'(s), P'(s))\big)ds\right\|_2 \\ &\leq \|P_0 - P_0'\| + 3L\int_0^t \big\|(Q(s), P'(s)) - (Q'(s), P'(s))\big\|_2 ds.\end{aligned}$$

Hence the continuous dependence follows by a Gronwall-type estimate.

A similar estimate gives that for $3Lt < 1$ the above map is a contraction when $(Q_0, P_0) = (Q_0', P_0')$, so that one deduce uniqueness. The convergence to the initial data follows from (8).

The estimates on $\ddot{x}, \dot{v}$ follow by differentiating the ODE (8) and Proposition 3.2, and the conservation of energy $H(t, \mathcal{L}^1 \llcorner_{(0,1)})$ directly by differentiating w.r.t. t (which is now allowed since $\dot{x}, \dot{v}$ are in $L^2(0,1)$. □

B An example of non-uniqueness

We present an example of non uniqueness for the ODE in dimension 1 with Hamiltonian

$$H(\mu) = \int \frac{v^2}{2} \mu(dxdv) + \int W(x - x')\mu \times \mu(dxdvdx'dv'),$$

where the potential W is not semiconvex. The measure μ will be purely atomic. It is not clear to us whether this example can be adapted to W semiconvex, where a solution can be constructed [12].

Let

$$\phi(x) = \begin{cases} x & |x| \leq 1, \\ \mathrm{sign}(x) & |x| > 1, \end{cases} \qquad \Phi(x) = \begin{cases} x^2/2 & |x| \leq 1, \\ |x| - 1/2 & |x| > 1. \end{cases}$$

and define

$$\psi(x) = \sum_{n \in \mathbb{N}} \frac{\phi(n^{16}x)}{n^{16}}, \qquad \Psi(x) = \sum_n \frac{\Phi(n^{16}x)}{n^{32}}. \tag{28}$$

Let μ_0 be the initial configuration

$$\mu_0(dx) = \delta_0(dx) + \sum_n \frac{1}{n^8} \delta_{n^3}(dx),$$

with speed 0 for all $n = 0, 1, \dots$.

Define

$$W(x) = \begin{cases} -\Phi(n^{16}(x - n^3))/n^{24} & |x - n^3| \leq 1/3, \\ \text{smooth } \sim n^{-8} & 1/3 < |x - n^3| < 1/2, \qquad n \in \mathbb{Z} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

which is explicitly

$$W(x) = \begin{cases} -n^8(x - n^3)^2/2 & |x - n^3| \leq n^{-16}, \\ (n^{16}|x| - 1/2)/n^{24} & n^{-16} < |x| \leq 1/3, \\ \text{smooth } \sim n^{-8} & 1/3 < |x - n^3| < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Its derivative is

$$W'(x) = \begin{cases} -\phi(n^{16}(x - n^3))/n^8 & |x - n^3| \leq 1/3, \\ \text{smooth } \sim n^{-8} & 1/3 < |x - n^3| < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

which is explicitly

$$W'(x) = \begin{cases} -n^8(x-n^3) & |x-n^3| \leq n^{-16}, \\ \mathrm{sign}(x)/n^8 & n^{-16} < |x| \leq 1/3, \\ \text{smooth } \sim n^{-8} & 1/3 < |x-n^3| < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Its second derivative is

$$W''(x) = \begin{cases} -n^8 & |x-n^3| \leq n^{-16}, \\ 0 & n^{-16} < |x| \leq 1/3, \\ \text{smooth } \sim n^{-8} & n^{-16} < |x-n^3| < 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

showing that it is not semiconvex.

We have

$$\int x^2 \mu_0(dx) = \sum_n \frac{1}{n^8}(n^3)^2 < \infty,$$

$$\int W(x-y)\mu_0(dx)\mu_0(dy) = \sum_n \frac{1}{n^8}\frac{-\Phi(0)}{n^{12}} = 0.$$

We have observe that the only solutions to

$$n^3 - m^3 = z^3, \quad m, n \in \mathbb{N}, z \in \mathbb{Z}$$

is when $n = m, z = 0$ or $m = 0, n = z$ or $n = 0, z = m$, so that if the positions $x_n(t)$ are such that

$$|x_n - n^3| < 1/4$$

then

$$W(x_n - x_m) \neq 0 \ \Leftrightarrow\ \exists k \in \mathbb{Z}\big(|x_n - x_m - k^3| < 1/2\big) \ \Leftrightarrow\ \exists k \in \mathbb{Z}\big(|n^3 - m^3 - k^3| < 1\big)$$

and the last inequality implies

$$\big(n = m \ \wedge\ k = 0\big) \ \vee \big(n = k \ \wedge\ m = 0\big) \ \vee\ \big(m = -k \ \wedge\ n = 0\big).$$

In particular, if the position remains inside $n^3 + [-1,1]/4$, then

$$\sum_{n,n'} m_n m_{n'} W(x_n - x_{n'}) = 2\sum_{n\geq 1} m_n W(x_n - x_0) \simeq \sum_n n^{-8-8} < \infty.$$

Let

$$q_n = x_n - n^3.$$

When

$$x_n = n^3 + q_n \in n^3 + [-1/4, 1/4],$$

the equation (8) for this case can be rewritten as system of second order ODEs: using $\dot{q}_n = v_n$, it is immediate to see that

$$\begin{cases} \ddot{q}_0 = \sum_{n=1}^{+\infty} m_i W'(q_n - q_0) \\ \ddot{q}_1 = m_0 W'(q_0 - q_1) \\ \ddot{q}_2 = m_0 W'(q_0 - q_2) \\ \vdots \\ \ddot{q}_n = m_0 W'(q_n - q_0) \\ \vdots \end{cases}$$

which is explicitly (being ϕ antisymmetric)

$$\begin{cases} \ddot{q}_0 = \sum_{n=1}^{+\infty} \frac{\phi(n^{16}(q_0 - q_n))}{n^{16}} \\ \ddot{q}_1 = \phi(q_1 - q_0) \\ \ddot{q}_2 = 2^{-8}\phi(2^{16}(q_2 - q_0)) \\ \vdots \\ \ddot{q}_n = n^{-8}\phi(n^{16}(q_n - q_0)) \\ \vdots \end{cases} \tag{29}$$

B.1 Non-uniqueness for the particle in the origin

Consider the ODE

$$\ddot{u} = \psi(u), \quad u(0) = u'(0) = 0,$$

where ψ is given in (28). Multiplying by $\dot{u}$ and integrating

$$\frac{\dot{u}^2}{2} = \Psi(u),$$

which has a solution not identically 0 iff

$$\int_0^\delta \frac{1}{\sqrt{\Psi(u)}} du < \infty.$$

We study the Hölder exponent of the function Ψ when $0 \leq u \ll 1$.

Using the explicit formulas for Ψ we have

$$\begin{aligned} \Psi(u) &= \sum_n n^{-32}\Phi(n^{16}u) \\ &= \left(\sum_{n^{16}u \leq 1} n^{-32}\right)\frac{(n^{16}u)^2}{2} + \sum_{n^{16}u > 1} n^{-32}\left(n^{16}u - \frac{1}{2}\right) \\ &= \left(\sum_{n^{16}u \leq 1} 1\right)\frac{u^2}{2} + \left(\sum_{n^{16}u > 1} n^{-16}\right)u - \frac{1}{2}\sum_{n^{16}u > 1} n^{-32}. \end{aligned}$$

We use now the estimates

$$\sum_{n^{16}u\leq 1} \sim \int_1^{u^{-1/16}} d\omega \sim u^{-1/16},$$

$$\sum_{n^{16}u>1} n^{-16} \sim \int_{u^{-1/16}}^{\infty} \omega^{-16} d\omega = \frac{1}{15} u^{15/16},$$

$$\sum_{n^{16}u>1} n^{-32} \sim \int_{u^{-1/16}}^{\infty} \omega^{-32} d\omega = \frac{1}{31} u^{31/16}.$$

Hence for $u \ll 1$ it holds

$$\Psi(u) \sim \left(\frac{1}{2} + \frac{1}{15} - \frac{1}{2}\frac{1}{31}\right) u^{31/16} = \frac{256}{465} u^{31/16}.$$

since

$$\int_0^{\delta} \frac{1}{\sqrt{\Psi(\omega)}} d\omega \sim \int_0^{\delta} \omega^{-31/32} d\omega \simeq \delta^{1/32}.$$

Thus there is not uniqueness, and the nonzero solution will behave like t^{32}.

B.2 Non uniqueness - part 2

Before we assumed that the other particles remain in $q_n = 0$ (i.e. $x_n = n^3$). The idea of this section is that they will approach q_0, so that the actual force in q_0 is larger.

Let $\Theta(t, u)$ be a continuous function such that

$$\frac{3}{2} \geq \Theta(t, u) \geq \Psi'(u) \sim u^{15/16}, \qquad u \in [0, 1/4],$$

and consider the ODE

$$\ddot{u} = \Theta(t, u) \quad \Rightarrow \quad \begin{cases} \dot{t} = 1, \\ \dot{u} = w, \\ \dot{w} = \Theta(t, u). \end{cases}$$

The forward-in-time invariant region $S \subset \mathbb{R}^3$ we consider is the region

$$S = \Big\{ 0 \leq u \leq 1/4,\ \sqrt{u}/2 \leq t \leq 2\zeta^{-1}(u),\ \dot{\zeta}(\zeta^{-1}(u)) \leq w \leq 2\sqrt{u} \Big\},$$

where $\zeta(t)$ is the graph of the non-zero solution to

$$\dot{u} = \frac{1}{2}\sqrt{\Psi(u)}, \quad u(0) = 0,$$

which we know to behave like

$$\zeta(t) \sim t^{32}, \qquad \dot{\zeta}(\zeta^{-1}(u)) \sim u^{31/32}$$

by the computation at the end of the previous section. The other bound is obtained by solving

$$\dot{u} = w, \quad \dot{w} = 2 \quad \Rightarrow \quad u = t^2, w = t.$$

The region is forward invariant because

$$w = 2\sqrt{u} \quad \Rightarrow \quad \frac{\dot w}{\dot u} = \frac{\Theta(t,u)}{2\sqrt{u}} \le \frac{3}{4\sqrt{u}} < \frac{1}{\sqrt{u}} = \partial_u(2\sqrt{u}),$$

$$w = \zeta(u) \quad \Rightarrow \quad \frac{\dot w}{\dot u} = \frac{\Theta(t,u)}{\zeta(u)} \ge \frac{\Psi'(u)}{\zeta(u)} > \frac{\Psi'(u)/8}{\zeta(u)} = \partial_u(\zeta(u)),$$

$$t = \frac{\sqrt{u}}{2} \quad \Rightarrow \quad \frac{\dot t}{\dot u} = \frac{1}{w} \ge \frac{1}{2\sqrt{u}} > \frac{1}{4\sqrt{u}} = \partial_u\biggl(\frac{\sqrt{u}}{2}\biggr),$$

$$t = 2\zeta^{-1}(u) \quad \Rightarrow \quad \frac{\dot t}{\dot u} = \frac{1}{w} \le \frac{1}{\zeta'(\zeta^{-1}(u))} < \frac{2}{\zeta'(\zeta^{-1}(u))} = \partial_u(2\zeta^{-1}(u)).$$

In all formulas above we are comparing the vector field $(1, w, \Theta(t,u))$ with the tangent to the boundary (whose slope is at the r.h.s. of the previous formulas).

The strict inequalities gives that the flow is entering the region S: in particular, every trajectory entering in S at some point $(t,x,w) \in \partial S$ is exiting S in some point in the interior of the region

$$E = \biggl\{ u = \frac{1}{4},\ \zeta'(\zeta^{-1}(1/4)) \le w \le 1,\ \frac{1}{4} \le t \le \zeta^{-1}(1/4) \biggr\}.$$

Lemma B.1. *If $\Psi'(u) \le \Theta(t,u) \le 3/2$, there is a trajectory starting from $(0,0)$ at $t = 0$ and reaching $u(\bar t) = 1/4$ inside S with $1/2 < \bar t < \zeta^{-1}(1/4) \sim 2^{1/16}$.*

Proof. consider a sequence of points $(t_n, x_n, w_n) \in \partial S$ converging to $(0,0,0)$, let γ_n be a trajectory starting from (t_n, x_n, w_n) inside S. Then, up to subsequences, the limit trajectory γ satisfies the statement. □

We will denote such a trajectory with $\hat q_0(t)$, and we can assume that it is defined for $t \in [0,1/2]$ and $\hat q_0(t) > 0$ for $t > 0$.

We next analyze the other components. The ODE for $|q_n| \le 1/3$ is

$$\ddot q_n = n^{-8} \phi\bigl(n^{16}(q_n - q_0(t))\bigr), \quad q_n(0) = \dot q_n(0) = 0. \tag{30}$$

We assume that the function $q_0(t)$ is given, and it is ≥ 0. Then the ODE above is rewritten as

$$\dot q_n = w_n,\ \dot w_n = n^{-8}\phi(n^{16}(q_n - q_0(t)),$$

and then the quarter plane $\{q_n, w_n \le 0\}$ is forward invariant for $t \in [0,1/3]$: indeed the vector field is of order n^{-8} and Lipschitz, and

$$q_n = 0 \quad \Rightarrow \quad \dot q_n \le 0,$$

$$w_n = 0, \quad \Rightarrow \quad \dot w_n \le 0.$$

We have used that $q_0 \ge 0$ and the uniqueness of the solution: hence

Lemma B.2. *For every $q_0(t) \ge 0$ there is a unique solution $q_n(t)$ to (30) such that*

$$-\mathcal{O}(n^{-8}) \le q_n(t), \dot q_n(t) \le 0.$$

Figure 4: two different conservative solutions of the Hamiltonian system (29), i.e. the stationary solution (red) and the solution $\bar{Q}$ of Proposition B.3 (green).

Finally, define the compact set

$$K = \mathrm{Lip}([0,1/3],[-1,1]) \times \big(\mathrm{Lip}([0,1/3],[-1,1])\big)^{\mathbb{N}} \subset (C^0([0,1/3],\mathbb{R}))^{\mathbb{N}_0}$$

with the product topology. Given $Q = \{q_n(t)\}_n \in K$, then construct $Q' = \{q'_n(t)\}_n \in K$ as the point whose coordinates are

$$q'_0 = \text{ a solution by Lemma B.1 with } \Theta(t,u) = \sum_n n^{-16}\phi(n^{16}(u - q_n(t))),$$

$$q'_n = \text{ the solution by Lemma B.2.}$$

Since $q_n \leq 0$ for $n \geq 1$, then

$$\Psi'(u) = \sum_n n^{-16}\phi(n^{16}u) \leq \Theta(t,u) = \sum_n n^{-16}\phi(n^{16}(u - q_n(t))) \leq \frac{3}{2}, \qquad 0 \leq u, -q_n(t) \leq \frac{1}{2},$$

so that the assumptions of Lemma B.1 are satisfied for $0 \leq t \leq \frac{1}{3}$.

It is fairly easy to see that $Q \mapsto Q'$ maps K into K.

Repeating the process countably many times, we obtain a family of point $Q_i = \{q_{n,i}\}_n \in K$: assume by compactness that

$$\lim_i q_{n,i}(t) = \bar{q}_n(t)$$

in C^0 up to subsequences. Then

$$\Theta_i(t, q_{0,i}) = \sum_n n^{-16}\phi(n^{16}(q-0,i(t) - q_{n,i}(t))) \to \bar{\Theta}(t,\bar{q}_0) = \sum_n n^{-16}\phi(n^{16}(\bar{q}_0(t) - \bar{q}_n(t)))$$

because the series is uniformly convergent and

$$\phi(n^{16}(q_0(t) - q_n(t)) \to \phi(n^{16}(\bar{q}_0(t) - \bar{q}_n(t)).$$

In particular, since each $q_{0,i}$ is a trajectory in S by Lemma B.1, we deduce

Proposition B.3. *The limit point* $\bar{Q} = \{\bar{q}_n(t)\}_n$ *is a non constant solution to* (29).

Figure B.2 depicts the two different conservative solutions.

C An non trivial example of uniqueness

Consider the space $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and assume

$$H(\mu) = \int \frac{p^2}{2}\mu + \frac{\epsilon}{2}\int\int \frac{q_1^2(q'_1)^2}{4}\mu \times \mu = \int \frac{p^2}{2}\mu + \frac{\epsilon}{2}\bigg(\int \frac{q_1^2}{2}\mu\bigg)^2,$$

with $0 < \epsilon \ll 1$. The Hamiltonian has quartic growth, but since we will consider solutions for $t \in [-1,2]$ such that

$$\mathrm{supp}(\mu) \subset \{|Q| \leq 12\} \times \mathbb{R}^3,$$

we can alter the function $W(q,q') = \epsilon q_1^2 (q_1')^2/4$ outside $B_{12}^{\mathbb{R}^3}(0)$ arbitrarily.

The Hamiltonian ODE in the second and third coordinates is

$$\dot{q}_i = p_i, \ \dot{p}_i = 0, \quad i = 2,3,$$

whose solution is

$$q_i(t) = p_i(0)t + q_i(0), \quad i = 2,3,$$

while the first component satisfies the ODE

$$\dot{q}_1 = p_1, \quad \dot{p}_1 = -\epsilon \bigg(\int \frac{(q_1')^2}{2} \mu(dq') \bigg) q_1. \tag{31}$$

For $\alpha \in [-1,1]$ consider the initial data for $i = 2,3$

$$(q_2,q_3)(\alpha,0) = (\alpha, -|\alpha|), \quad (p_2,p_3)(\alpha,0) = (-\mathrm{sign}(\alpha), 1).$$

The solutions are then

$$(q_2,q_3)(\alpha,t) = \big(\alpha - \mathrm{sign}(\alpha)t, t - |\alpha|\big),$$

and then the unique intersection points of the trajectories $q(\alpha), q(\alpha')$ occur only for

$$\alpha' = -\alpha, \quad t = |\alpha|, \quad (q_2,q_3)(\alpha,|\alpha|) = (0,0).$$

Claim 1: *there are initial data at $t = -1$ such that the conservative solution $(Q(t),P(t))$ satisfies*

$$q_1(\alpha,|\alpha|) = q_1(-\alpha,|\alpha|) = 0, \quad p_1(\alpha,|\alpha|) = -p_1(-\alpha,|\alpha|) = -1.$$

Proof. It is a standard contraction argument for the map

$$\big\{q_1(\alpha,t), p_1(\alpha,t)\big\}_\alpha \mapsto \bigg\{ \int_{|\alpha|}^t p_1(\alpha,\tau)d\tau, -\mathrm{sign}(\alpha) - \epsilon \int_{|\alpha|}^t \bigg(\int \frac{p_1(\alpha',\tau)^2}{2} \mu(d\alpha') \bigg) q_1(\alpha,\tau)d\tau \bigg\},$$

which is a contraction for $\epsilon \ll 1$ and $t \in [-1.2]$. □

The values $\{Q(\alpha,-1), P(\alpha,-1)\}$ are the initial data for the dissipative solutions we are going to study.

When we consider a possible dissipative solution η, the projection of the motion on the component 2,3 is exactly a sticky particles system, or more precisely it is a dissipative solution to the sticky particle PDE. The third component gives

$$\mathbb{P}_t \bigg(p_3(\alpha,0) - \int_0^t \nabla_x H(q_3(s),p_3(s))ds \bigg) = \mathbb{P}_t(1) = 1,$$

so that the third component of the trajectories of the dissipative solution is again

$$q_3(t) = t - |\alpha|.$$

In particular, only the particles $(|\alpha|, -|\alpha|)$ can interact, and only at time $|\alpha|$, and no additional interactions can occur at a later time.

We can thus write the dissipative solution with the parametrization $q(|\alpha|,\beta), p(|\alpha|,\beta)$, $\beta \in [0,1]$, with the measure $\mathcal{L}^2(d|\alpha| d\beta)$, and the projection $\mathbb{P}_t$ as $\mathbb{P}_{|\alpha|}$ acting on $L^2(d\beta)$. The function $\alpha \mapsto \mathbb{P}_\alpha$ can be assumed Borel, in the sense that for every $f \in L^2(d\beta)$ the function $\alpha \mapsto \mathbb{P}_\alpha(f)$ is Borel.

Let $(\hat{Q}, \hat{P})(|\alpha|,\beta)$ be a dissipative solution with initial data $\{Q(\alpha,-1), P(\alpha,-1)\}$, which in the parametrization $(|\alpha|,\beta)$ corresponds to

$$(Q,P)(|\alpha|,\beta,-1) = \begin{cases} (Q,P)(-|\alpha|,-1) & \beta \in [0,1/2], \\ (Q,P)(|\alpha|,-1) & \beta \in (1/2,1]. \end{cases}$$

Let $t_0 \in [1,2]$ be the first time such that

$$\forall t > t_0 \bigg(\int_{t_0}^t \bigg[\int \big|(\mathbb{I} - \mathbb{P}_{|\alpha|}) Q(|\alpha|,\beta,\tau)\big|^2 d|\alpha| d\beta \bigg] d\tau > 0 \bigg).$$

Here and in the following (Q,P) is the conservative solution, while $(\hat{Q},\hat{P})$ is the dissipative one (both parametrized by $|\alpha|,\beta$).

As an approximation for the dissipative solution, we define

$$(\check{Q},\check{P})(|\alpha|,\beta,t) = \begin{cases} (Q,P)(|\alpha|,\beta) & t \leq |\alpha|, \\ \mathbb{P}_{|\alpha|}(Q,P)(|\alpha|,\beta) & t > |\alpha|, \end{cases}$$

i.e. the trajectories obtained by patching together the conservative solution before the merging time $|\alpha|$, and its projection after the the merging time. This is not a solution, since one has

$$\ddot{\check{q}}_1(|\alpha|,\beta) = -\epsilon \frac{\|q_1\|_2^2}{2} \check{q}_1(|\alpha|,\beta) \neq -\epsilon \frac{\|\check{q}_1\|_2^2}{2} \check{q}_1(|\alpha|,\beta). \tag{32}$$

In particular, for $t > t_0$ it holds

$$\int_{t_0}^t \frac{\|\check{q}_1\|_2^2}{2} - \frac{\|q_1\|_2^2}{2} d\tau < 0.$$

The contradiction we arrive is exactly in the inequality above, which will imply that the particles $q_1(|\alpha|, \beta \in [0,1/2]), q_1(|\alpha|, \beta \in (1/2,1])$ with arrive late at the merging time $t = |\alpha|$.

Claim 2: *The correction $\delta q_1, \delta p_1$ to $\check{q}_1, \check{p}_1$ satisfies*

$$\dot{\delta q_1} = \delta p_1, \quad \dot{\delta p_1} = -\epsilon \bigg(\int \frac{(\check{q}_1 + \delta q_1)^2}{2} \mathcal{L}^2 \bigg) \delta q_1 + \epsilon \bigg[\frac{\|q_1\|_2^2}{2} - \int \frac{(\check{q}_1 + \delta q_1)^2}{2} \mathcal{L}^2 \bigg] \check{q}_1, \tag{33}$$

with initial data $(0,0)$.

Proof. Just substitute and use (32). □

We next use the following simple estimate: if

$$\dot{x} = v, \ \dot{v} = a(t)x + b(t), \qquad x(0), v(0) = 0,$$

then for every $\delta > 0$ there exists $\bar{t}$ such that for $t \in [0, \bar{t}]$

$$|x(t)| \leq (1+\delta) \int_0^t (t-\tau)|b(\tau)|d\tau, \quad |v(t)| \leq (1+\delta) \int_0^t |b(\tau)|d\tau. \tag{34}$$

Moreover $\bar{t} = \frac{\delta}{3(1+\|a\|_\infty)}$ suffices.

Claim 3: *It holds*

$$\int \big|\delta q_1(t)\big| \mathcal{L}^2 \leq 2\epsilon \int_{t_0}^t \bigg(\frac{\|q_1\|_2^2}{2} - \frac{\|\check{q}_1\|_2^2}{2} \bigg) d\tau.$$

Proof. The estimate (34) applied to (33) gives

$$|\delta q_1(|\alpha|, \beta, t)| \leq (1+\delta)\epsilon \int_{t_0}^t \bigg| \frac{\|q_1\|_2^2}{2} - \int \frac{(\check{q}_1 + \delta q_1)^2}{2} \mathcal{L}^2 \bigg| |\check{q}_1(|\alpha, \beta, \tau)| d\tau,$$

for

$$0 \leq t - t_0 \leq \bar{t} = \frac{\delta}{3(1 + \epsilon \sup_{\tau \in [t_0, t]} \int \frac{(\check{q}_1(\tau) + \delta q_1(\tau))^2}{2} \mathcal{L}^2)}.$$

It is an easy computation to show that the first equation gives the claim, if $\epsilon \ll 1$ and in particular the choice $\bar{t} = 1/4$ can be allowed. □

With the above claim, we obtain that

$$\int_{t_0}^t \bigg(\frac{\|q_1(\tau)\|_2^2}{2} - \int \frac{(\check{q}_1(\tau) + \delta q_1(\tau))^2}{2} \mathcal{L}^2 \bigg) d\tau = \big(1 + \mathcal{O}(\epsilon t)\big) \int_{t_0}^t \bigg(\frac{\|q_1(\tau)\|_2^2}{2} - \frac{\|\check{q}_1(\tau)\|_2^2}{2} \bigg) d\tau > 0$$

for $0 < t - t_0 < \bar{t}$, and then using again (33) we conclude that $\delta q_1 < 0$ in a small positive time interval (t_0, t_1). This will implies that for $t_0 < |\alpha| < t_1$ the particles solving the ODE (31) (i.e. the ones which have not yet interacted) will have

$$q_1(|\alpha|, \beta \in [0, 1/2], |\alpha|) < 0 < q_1(|\alpha|, \beta \in (1/2, 1], |\alpha|),$$

contradicting the assumption that they are interacting, i.e. $q_1(|\alpha|, \beta \in [0, 1/2], |\alpha|) = q_1(|\alpha|, \beta \in (1/2, 1], |\alpha|)$.

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