hep-th/0611229
Bicocca-FT-06-18
LPTENS-06/54
SISSA 65/2006/EP

Counting BPS Baryonic Operators in CFTs with Sasaki-Einstein duals

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Abstract

We study supersymmetric D3 brane configurations wrapping internal cycles of type II backgrounds $AdS_5 \times H$ for a generic Sasaki-Einstein manifold H. These configurations correspond to BPS baryonic operators in the dual quiver gauge theory. In each sector with given baryonic charge, we write explicit partition functions counting all the BPS operators according to their flavor and R-charge. We also show how to extract geometrical information about H from the partition functions; in particular, we give general formulae for computing volumes of three cycles in H.

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1 Introduction

The study of BPS states in quantum field theory and in string theory is clearly a very important topic. These states are generically protected against quantum corrections and contain information regarding the strong coupling behaviour of supersymmetric field theories and superstring theories. In the past years they were especially important in the study of strong weak dualities, like the AdS/CFT conjecture [1] which gives a connection between the BPS operators in conformal field theories and BPS states in string theory.

In this paper we discuss the set of one half BPS states in string theory realized as $D3$ branes wrapped on (generically non trivial) three cycles in the supergravity background $AdS_5 \times H$, where H is a Sasaki-Einstein manifold [2, 3]. These states are holographically dual to baryonic BPS operators in $\mathcal{N} = 1$ four dimensional CFTs [4], which are quiver gauge theories.

Recently there has been a lot of work in order to generalize the basic AdS/CFT correspondence to the case of a generic Sasaki-Einstein horizon H. After the discovery of new infinite classes of non compact CY metrics [5–8] and the construction of their dual $\mathcal{N} = 1$ supersymmetric CFT [9–12] [1] there has been a lot of effort to understand the correspondence in the case of $\mathcal{N} = 1$ CFT dual to toric Calabi-Yau manifolds. The correspondence in the toric case is now completely clear [10, 12, 14–22]. The non toric case is still less understood: there exist studies on generalized conifolds [23, 24], del Pezzo series [25–27], and more recently there was a proposal to construct new non toric examples [28].

There has been some parallel interest in counting certain BPS states in the CFTs [29–34]. The result was the construction of a partition function counting the set of mesonic BPS gauge invariant operators with flavor quantum numbers. In the case of scalar operators it was shown [33, 35] that the partition function contains a lot of information regarding the geometry of the CY singularity. For example, using the partition function, we can compute the volume of H and the algebraic equations defining the singularity. The existing countings focus on the BPS mesonic gauge invariant sector of the CFT. Geometrically this corresponds to consider giant graviton configurations [36]: BPS $D3$ branes wrapped on trivial three cycles in H. From the partition function of these string theory objects one can compute the volumes of the total space H but is unable to gain any information regarding the volume of the three cycles in H. The volume of the total space and of the three cycles are duals to the central charge and the R charges of the baryonic operators respectively [4, 37] and hence are very important in the AdS/CFT correspondence. In this paper we push this investigation further: we succeed in counting BPS states charged under the baryonic charges of the field theory and we extract from this baryonic partition function the volume of the three cycles. We will mostly concentrate on the toric case but our procedure seems adaptable to the non toric case as well.

We will use homomorphic surfaces to parameterize the supersymmetric BPS configurations of $D3$ brane wrapped in H, following results in [38, 39] [2]. In the case where X is a toric variety we have globally defined homogeneous coordinates x_i which are charged under the baryonic charges of the theory and which we can use to parametrize these surfaces. We will quantize configurations of D3 branes wrapped on these surfaces and we will find the Hilbert space of BPS states using a prescription found by Beasley [39]. The complete BPS Hilbert space factorizes in sectors with definite baryonic charges. Using toric geometry tools, we can assign to each sector a convex polyhedron P. The BPS operators in a given sector are in one-to-one

[1] See [13], and references therein, for an overview of analogous results for non-conformal fields theories.

[2] See [40, 41] for some recent developments in wrapping branes on non trivial three cycles inside toric singularities.

correspondence with symmetrized products of N (number of colors) integer points in P. The corresponding BPS operators in field theory are constructed by taking the basic baryons and replacing the N constituents elementary chiral superfields with opportune sequences of chiral superfields which are associated to points in P. We will write a partition function Z_D counting the integer points in P_D and a partition function $Z_{D,N}$ counting the integer points in the symmetric product of P_D. The most interesting fact is that $Z_{D,N}$ counts the BPS states with a given set of baryonic and global charges and hence it is the full baryonic partition function. From Z_D, taking a suitable limit, we will be able to compute the volume of the three cycles in H. Although we mainly focus on the toric case we propose a general formula for the computation of the volume of the three cycles valid for every type of conical CY singularity.

2 A short review of toric geometry

In this section we summarize some basic topics of toric geometry; in particular we review divisors and line bundles on toric varieties that will be very useful for the complete understanding of the paper. Very useful references on toric geometry are [42, 43].

A toric variety V_Σ is defined by a fan Σ: a collection of strongly convex rational polyhedral cones in the real vector space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ (N is an n dimensional lattice $N \simeq \mathbb{Z}^n$). Some examples are presented in Figure 1.

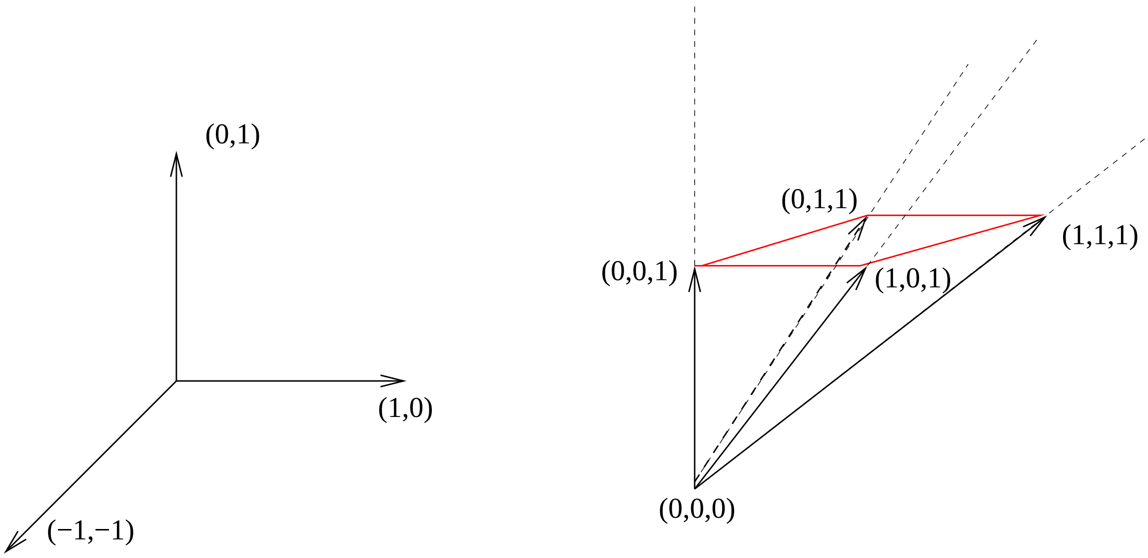


Figure 1: On the left: the fan for $\mathbb{P}^2$ with three maximal cones of dimension two which fill completely $\mathbb{R}^2$; there are three one dimensional cones in $\Sigma(1)$ with generators $\{(1,0),(0,1),(-1,-1)\}$. On the right: the fan for the conifold with a single maximal cone of dimension three; there are four one dimensional cones in $\Sigma(1)$ with generators $\{(0,0,1),(1,0,1),(1,1,1),(0,1,1)\}$.

We define the variety V_Σ as a symplectic quotient [42, 43]. Consider the one dimensional cones of Σ and a minimal integer generator n_i of each of them. Call the set of one dimensional cones $\Sigma(1)$. Assign a "homogeneous coordinate" x_i to each $n_i \in \Sigma(1)$. If $d = \dim\Sigma(1)$, x_i span $\mathbb{C}^d$. Consider the group

$$G = \{(\mu_1, ..., \mu_d\} \in (\mathbb{C}^*)^d | \prod_{i=1}^{d} \mu_i^{<m,n_i>} = 1\,, \quad m \in \mathbb{Z}^3\}\,, \tag{2.1}$$

which acts on x_i as

$$(x_1, ..., x_d) \to (\mu_1 x_1, ..., \mu_d x_d)\,.$$

G is isomorphic, in general, to $(\mathbb{C}^*)^{d-n}$ times a discrete group. The continuous part $(\mathbb{C}^*)^{d-n}$ can be described as follows. Since $d \geqslant n$ the n_i are not linearly independent. They determine $d - n$ linear relations:

$$\sum_{i=1}^{d} Q_i^{(a)} n_i = 0 \tag{2.2}$$

with $a = 1, ..., d - n$ and $Q_i^{(a)}$ generate a $(\mathbb{C}^*)^{d-n}$ action on $\mathbb{C}^d$:

$$(x_1, ..., x_d) \to (\mu^{Q_1^{(a)}} x_1, ..., \mu^{Q_d^{(a)}} x_d) \tag{2.3}$$

where $\mu \in \mathbb{C}^*$.

For each maximal cone $\sigma \in \Sigma$ define the function $f_\sigma = \prod_{n_i \notin \sigma} x_i$ and the locus S as the intersection of all the hypersurfaces $f_\sigma = 0$. Then the toric variety is defined as:

$$V_\Sigma = (\mathbb{C}^d - S)/G$$

There is a residual $(\mathbb{C}^*)^n$ complex torus action acting on V_Σ, from which the name *toric variety*. In the following, we will denote with $T^n \equiv U(1)^n$ the real torus contained in $(\mathbb{C}^*)^n$.

In all the examples in this paper $G = (\mathbb{C}^*)^{d-n}$ and the previous quotient is interpreted as a symplectic reduction. The case where G contains a discrete part includes further orbifold quotients. These cases can be handled similarly to the ones discussed in the main text.

Using these rules to construct the toric variety, it is easy to recover the usual representation for $\mathbb{P}^n$:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\{x \sim \mu x\}$$

where the the minimal integer generators of $\Sigma(1)$ are $n_i = \{e_1, ..., e_n, -\sum_{k=1}^{n} e_k\}$, $d = n + 1$ and $Q = (1, ..., 1)$ (see Figure 1 for the case $n = 2$).

In this paper we will be interested in affine toric varieties, where the fan is a single cone $\Sigma = \sigma$. In this case S is always the null set. It is easy, for example, to find the symplectic quotient representation of the conifold:

$$C(T^{1,1}) = \mathbb{C}^4/(1, -1, 1, -1)$$

where $d = 4$, $n = 3$, $n_1 = (0, 0, 1)$, $n_2 = (1, 0, 1)$, $n_3 = (1, 1, 1)$, $n_4 = (0, 1, 1)$ and we have written $(1, -1, 1, -1)$ for the action of $\mathbb{C}^*$ with charges $Q = (1, -1, 1, -1)$.

This type of description of a toric variety is the easiest one to study divisors and line bundles. Each $n_i \in \Sigma(1)$ determines a T-invariant divisor D_i corresponding to the zero locus $\{x_i = 0\}$ in V_Σ. T-invariant means that D_i is mapped to itself by the torus action $(\mathbb{C}^*)^n$ (for simplicity we will call them only divisors from now on). The d divisors D_i are not all independent but satisfy the n basic equivalence relations:

$$\sum_{i=1}^{d} < e_k, n_i > D_i = 0 \tag{2.4}$$

where e_k with $k = 1, ..., n$ is the orthonormal basis of the dual lattice $M \sim \mathbb{Z}^n$ with the natural paring: for $n \in N$, $m \in M$ $< n, m >= \sum_{i=1}^{n} n_i m_i \in \mathbb{Z}$. Given the basic divisors D_i the generic divisor D is given by the formal sum $D = \sum_{i=1}^{d} c_i D_i$ with $c_i \in \mathbb{Z}$. Every divisor D determines a line bundle $\mathcal{O}(D)$ [3].

There exists a simple recipe to find the holomorphic sections of the line bundle $\mathcal{O}(D)$. Given the c_i, the global sections of $\mathcal{O}(D)$ can be determined by looking at the polytope (a convex rational polyhedron in $M_\mathbb{R}$):

$$P_D = \{u \in M_\mathbb{R} | < u, n_i > \quad \geq \quad - c_i \ , \ \forall i \in \Sigma(1)\} \tag{2.5}$$

where $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. Using the homogeneous coordinate x_i it is easy to associate a section χ^m to every point m in P_D:

$$\chi^m = \prod_{i=1}^{d} x_i^{<m,n_i>+c_i}. \tag{2.6}$$

Notice that the exponent is equal or bigger than zero. Hence the global sections of the line bundle $\mathcal{O}(D)$ over V_Σ are:

$$H^0(V_\Sigma, \mathcal{O}_{V_\Sigma}(D)) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m \tag{2.7}$$

At this point it is important to make the following observation: all monomials χ^m have the same charges under the $(\mathbb{C}^*)^{d-n}$ described at the beginning of this Section (in the following these charges will be identified with the baryonic charges of the dual gauge theory). Indeed, under the $(\mathbb{C}^*)^{d-n}$ action we have:

$$\chi^m \to \prod_{i=1}^{d} (\mu^{<m, Q_i^{(a)} n_i > + Q_i^{(a)} c_i}) x_i^{<m,n_i>+c_i} = \mu^{\sum_{i=1}^{d} Q_i^{(a)} c_i} \chi^m \tag{2.8}$$

where we have used equation (2.2). Similarly, all the sections have the same charge under the discrete part of the group G. This fact has an important consequence. The generic polynomial

$$f = \sum a_m \chi^m \in H^0(V_\Sigma, \mathcal{O}_{V_\Sigma}(D))$$

[3]The generic divisor D on an affine cone is a Weil divisor and not a Cartier divisor [43]; for this reason the map between divisors and line bundles is more subtle, but it can be easily generalized using the homogeneous coordinate ring of the toric variety V_Σ [44] in a way that we will explain. With an abuse of language, we will continue to call the sheaf $\mathcal{O}(D)$ the line bundle associated with the divisor D.

is not a *function* on V_Σ, since it is not invariant under the $(\mathbb{C}^*)^{d-n}$ action (and under possible discrete orbifold actions). However, it makes perfectly sense to consider the zero locus of f. Since all monomials in f have the same charge under $(\mathbb{C}^*)^{d-n}$, the equation $f = 0$ is well defined on V_Σ and defines a divisor [4].

2.1 A simple Example

After this general discussion, let us discuss an example to clarify the previous definitions.

Consider the toric variety $\mathbb{P}^2$. The fan Σ for $\mathbb{P}^2$ is generated by:

$$n_1 = e_1 \; n_2 = e_2 \; n_3 = -e_1 - e_2 \tag{2.9}$$

The three basic divisors D_i correspond to $\{x_1 = 0\}$, $\{x_2 = 0\}$, $\{x_3 = 0\}$, and they satisfy the following relations (see equation (2.4)):

$$\begin{aligned} D_1 - D_3 &= 0 \\ D_2 - D_3 &= 0 \end{aligned}$$

and hence $D_1 \sim D_2 \sim D_3 \sim D$. All line bundles on $\mathbb{P}^2$ are then of the form $\mathcal{O}(nD)$ with an integer n, and are usually denoted as $\mathcal{O}(n) \to \mathbb{P}^2$. It is well known that the space of global holomorphic sections of $\mathcal{O}(n) \to \mathbb{P}^2$ is given by the homogeneous polynomial of degree n for $n \geq 0$, while it is empty for negative n. We can verify this statement using the general construction with polytopes.

Consider the line bundle $\mathcal{O}(D_1)$ associated with the divisor D_1. In order to construct its global sections we must first determine the polytope P_{D_1} $(c_1 = 1, c_2 = c_3 = 0)$:

$$P_{D_1} = \{u_1 \geqslant -1,\; u_2 \geqslant 0,\; u_1 + u_2 \leqslant 0\} \tag{2.10}$$

Then, using (2.6), it easy to find the corresponding sections:

$$\{x_1,\; x_2,\; x_3\} \tag{2.11}$$

These are the homogeneous monomials of order one over $\mathbb{P}^2$. Indeed we have just constructed the line bundle $\mathcal{O}(1) \to \mathbb{P}^2$ (see Figure 2).

Consider as a second example the line bundle $\mathcal{O}(D_1 + D_2 + D_3)$. In this case the associated polytope is:

$$P_{D_1+D_2+D_3} = \{u_1 \geqslant -1,\; u_2 \geqslant -1,\; u_1 + u_2 \leqslant 1\} \tag{2.12}$$

Using (2.6) it is easy to find the corresponding sections:

$$\{x_1^3,\; x_1^2 x_2,\; x_1 x_2 x_3, ...\} \tag{2.13}$$

These are all the homogeneous monomials of degree 3 over $\mathbb{P}^2$; we have indeed constructed the line bundle $\mathcal{O}(3) \to \mathbb{P}^2$ (see Figure 2).

[4] In this way, we can set a map between linearly equivalent divisors and sections of the sheaf $\mathcal{O}_{V_\Sigma}(D)$ generalizing the usual map in the case of standard line bundles.

The examples of polytopes and line bundles presented in this Section are analogous to the ones that we will use in the following to characterize the BPS baryonic operators. The only differences (due to the fact that we are going to consider affine toric varieties) is that the polytope P_D will be a non-compact rational convex polyhedron, and the space of sections will be infinite dimensional.

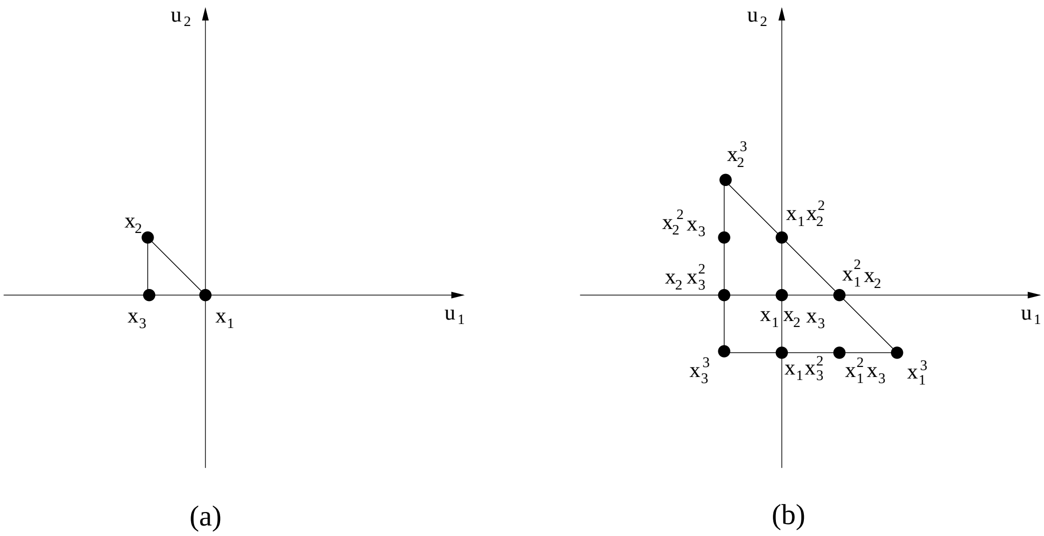


Figure 2: (a) The polytope associated to the line bundle $\mathcal{O}(1) \rightarrow \mathbb{P}^2$. (b) The polytope associated to the line bundle $\mathcal{O}(3) \rightarrow \mathbb{P}^2$.

3 BPS D3 brane configurations

In this Section we discuss Beasley's prescription [39] for determining the BPS Hilbert space corresponding to supersymmetric D3 brane configurations. We generalize the example of the conifold presented in [39] to the case of a generic toric Calabi-Yau cone.

3.1 Motivations

Consider the supersymmetric background of type IIB supergravity $AdS_5 \times H$ with H a Sasaki-Einstein manifold. This geometry is obtained by taking the near horizon geometry of a stack of N $D3$ on the isolated Goreinstein singularity of a local Calabi-Yau three-fold given by the real cone $C(H)$ over the base H. The D3 branes fill the four dimensional Minkowski space-time M_4 in $M_4 \times C(H)$.

The dual superconformal field theory is a quiver gauge theory: an $\mathcal{N} = 1$ supersymmetric quantum field theory with gauge group $SU(N_1) \times ...SU(N_k)$ and chiral superfields that transform under the fundamental of a gauge group and the anti-fundamental of another gauge group. Due to the presence of $SU(N)$ type groups these theories have generically baryonic like operators inside their spectrum and these are the objects we are interested in.

Let us take the field theory dual to the conifold singularity as a basic example. The theory has gauge group $SU(N)\times SU(N)$ and chiral superfields A_1, A_2 that transform under the fundamental of the first gauge group and under the anti-fundamental of the second one, and B_1, B_2 that transform under the conjugate representation. There exists also a non-abelian global symmetry $SU(2)\times SU(2)$ under which the A fields transform as $(2,1)$ and the B as $(1,2)$. The superpotential is $W=\epsilon_{ij}\epsilon_{pq}A_iB_pA_jB_q$. It is known that this theory has one baryonic charge and that the A_i fields have charge one under this symmetry and the B_i fields have charge minus one. Hence one can build the two basic baryonic operators:

$$\begin{aligned}\epsilon^1_{p_1,...,p_N}\epsilon_2^{k_1,...,k_N}(A_{i_1})^{p_1}_{k_1}...(A_{i_N})^{p_N}_{k_N} &= (\det A)_{(i_1,...,i_N)}\\ \epsilon^1_{p_1,...,p_N}\epsilon_2^{k_1,...,k_N}(B_{i_1})^{p_1}_{k_1}...(B_{i_N})^{p_N}_{k_N} &= (\det B)_{(i_1,...,i_N)}\end{aligned} \tag{3.1}$$

These operators are clearly symmetric in the exchange of the A_i and B_i respectively, and transform under $(N+1,1)$ and $(1,N+1)$ representation of $SU(2)\times SU(2)$. The important observation is that these are the baryonic operators with the smallest possible dimension: $\Delta_{\det A,\det B}=N\Delta_{A,B}$. One can clearly construct operator charged under the baryonic symmetry with bigger dimension in the following way. Defining the operators [39, 45] [5]

$$A_{I;J}=A_{i_1}B_{j_1}...A_{i_m}B_{j_m}A_{i_{m+1}} \tag{3.2}$$

the generic type A baryonic operator is:

$$\epsilon^1_{p_1,...,p_N}\epsilon_2^{k_1,...,k_N}(A_{I_1;J_1})^{p_1}_{k_1}...(A_{I_N;J_N})^{p_N}_{k_N}\,. \tag{3.3}$$

One can clearly do the same with the type B operators.

Using the tensor relation

$$\epsilon_{\alpha_1...\alpha_N}\epsilon^{\beta_1...\beta_N}=\delta^{\beta_1}_{[\alpha_1}...\delta^{\beta_N}_{\alpha_N]}\,, \tag{3.4}$$

depending on the symmetry of (3.3), one can sometimes factorize the operator in a basic baryon times operators that are neutral under the baryonic charge [39,45]. It is a notorious fact that the AdS/CFT correspondence maps the basic baryonic operators (3.1) to static $D3$ branes wrapping specific three cycles of $T^{1,1}$ and minimizing their volumes. The volumes of the $D3$ branes are proportional to the dimension of the dual operators in CFT. Intuitively, the geometric dual of an operator (3.3) is a fat brane wrapping a three cycle, not necessarily of minimal volume, and moving in the $T^{1,1}$ geometry (we will give more rigorous arguments below). If we accept this picture the factorizable operators in field theory can be interpreted in the geometric side as the product of gravitons/giant gravitons states with a static $D3$ brane wrapped on some cycle, and the non-factorisable ones are interpreted as excitation states of the basic $D3$ branes or non-trivial brane configurations.

[5] which are totally symmetric in the $SU(2)\times SU(2)$ indices due to the F-term relations $A_iB_pA_j=A_jB_pA_i$, $B_pA_iB_q=B_qA_iB_p$.

What we would like to do is to generalize this picture to a generic conical CY singularity. Using a clever parametrization of the possible $D3$ brane BPS configurations in the geometry found in [38, 39], we will explain how it is possible to characterize all the baryonic operators in the dual $SCFT$, count them according to their charges and extract geometric information regarding the cycles.

3.2 Supersymmetric D3 brane configurations

Consider supersymmetric $D3$ branes wrapping three-cycles in H. There exists a general characterization of these types of configurations [38, 39] that relates the $D3$-branes wrapped on H to holomorphic four cycles in $C(H)$. The argument goes as follows. Consider the euclidean theory on $\mathbb{R}^4 \times C(H)$. It is well known that one $D3$-brane wrapping a holomorphic surface S in $C(H)$ preserves supersymmetry. If we put N $D3$-branes on the tip of the cone $C(H)$ and take the near horizon limit the supergravity background becomes $Y_5 \times H$ where Y_5 is the euclidean version of AdS_5. We assume that S intersects H in some three-dimensional cycle C_3. The BPS $D3$ brane wrapped on S looks like a point in $\mathbb{R}^4$ and like a line in Y_5: it becomes a brane wrapped on a four-dimensional manifold in $\gamma \times H$ where γ is the geodesic in Y_5 obtained from the radial direction in $C(H)$. Using the $SO(5,1)$ global symmetry of Y_5 we can rotate γ into any other geodesic in Y_5. For this reason when we make the Wick rotation to return to Minkowski signature (this procedure preserves supersymmetry) we may assume that γ becomes a time-like geodesic in AdS_5 spacetime. In this way we have produced a supersymmetric $D3$ bra! ne wrapped on a three cycle in H which moves along γ in AdS_5. Using the same argument in the opposite direction, we realize also that any supersymmetric $D3$ brane wrapped on H can be lifted to a holomorphic surface S in $C(H)$.

Due to this characterization, we can easily parametrize the classical configuration space $\mathcal{M}_{cl}$ of supersymmetric $D3$ brane using the space of holomorphic surfaces in $C(H)$ without knowing the explicit metric on the Sasaki-Einstein space H (which is generically unknown!).

The previous construction characterizes all kind of supersymmetric configurations of wrapped D3 branes. These include branes wrapping trivial cycles and stabilized by the combined action of the rotation and the RR flux, which are called giant gravitons in the literature [36]. Except for a brief comment on the relation between giant gravitons and dual giant gravitons, we will be mostly interested in D3 branes wrapping non trivial cycles. These correspond to states with non zero baryonic charges in the dual field theory. The corresponding surface D in $C(H)$ is then a non trivial divisor, which, modulo subtleties in the definition of the sheaf $\mathcal{O}(D)$, can be written as the zero locus of a section of $\mathcal{O}(D)$

$$\chi = 0 \qquad\qquad \chi \in H^0(X, \mathcal{O}(D)) \tag{3.5}$$

3.2.1 The toric case

The previous discussion was general for arbitrary Calabi-Yau cones $C(H)$. ¿From now on we will mostly restrict to the case of an affine toric Calabi-Yau cone $C(H)$.

For this type of toric manifolds the fan Σ described in Section 2 is just a single cone σ, due to the fact that we are considering a singular affine variety. Moreover, the Calabi-Yau nature of the singularity requires that all the generators of the one dimensional cone in $\Sigma(1)$ lie on a plane; this is the case, for example, of the conifold pictured in Figure 1. We can then characterize the variety with the convex hull of a fixed number of integer points in the plane: the toric diagram (Figure 3). For toric varieties, the equation for the D3 brane configuration can be written quite explicitly using homogeneous coordinates. As explained in Section 2, we can associate to every vertex of the toric diagram a global homogeneous coordinate x_i. Consider a divisor D. All the supersymmetric configurations of D3 branes corresponding to surfaces linearly equivalent to D can be written as the zero locus of the generic section of $H^0(V_\Sigma, \mathcal{O}_{V_\Sigma}(D))$

$$P(x_1, x_2, ..., x_d) \equiv \sum_{m\in P_D\cap M} h_m \chi^m = 0 \qquad (3.6)$$

As discussed in Section 2, the sections take the form of the monomials (2.6)

$$\chi^m = \prod_{i=1}^{d} x_i^{<m,n_i>+c_i}$$

and there is one such monomial for each integer point $m \in M$ in the polytope P_D associated with D as in equation (2.5)

$$\{u \in M_{\mathbb{R}} | < u, n_i > \ \geq \ - c_i \ , \ \forall i \in \Sigma(1)\}$$

As already noticed, the x_i are only defined up to the rescaling (2.3) but the equation $P(x_1, ...x_d) = 0$ makes sense since all monomials have the same charge under $(\mathbb{C}^*)^{d-3}$ (and under possible discrete orbifold actions). Equation (3.6) generalizes the familiar description of hypersurfaces in projective spaces $\mathbb{P}^n$ as zero locus of homogeneous polynomials. In our case, since we are considering affine varieties, the polytope P_D is non-compact and the space of holomorphic global sections is infinite dimensional.

We are interested in characterizing the generic supersymmetric $D3$ brane configuration with a fixed baryonic charge. We must therefore understand the relation between divisors and baryonic charges: it turns out that there is a one-to one correspondence between baryonic charges and classes of divisors modulo the equivalence relation (2.4) [6]. We will understand this point by analyzing in more detail the $(\mathbb{C}^*)^{d-3}$ action defined in Section 2.

3.2.2 The assignment of charges

To understand the relation between divisors and baryonic charges, we must make a digression and recall how one can assign $U(1)$ global charges to the homogeneous coordinates associated to a given toric diagram [10, 12, 19, 20].

Non-anomalous $U(1)$ symmetries play a very important role in the dual gauge theory and it turns out that we can easily parametrize these global symmetries directly

[6] In a fancy mathematical way, we could say that the baryonic charges of a D3 brane configuration are given by an element of the Chow group $A_2(C(H))$.

from the toric diagram. In a sense, we can associate field theory charges directly to the homogeneous coordinates.

For a background with horizon H, we expect $d-1$ global non-anomalous symmetries, where d is number of vertices of the toric diagram [7]. We can count these symmetries by looking at the number of massless vectors in the AdS dual. Since the manifold is toric, the metric has three isometries $U(1)^3 \equiv T^3$, which are the real part of the $(\mathbb{C}^*)^3$ algebraic torus action. One of these, generated by the Reeb vector, corresponds to the R-symmetry while the other two give two global flavor symmetries in the gauge theory. Other gauge fields in AdS come from the reduction of the RR four form on the non-trivial three-cycles in the horizon manifold H, and there are $d-3$ three-cycles in homology [12]. On the field theory side, these gauge fields correspond to baryonic symmetries. Summarizing, the global non-anomalous symmetries are:

$$U(1)^{d-1} = U(1)_F^2 \times U(1)_B^{d-3} \tag{3.7}$$

In this paper we use the fact that these $d-1$ global non-anomalous charges can be parametrized by d parameters $a_1, a_2, \ldots, a_d$, each associated with a vertex of the toric diagram (or a point along an edge), satisfying the constraint:

$$\sum_{i=1}^{d} a_i = 0 \tag{3.8}$$

The $d-3$ baryonic charges are those satisfying the further constraint [12]:

$$\sum_{i=1}^{d} a_i n_i = 0 \tag{3.9}$$

where n_i are the vectors of the fan: $n_i = (y_i, z_i, 1)$ with (y_i, z_i) the coordinates of integer points along the perimeter of the toric diagram. The R-symmetries are parametrized with the a_i in a similar way of the other non-baryonic global symmetry, but they satisfy the different constraint

$$\sum_{i=1}^{d} a_i = 2 \tag{3.10}$$

due to the fact that the terms in the superpotential must have R-charges equal to two.

Now that we have assigned trial charges to the vertices of a toric diagram and hence to the homogeneous coordinates x_i, we can return to the main problem of identifying supersymmetric D3 branes with fixed baryonic charge. Comparing equation (2.2) with equation (3.9), we realize that the baryonic charges a_i in the dual field theory are the charges $Q_i^{(a)}$ of the action of $(\mathbb{C}^*)^{d-3}$ on the homogeneous coordinates x_i. We can now assign a baryonic charge to each monomials made with the homogeneous

[7] More precisely, d is the number of integer points along the perimeter of the toric diagram. Smooth horizons have no integer points along the sides of the toric diagram except the vertices, and d coincides with the number of vertices. Non smooth horizons have sides passing through integer points and these must be counted in the number d.

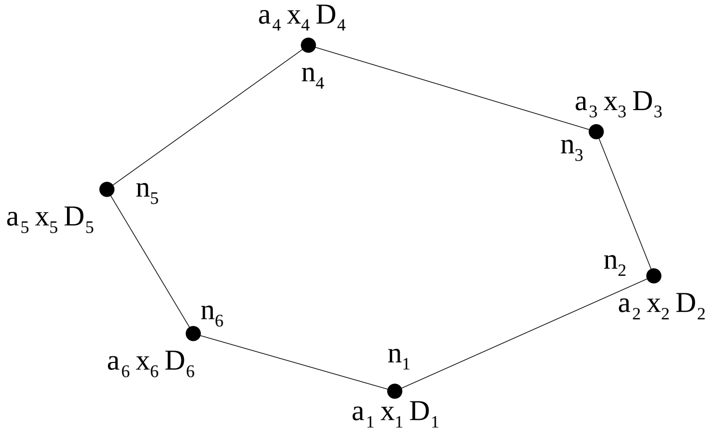


Figure 3: A generic toric diagram with the associated trial charges a_i, homogeneous coordinates x_i and divisors D_i.

coordinates x_i. All terms in the equation (3.6) corresponding to a D3 brane wrapped on D are global sections χ^m of the line bundle $\mathcal{O}(\sum_{i=1}^d c_i D_i)$ and they have all the same $d-3$ baryonic charges $B^a = \sum_{i=1}^d Q_i^{(a)} c_i$; these are determined only in terms of the c_i defining the corresponding line bundle (see equation (2.8)).

Using this fact we can associate a divisor in $C(H)$ to every set of $d-3$ baryonic charges. The procedure is as follows. Once we have chosen a specific set of baryonic charges B^a we determine the corresponding c_i using the relation $B^a = \sum_{i=1}^d c_i Q_i^{(a)}$. These coefficients define a divisor $D = \sum_{i=1}^d c_i D_i$. It is important to observe that, due to equation (2.2), the c_i are defined only modulo the equivalence relation $c_i \sim c_i + < m, n_i >$ corresponding to the fact that the line bundle $\mathcal{O}(\sum_{i=1}^d c_i D_i)$ is identified only modulo the equivalence relations (2.4) $D \sim D + \sum_{i=1}^d < m, n_i > D_i$. We conclude that baryonic charges are in one-to-one relation with divisors modulo linear equivalence.

For simplicity, we only considered continuous baryonic charges. In the case of varieties which are orbifolds, the group G in equation (2.1) contains a discrete part. In the orbifold case, the quantum field theory contains baryonic operators with discrete charges. This case can be easily incorporated in our formalism.

3.2.3 The quantization procedure

Now we want to quantize the classical configuration space $\mathcal{M}_{cl}$ using geometric quantization [46] following [39].

Considering that P and λP with $\lambda \in \mathbb{C}^*$ vanish on the same locus in $C(H)$, it easy to understand that the various distinct surfaces in $C(H)$ correspond to a specific set of h_m with $m \in P_D \cap M$ modulo the equivalence relation $h_m \sim \lambda h_m$: this is a point in the infinite dimensional space $\mathbb{CP}^\infty$ in which the h_m are the homogeneous coordinates. Thus we identify the classical configurations space $\mathcal{M}_{cl}$ of supersymmetric $D3$ brane associated to a specific line bundle $\mathcal{O}(D)$ as $\mathbb{CP}^\infty$ with homogeneous coordinates h_m.

A heuristic way to understand the geometric quantization is the following. We can think of the $D3$ brane as a particle moving in $\mathcal{M}_{cl}$ and we can associate to it a wave function Ψ taking values in some holomorphic line bundle $\mathcal{L}$ over $\mathbb{CP}^{\infty}$. The reader should not confuse the line bundle $\mathcal{L}$, over the classical configuration space $\mathcal{M}_{cl}$ of wrapped D3 branes, with the lines bundles $\mathcal{O}(D)$, which are defined on $C(H)$. Because all the line bundles $\mathcal{L}$ over a projective space are determined by their degree (i.e. they are of the form $\mathcal{O}(\alpha)$) we have only to find the value of α. This corresponds to the phase picked up by the wavefunction Ψ when the $D3$ brane goes around a closed path in $\mathcal{M}_{cl}$. Moving in the configuration space $\mathcal{M}_{cl}$ corresponds to moving the $D3$ brane in H. Remembering that a $D3$ brane couples to the four form field C_4 of the supergravity and that the backgrounds we are considering are such that $\int_H F_5 = N$, it was argued in [39] that the wavefunction Ψ picks up the phase $e^{2\pi i N}$ and $\alpha = N$. For this reason $\mathcal{L} = \mathcal{O}(N)$ and the global holomorphic sections of this line bundle over $\mathbb{CP}^{\infty}$ are the degree N polynomials in the homogeneous coordinates h_m.

Since the BPS wavefunctions are the global holomorphic sections of $\mathcal{L}$, we have that the BPS Hilbert space $\mathcal{H}_D$ is spanned by the states:

$$|h_{m_1}, h_{m_2}, ..., h_{m_N} > \tag{3.11}$$

This is Beasley's prescription.

We will make a correspondence in the following between h_m and certain operators in the field theory with one (or more) couple of free gauge indices

$$h_m \, , \qquad\qquad (O_m)_{\alpha\beta} \text{ (or } (\mathrm{O_m})_{\alpha_1\beta_1 \ldots \alpha_k\beta_k}) \tag{3.12}$$

where the O_m are operators with fixed baryonic charge. The generic state in this sector $|h_{m_1}, h_{m_2}, ..., h_{m_N} >$ will be identify with a gauge invariant operator obtained by contracting the O_m with one (or more) epsilon symbols. The explicit example of the conifold is discussed in details in [39]: the homogeneous coordinates with charges $(1, -1, 1, -1)$ can be put in one-to-one correspondence with the elementary fields (A_1, B_1, A_2, B_2) which have indeed baryonic charge $(1, -1, 1, -1)$. The use of the divisor D_1 (modulo linear equivalence) allows to study the BPS states with baryonic charge $+1$. It is easy to recognize that the operators $A_{I,J}$ in equation (3.2) have baryonic charge $+1$ and are in one-to-one correspondence with the sections of $\mathcal{O}(D_1)$

$$\sum_{m \in P_{D_1} \cap M} h_m \chi^m = h_1 x_1 + h_3 x_3 + ...$$

The BPS states $|h_{m_1}, h_{m_2}, ..., h_{m_N} >$ are then realized as all the possible determinants, as in equation (3.3).

3.3 Comments on the relation between giant gravitons and dual giant gravitons

Among the $\mathcal{O}(D)$ line bundles there is a special one: the bundle of holomorphic functions $\mathcal{O}$ [8]. It corresponds to the supersymmetric $D3$ brane configurations wrapped on homologically trivial three cycles C_3 in H (also called giant gravitons [36]).

[8] Clearly there are other line bundles equivalent to $\mathcal{O}$. For example, since we are considering Calabi-Yau spaces, the canonical divisor $K = -\sum_{i=1}^{d} D_i$ is always trivial and $\mathcal{O} \sim \mathcal{O}(\sum_{i=1}^{d} D_i)$.

When discussing trivial cycles, we can parameterize holomorphic surfaces just by using the embedding coordinates [9]. Our discussion here can be completely general and not restricted to the toric case. Consider the general Calabi-Yau algebraic affine varieties V that are cone over some compact base H (they admit at least a $\mathbb{C}^*$ action $V = C(H)$). These varieties are the zero locus of a collection of polynomials in some $\mathbb{C}^k$ space. We will call the coordinates of the $\mathbb{C}^k$ the embedding coordinates z_j, with $j = 1, ..., k$. The coordinate rings $\mathbb{C}[V]$ of the varieties V are the restriction of the polynomials in $\mathbb{C}^k$ of arbitrary degree on the variety V:

$$\mathbb{C}[V] = \frac{\mathbb{C}[z_1, ...z_k]}{\{p_1, ..., p_l\}} = \frac{\mathbb{C}[z_1, ...z_k]}{\mathbb{I}[V]} \tag{3.13}$$

where $\mathbb{C}[z_1, ..., z_k]$ is the $\mathbb{C}$-algebra of polynomials in k variables and p_j are the defining equations of the variety V. We are going to consider the completion of the coordinate ring (potentially infinity polynomials) whose generic element can be written as the (infinite) polynomial in $\mathbb{C}^k$

$$P(z_1, ...z_k) = c + c_i z_i + c_{ij} z_i z_j + ... = \sum_I c_I z_I \tag{3.14}$$

restricted by the algebraic relations $\{p_1 = 0, ..., p_l = 0\}$[10].

At this point Beasley's prescription says that the BPS Hilbert space of the giant gravitons $\mathcal{H}_g$ is spanned by the states

$$|c_{I_1}, c_{I_2}, ..., c_{I_N} > \tag{3.15}$$

These states are holomorphic polynomials of degree N over $\mathcal{M}_{cl}$ and are obviously symmetric in the c_{I_i}. For this reason we may represent (3.15) as the symmetric product:

$$Sym(|c_{I_1} > \otimes |c_{I_2} > \otimes ... \otimes |c_{I_N} >) \tag{3.16}$$

Every element $|c_{I_i} >$ of the symmetric product is a state that represents a holomorphic function over the variety $C(H)$. This is easy to understand if one takes the polynomial $P(z_1, ...z_k)$ and consider the relations among the c_I induced by the radical ring $\mathbb{I}[V]$ (in a sense one has to quotient by the relations generated by $\{p_1 = 0, ..., p_l = 0\}$). For this reason the Hilbert space of giant gravitons is:

$$\mathcal{H}_g = \bigotimes^{N\ Sym} \mathcal{O}_{C(H)} \tag{3.17}$$

Obviously, we could have obtained the same result in the toric case by applying the techniques discussed in this Section. Indeed if we put all the c_i equal to zero the polytope P_D reduces to the dual cone $\mathcal{C}^*$ of the toric diagram, whose integer points corresponds to holomorphic functions on $C(H)$ [35].

[9] These can be also expressed as specific polynomials in the homogeneous coordinates such as that their total baryonic charges are zero.

[10] There exist a difference between the generic baryonic surface and the mesonic one: the constant c. The presence of this constant term is necessary to represent the giant gravitons, but if we take for example the constant polynomial $P = c$ this of course does not intersect the base H and does not represent a supersymmetric $D3$ brane. However it seems that this is not a difficulty for the quantization procedure [32, 39]

In a recent work [34] it was shown that the Hilbert space of a dual giant graviton[11] $\mathcal{H}_{dg}$ in the background space $AdS_5 \times H$, where H is a generic Sasaki-Einstein manifold, is the space of holomorphic functions over the cone $C(H)$. At this point it is easy to understand why the counting of $1/8$ BPS states of giant gravitons and dual giant gravitons give the same result [31, 32]: the counting of $1/8$ BPS mesonic state in field theory. Indeed:

$$\mathcal{H}_g = \bigotimes^{N\ Sym} \mathcal{H}_{dg} \tag{3.18}$$

4 Flavor charges of the BPS baryons

In the previous Section, we discussed supersymmetric $D3$ brane configurations with specific baryonic charge. Now we would like to count, in a sector with given baryonic charge, the states with a given set of flavor charges $U(1) \times U(1)$ and R-charge $U(1)_R$. The generic state of the BPS Hilbert space (3.11) is, by construction, a symmetric product of the single states $|h_m>$. These are in a one to one correspondence with the integer points in the polytope P_D, which correspond to sections χ^m. As familiar in toric geometry [42, 43], a integer point $m \in M$ contains information about the charges of the T^3 torus action, or, in quantum field theory language, about the flavor and R charges.

Now it is important to realize that, as already explained in Section 3, the charges a_i that we can assign to the homogeneous coordinates x_i contain information about the baryonic charges (we have already taken care of them) but also about the flavor and R charges in the dual field theory. If we call f_i^k with $k = 1, 2$ the two flavor charges and R_i the R-charge, the section χ^m has flavor charges (compare equation (2.6)):

$$f_m^k = \sum_{i=1}^{d} (< m, n_i > + c_i) f_i^k \tag{4.1}$$

and R-charge:

$$R_m = \sum_{i=1}^{d} (< m, n_i > + c_i) R_i \tag{4.2}$$

It is possible to refine the last formula. Indeed the R_i, which are the R-charges of a set of elementary fields of the gauge theory [12, 19] [12], are completely determined by the Reeb vector of H and the vectors n_i defining the toric diagram [17]. Moreover, it is possible to show that $\sum_{i=1}^{d} n_i R_i = \frac{2}{3} b$ [19], where b specifies how the Reeb vector lies inside the T^3 toric fibration. Hence:

$$R_m = \frac{2}{3} < m, b > + \sum_{i=1}^{d} c_i R_i \tag{4.3}$$

[11] A dual giant graviton is a D3 brane wrapped on a three-sphere in AdS_5

[12] The generic elementary field in the gauge theory has an R-charge which is a linear combination of the R_i [19].

This formula generalizes an analogous one for mesonic operators [28]. Indeed if we put all the c_i equal to zero the polytope P_D reduces to the dual cone $\mathcal{C}^*$ of the toric diagram [35]. We know that the elements of the mesonic chiral ring of the CFT correspond to integer points in this cone and they have R-charge equal to $\frac{2}{3} < m, b >$. In the case of generic c_i, the right most factor of (4.3) is in a sense the background R charge: the R charge associated to the fields carrying the non-trivial baryonic charges. In the simple example of the conifold discussed in subsection 3.1, formula (4.3) applies to the operators (3.2) where the presence of an extra factor of A takes into account the background charge. In general the R charge (4.3) is really what we expect from an operator in field theory that is given by elementary fields with some baryonic charges dressed by "mesonic insertions".

The generic baryonic configuration is constructed by specifying N integer points m_ρ in the polytope P_D. Its R charge R_B is

$$R_B = \frac{2}{3}\sum_{\rho=1}^{N} < m_\rho, b > + N\sum_{i=1}^{d} c_i R_i \tag{4.4}$$

This baryon has N times the baryonic charges of the associated polytope. Recalling that at the superconformal fixed point dimension Δ and R-charge of a chiral superfield are related by $R = 2\Delta/3$, it is easy to realize that the equation (4.4) is really what is expected for a baryonic object in the dual superconformal field theory. Indeed if we put all the m_ρ equal to zero we have (this means that we are putting to zero all the mesonic insertions)

$$\Delta_B = N\sum_{i=1}^{d} c_i \Delta_i \tag{4.5}$$

This formula can be interpreted as follows. The elementary divisor D_i can be associated with (typically more than one) elementary field in the gauge theory, with R charge R_i. By taking just one of the c_i different from zero in formula (4.5), we obtain the dimension of a baryonic operator in the dual field theory: take a fixed field, compute its determinant and the dimension of the operator is N times the dimension of the individual fields. These field operators correspond to $D3$ branes wrapped on the basic divisors D_i and are static branes in the $AdS_5 \times H$ background [13]. They wrap the three cycles C_3^i obtained by restricting the elementary divisors D_i at $r = 1$. One can also write the R_i in terms of the volume of the Sasaki-Einstein space H and of the volume of C_3^i [4]:

$$R_i = \frac{\pi \mathrm{Vol}(C_3^i)}{3\mathrm{Vol}(H)} \tag{4.6}$$

Configurations with more than one non-zero c_i in equation (4.5) correspond to basic baryons made with elementary fields whose R-charge is a linear combination of the R_i (see [19]) or just the product of basic baryons.

[13] The generic configuration of a $D3$ brane wrapped on a three cycle C_3 in H is given by a holomorphic section of $\mathcal{O}(D)$ that is a non-homogeneous polynomial under the R-charge action. For this reason, and holomorphicity, it moves around the orbits of the Reeb vector [38, 39]. Instead the configuration corresponding to the basic baryons is given by the zero locus of a homogeneous monomial (therefore, as surface, invariant under the R-charge action), and for this reason it is static.

The generic baryonic configuration has N times the R-charge and the global charges of the basic baryons (static branes which minimize the volume in a given homology class) plus the charges given by the fattening and the motion of the three cycle inside the geometry (the mesonic fluctuations on the $D3$ brane or "mesonic insertions" in the basic baryonic operators in field theory). It is important to notice that the BPS operators do not necessarily factorize in a product of basic baryons times mesons [14].

4.1 Setting the counting problem

In Section 6 we will count the baryonic states of the theory with given baryonic charges (polytope P_D) according to their R and flavor charges. Right now we understand the space of classical supersymmetric $D3$ brane configurations $\mathcal{N}$ as a direct sum of holomorphic line bundles over the variety $C(H)$:

$$\mathcal{N} = \bigoplus_{c_i \sim c_i + <m,n_i>} \mathcal{O}\Big(\sum_i^d c_i D_i\Big) \tag{4.7}$$

where the c_i specify the baryonic charges. We have just decomposed the space $\mathcal{N}$ into sectors according to the grading given by the baryonic symmetry. Geometrically, this is just the decomposition of the homogeneous coordinate ring of the toric variety under the grading given by the action of $(\mathbb{C}^*)^{d-3}$. Now, we want to introduce a further grading. Inside every line bundle there are configurations with different flavor and R charges.

Once specified the baryonic charges, the Hilbert space of BPS operators is the N order symmetric product of the corresponding line bundle. Hence the $1/2$ BPS Hilbert space is also decomposed as:

$$\mathcal{H} = \bigoplus_{c_i \sim c_i + <m,n_i>} \mathcal{H}_D \tag{4.8}$$

We would like to count the baryonic operators of the dual $SCFT$ with a given set of flavor and R charges. We can divide this procedure into three steps:

- find a way to count the global sections of a given holomorphic line bundle (a baryonic partition function Z_D);
- write the total partition function for the N-times symmetric product of the polytope P_D (the partition function $Z_{D,N}$). This corresponds to find how many combinations there are with the same global charges a_B^k (with $k = 1, 2$ for the flavor charges and $k = 3$ for the R charge) for a given baryonic state: the possible set of m_ρ such that:

$$a_B^k - N \sum_{i=1}^d c_i a_i^k = \sum_{m_\rho \in P_D \cap M} \sum_{i=1}^d < m_\rho, n_i > a_i^k \, . \tag{4.9}$$

[14] In the simple case of the conifold this is due to the presence of two fields A_i with the same gauge indices; only baryons symmetrized in the indices i factorize [39, 45]. In more general toric quiver gauge theories it is possible to find different strings of elementary fields with the same baryonic charge connecting a given pair of gauge groups; their existence prevents the generic baryons from being factorizable.

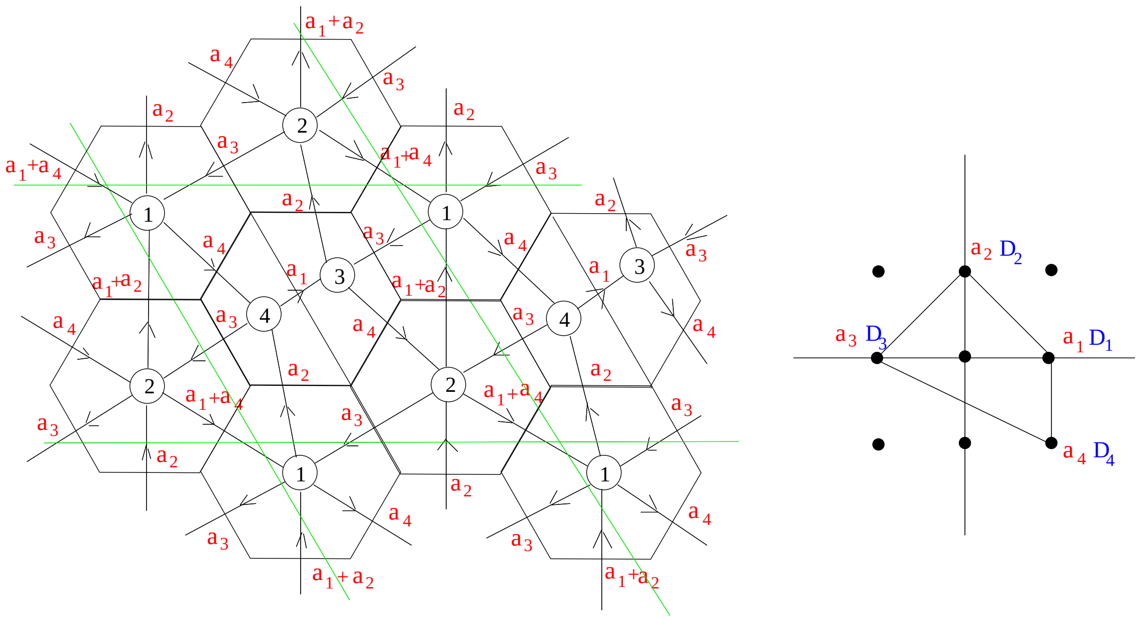
- write the complete BPS partition function of the field theory by summing over all sectors with different baryonic charges. Eventually we would also like to write the complete BPS partition function of the field theory including all the d charges at a time: $d-3$ baryonic, 2 flavor and 1 R charges [47].

In the following Sections, we will solve completely the first two steps. The third step is complicated by the fact that the correspondence between the homogeneous coordinates and fields carrying the same $U(1)$ charges is not one to one. From the gravity side of the AdS/CFT correspondence one can explain this fact as follow [12]. The open strings attached to a $D3$ brane wrapped on the non-trivial three cycles corresponding to the basic baryons in the dual field theory have in general many supersymmetric vacuum states. This multiplicity of vacua corresponds to the fact that generically the first homotopy group of the three cycles $\pi_1(C_3)$ is non-trivial and one can turn on a locally flat connection with non-trivial Wilson lines. The different Wilson lines give the different open string vacua and these are associated with different elementary fields X_{ij} (giving rise to basic baryons $\det X_{ij}$ with the same global charges). One has then to include non trivial multiplicities for the $Z_{D,N}$ when computing the complete BPS partition function. The determination of the partition function depending on all the d charges is left for future work [47].

5 Comparison with the field theory side

At this point it is probably worthwhile to make a more straight contact with the field theory. Before doing this we very briefly review some results about the AdS/CFT correspondence for the class of toric geometries. The gauge theory dual to a given toric singularity is completely identified by the *dimer configuration*, or *brane tiling* (Figure 5) [16, 21, 22]. This is a bipartite graph drawn on a torus T^2: it has an equal number of white and black vertices and links connect only vertices of different colors. In the dimer the faces represent $SU(N)$ gauge groups, oriented links represent chiral bifundamental multiplets and nodes represent the superpotential: the trace of the product of chiral fields around a node gives a superpotential term with sign + or - according to whether the vertex is a white one or a black one. By applying Seiberg dualities to a quiver gauge theory we can obtain different quivers that flow in the IR to the same CFT: to a toric diagram we can associate different quivers/dimers describing the same physics. It turns out that one can always find phases where all the gauge groups have the same number of colors; these are called *toric phases*. Seiberg dualities keep constant the number of gauge groups F, but may change the number of fields E, and therefore the number of superpotential terms $V = E - F$. The toric phases having the minimal set of fields are called *minimal toric phases*.

There is a general recipe for assigning baryonic, flavors and R charges to the elementary fields in the CFT [10, 12, 19, 20]. As described in the last part of the Section 4, due to the non-trivial homotopy of the cycles the baryonic ground state of the $D3$ branes in H has multiplicity and this fact gives in the $SCFT$ many fields with the same charges. A method to compute the charges and the multiplicity of the elementary fields was given in [10, 12, 19]. As explained in Section 2 one can



(1) (2)

Figure 4: (1) Dimer configuration for the field theory dual to $C(Y^{2,1})$ with a given assignment of charges a_i and the orientation given by the arrows connecting the gauge groups. We have drawn in green the bounds of the basic cell. For notational simplicity we have not indicated with different colors the vertices; the dimer is a bipartite graph and this determines an orientation. (2) Toric diagram for the singularity $C(Y^{2,1})$.

associate with each vector n_i a chiral field with trial global charge a_i. We call C the set of all the unordered pairs of vectors in the (p,q) web (the (p,q) web is the set of vectors perpendicular to the edges of the toric diagram and with the same length as the corresponding edge); we label an element of C with the ordered indexes (i,j), with the convention that the vector v_i can be rotated to v_j in the counter-clockwise direction with an angle $\leq 180^o$. With our conventions $|\langle v_i, v_j\rangle| = \langle v_i, v_j\rangle$, where with $\langle\ ,\ \rangle$ we mean the determinant of the 2×2 matrix. Associate with any element of C the divisor

$$D_{i+1} + D_{i+2} + \dots D_j \tag{5.1}$$

and a type of chiral field in the field theory with multiplicity $\langle v_i, v_j\rangle$ and global charge equal to $a_{i+1} + a_{i+2} + \dots a_j$. The indexes i, j are always understood to be defined modulo d.

With this machinery in our hands we can analyze the the field theory operators corresponding to the $D3$ brane states analyzed in the past Sections. The first thing to understand is the map between the section χ^m of the line bundle we are considering and the field theory operators. We would like to associate to every point in the polytope P_D a sequence of contractions of elementary fields modulo F-term equivalence.

Take two gauge group U, V in the dimer and draw an oriented path P connecting them (an oriented path in the dimer is a sequence of chiral fields oriented in the same way and with the gauge indices contracted along the path). The global charges of P is the sum of the charges of the fields contained in P and can be schematically written as:

$$\sum_{i=1}^{d} c_i a_i \tag{5.2}$$

with some integers c_i. Draw another oriented path Q connecting the same gauge groups. Consider now the closed non-oriented path $Q - P$; as explained in [48] the charges for a generic non-oriented closed path can be written as (for non-oriented paths one has to sum the charges of the fields with the same orientation and subtract the charges of the fields with the opposite orientation):

$$\sum_{i=1}^{d} < m, n_i > a_i \tag{5.3}$$

with m a three dimensional integer vector. Hence the charges for a generic path Q connecting two gauge groups are:

$$\sum_{i=1}^{d} (< m, n_i > + c_i) a_i \, . \tag{5.4}$$

Because the path Q is oriented we have just summed the charges along it and the coefficients of the a_i are all positive:

$$< m, n_i > + c_i \geqslant 0 \, . \tag{5.5}$$

The freedom to change P corresponds to the fact that the c_i are defined modulo the equivalence $c_i \sim c_i + < m, n_i >$. Observe that (5.5) is the same condition on the exponents for the homogeneous coordinates x_i in the global sections χ^m (2.5), and the equivalence relation $c_i \sim c_i + < m, n_i >$ corresponds to the equivalence relation on the divisors $D \sim D + \sum_{i=1}^{d} < m, n_i > D_i$. Hence we realize that to every path connecting a couple of gauge groups we can assign a point in a polytope specified by the c_i modulo the equivalence relations.

Now that we have a concrete link between the paths in the dimer and the integer points in the polytope we want to show that this map is well defined: namely we map F-term equivalent operators to the same point in the polytope and to a point in the polytope corresponds only one operator in field theory modulo F terms relations: the injectivity of the map. The first step is easy to demonstrate: paths that are F-term equivalents have the same set of $U(1)$ charges and are mapped to the same point m in the polytope P_D. Conversely if paths connecting two gauge groups are mapped to the same point m it means that they have the same global charges. The path $P - P'$ is then a closed unoriented path with charge 0. As shown in [48, 49] P and P' are then homotopically equivalent [15]. Now we can use the Lemma 5.3.1 in [49] that say: "in a consistent tiling, paths with the same R-charge are F-term equivalent if and

[15] Indeed it is possible to show that m_1 and m_2 of a closed path are its homotopy numbers around the dimer.

only if they are homotopic" to conclude that paths mapped in the same point m in P_D are F-term equivalent. Moreover if we suppose that the map is surjective on P_D, than we conclude that to every point in P_D we have only one field theory operator in the chiral ring of the CFT [16].

Summarizing we have shown that, once we have chosen a pair of gauge groups (U, V), all the paths P linking the gauge groups U and V can be mapped to points in a polytope P_D defined by the charge $\sum_{i=1}^{d} c_i a_i$ of the shortest path ($m = 0$) connecting the two groups. Clearly in general there will be many couples of gauge groups with the same c_i and hence the same polytope P_D. For this reason we say that the polytope P_D has a multiplicity $\#_D$. The important point is that for every single polytope in the family of P_D the map between the points and the paths/operators is one to one modulo F-term relations.

The proposal for finding the gauge invariant operator dual to the BPS state $|h_{m_1}, ..., h_{m_N} >$ is the following[17]. Starting from the BPS state $|h_{m_1}, ..., h_{m_N} >$ in the gravity side to every h_{m_j}, with section χ^{m_j}, we associate an operator in field theory in the way we have just described (from now on we will call these paths the building blocks of the baryons). We assume that in this construction we encountered only single connected paths in the dimer connecting different gauge groups; the general case will be discussed in section 5.2. The building blocks are then operators with just two free gauge indices $(O_m)_{\alpha\beta}$. Then we take all the operators connecting the same gauge groups and we construct a gauge invariant operator by contracting all the N free indices of one gauge group with its epsilon and all the N free indices of the other gauge group with its own epsilon. The field theory operator we have just constructed has clearly the same global charges of the corresponding state in the string theory side and due to the epsilon contractions is symmetric in the permutation of the field theory building blocks like the string theory state. The number of baryonic operators (modulo F-term relations) with flavors and R charges associated with m can be obtained by multiplying in all possible ways N sections χ^{m_i} with $\sum_{i=1}^{N} m_i = m$ and by multiplying for the multiplicity $\#_D$ of the polytope.

As an example of this construction, we now discuss the baryonic building blocks associated with a line bundle over the Calabi-Yau cone $C(Y^{2,1})$.

5.1 Building blocks for $\mathcal{O}(D_3) \to C(Y^{2,1})$

Let us explain the map between the homogeneous coordinates and field theory operators in a simple example: $C(Y^{2,1})$. The cone over $Y^{2,1}$ has four divisors with three equivalence relations (see Figure 5) and hence we have the assignment:

$$D_1 = 3D \qquad D_2 = -2D \qquad D_3 = D \qquad D_4 = -2D \tag{5.6}$$

We want to construct the building blocks of the BPS operators with baryonic charge equal to one. Because the $(\mathbb{C}^*)^{d-3} = \mathbb{C}^*$ action is specified by the charge:

$$Q = (+3, -2, +1, -2) \tag{5.7}$$

[16] This uniqueness is only up to the multiplicity of the fields with the same charges (i.e paths with the same charges linking different gauge groups) due to the fact that the paths are open.

[17] This is just a simple generalization of the one in [39]

we choose $c_3 = 1$ and $c_1 = c_2 = c_4 = 0$. Hence:

$$P_{D_3} = \{m\epsilon M | m_1 + m_3 \geq 0, m_2 + m_3 \geq 0, -m_1 + m_3 \geq -1, m_1 - m_2 + m_3 \geq 0\} \quad (5.8)$$

We can now easily construct the sections of the corresponding line bundle $\mathcal{O}(D_3)$ and try to match these with the BPS operators, which are just the open paths in the dimer (see Figure 5) with the same trial charges a_i of the polynomial in the homogeneous variables. Looking at the dimer of $Y^{2,1}$ we immediately realize that there are three distinct pairs of gauge groups with charge $\sum_{i=1}^{d} c_i a_i = a_3$: $(2,1)$, $(4,2)$, $(1,3)$. Hence the multiplicity of P_{D_3} is $\#_{D_3} = 3$ and for every point in the polytope we have three different operators in the field theory side, corresponding to paths in the dimer connecting the three different pairs of gauge groups. In Table 1 we match the sections in the geometry side with the operators in the field theory side for few points in the polytope P_{D_3}.

(m_1, m_2, m_3)	sections	charges	$(2,1)$	$(4,2)$	$(1,3)$
(0,0,0)	x_3	a_3	$X^{dl}_{21} = Y$	$X^{dl}_{42} = Y$	$X^{dl}_{13} = Y$
(1,0,0)	$x_1 x_4$	$a_1 + a_4$	$X^{dr}_{21} = V$	$X^{ur}_{43} X^{dr}_{32} = ZU$	$X^{dr}_{14} X^{ur}_{43} = UZ$
(1,1,0)	$x_1 x_2$	$a_1 + a_2$	$X^{u}_{21} = V$	$X^{ur}_{43} X^{u}_{32} = ZU$	$X^{u}_{14} X^{ur}_{43} = UZ$
(-1,0,1)	$x_2 x_3^3$	$a_2 + 3a_3$	$X^{dl}_{21} X^{u}_{14} X^{dl}_{42} X^{dl}_{21}$ $= YUYY$	$X^{dl}_{42} X^{dl}_{21} X^{u}_{14} X^{dl}_{42}$ $= YYUY$	$X^{dl}_{13} X^{u}_{32} X^{dl}_{21} X^{dl}_{13}$ $= YUYY$
(-1,-1,1)	$x_3^3 x_4$	$3a_3 + a_4$	$X^{dl}_{21} X^{dl}_{13} X^{dr}_{32} X^{dl}_{21}$ $= YYUY$	$X^{dl}_{42} X^{dl}_{21} X^{dr}_{14} X^{dl}_{42}$ $= YYUY$	$X^{dl}_{13} X^{dr}_{32} X^{dl}_{21} X^{dl}_{13}$ $= YUYY$
...	...	...	...	...	...

Table 1: Few sections of $\mathcal{O}(D_3)$ and the corresponding field theory operators of baryon number 1. We write: the point m in the polytope; the corresponding section (x_i are the homogeneous coordinates); its charges; the three corresponding gauge operators (X are the fundamental fields): we used the label u, l, d, r for *up*, $left$, *down*, *right*, to specify the field direction and, for comparison with the literature, the field is also written using the notations commonly adopted for $Y^{p,q}$ [9].

As a simple check of the correspondence one can observe, by looking at the dimer (Figure 5), that in the fourth and fifth lines of Table 1 one can assign different field operators to the same sections, but it is easy to realize that these are related by F-term equations. Hence in this simple case the correspondence between geometry and field theory is manifest.

The gauge invariant operators are obtained by taking all these operators connecting the same gauge groups and by contracting all the N free indices of one gauge group with its epsilon and all the N free indices of the other gauge group with its own epsilon. All operators come in triples with the same quantum numbers.

5.2 Comments on the general correspondence

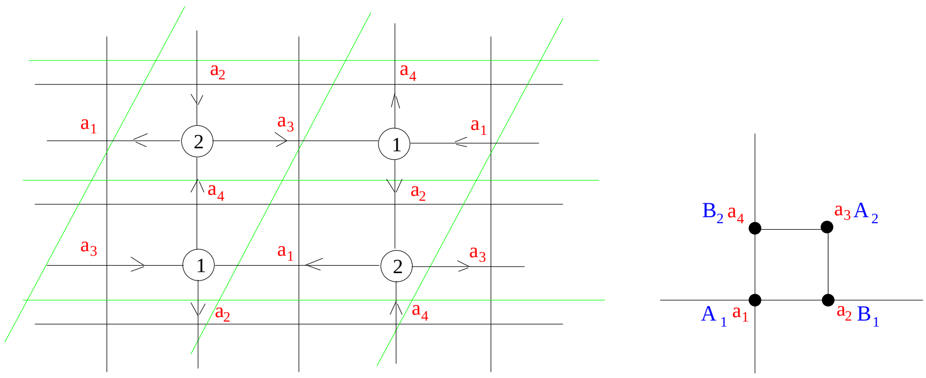
In the generic case there are various subtleties related to the construction of gauge invariant operators and the multiplicity of the polytope. Let us consider the simplest example: the conifold. Suppose we want to classify the BPS operators with baryonic charges equal to $2N$. Following our recipe we must find the pairs of gauge groups with, for example, $c_1 = 2$ and hence charge $2a_1$. However we can not find a single path in the dimer (Figure 5.2) connecting the two gauge groups with charge $2a_1$, but in field theory we clearly have baryonic operators with charge $2N$, for example $\det A_1 \cdot \det A_1$. Clearly all the products of two baryonic operators with baryon number N give a baryonic operator with baryon number $2N$. In a sense in the conifold all the operators in sectors with baryonic charge with absolute value bigger than one are factorized [39, 45].

To take into account this fact we can modify a little our prescription for constructing the gauge invariants. For every set of c_i we determine all the possible paths and multi-paths with the corresponding charges. For example, to the section χ^m in the polytope P_{2D_1} for the conifold we assign two paths connecting the two gauge groups with charges $\sum_{i=1}^{d} < m^{(1)}, n_i > +a_1$ and $\sum_{i=1}^{d} < m^{(2)}, n_i > +a_1$ with $m = m^{(1)} + m^{(2)}$ and therefore a building block consisting of two operators $(O_{m^{(1)}})_{\alpha_1\beta_1}(O_{m^{(2)}})_{\alpha_2\beta_2}$.

The second step is to construct the related gauge invariant operators. Out of these building blocks we cannot construct a single determinant because we don't have an epsilon symbol with $2N$ indices, but we can easily construct a product operator using four epsilons.

We expect, based on Beasley's prescription, a one to one correspondence between the points in the N times symmetric product of the polytope P_{2D_1} and the baryonic operators with baryonic charge $+2$ in field theory (modulo the multiplicity of the polytope). Naively, it would seem that, with the procedure described above, we have found many more operators. Indeed the procedure was plagued by two ambiguities: in the construction of the building blocks, it is possible to find more than a pair of paths corresponding to the same m (and thus the same $U(1)$ charges) that are not F-term equivalent; in the construction of the gauge invariants we have the ambiguity on how to distribute the operators between the two determinants. The interesting fact is that these ambiguities seem to disappear when we consider the final results for gauge invariant operators, due to the F-term relations and the properties of the epsilon symbol. One can indeed verify, at least in the case of P_{2D_1} and various values of N, there is exactly a one to one correspondence between the points in the N times symmetric product of the polytope and the baryonic operators in field theory. The fact that the correct correspondence is between the states $|h_{m_1}, ..., h_{m_N} >$ and baryonic gauge invariant operators is what we expect. The sections in the geometry are not states of the string theory and the paths/operators are not gauge invariant operators. What the AdS/CFT correspondence tells us is that there exist a one to one relation between states in string theory and gauge invariant operators in field theory, and this is the apparent one to one relation between the points (with multiplicity) in the symmetric product of a given polytope and the baryonic operators constructed using the building blocks and the epsilon symbols.

Once we have understood this simple example we can try to generalize the correspondence between states in string theory and gauge invariant operators to the case of a generic assignment of baryonic charges compatible with the geometry. As a first step we can observe that for a polytope P_D associated with a divisor D such that $D = E + F$, where E and F are other two divisors, we have the equation



(1) (2)

Figure 5: (1) Dimer configuration for the field theory dual to $C(T^{1,1})$ with a given assignment of charges a_i and the orientation given by the arrows linking the gauge groups. We have drawn in green the bounds of the basic cell. (2) Toric diagram for the singularity $C(T^{1,1})$.

$P_D = P_E + P_F$ [43]. This means that we can write every point m in P_D as a sum $m = m_1 + m_2$ where $m_1 \in P_E$ and $m_2 \in P_F$. Hence given a couple of gauge groups that are not directly connected (we mean that doesn't exist a chiral field in the fundamental of one gauge group and the anti-fundamental of the other gauge group, but we need at least two chiral fields to connect the two gauge groups) we can decompose the paths connecting them in shorter paths.

A proposal for a general recipe for the multiplicity of polytopes and the construction of gauge invariant operators is the following. Chose the baryonic charges of the operators you are interested in and the corresponding set of c_i. Look at the dimer and find all the possible paths linking two neighbouring gauge groups such that they can be composed to give a connected or not connected path with charges $\sum_{i=1}^{d} c_i a_i$. The multiplicity of the polytope $P_{c_i D_i}$ can be extracted from the number of ways in which you can compose the elementary paths to form a multi-path with the baryonic charges of the polytope. ¿From the multi-path one construct the gauge invariant operators by contracting the free gauge indices with the appropriate number of epsilon symbols.

This is clearly only a preliminary discussion, but fully understanding the multiplicity of a polytope is of fundamental importance if one wants to write the complete partition function of a given $SCFT$ [47].

6 Counting BPS baryonic operators

In this Section, as promised, we count the number of BPS baryonic operators in the sector of the Hilbert space $\mathcal{H}_\mathcal{D}$, associated with a divisor D. All operators in $\mathcal{H}_\mathcal{D}$ have fixed baryonic charges. Their number is obviously infinite, but, as we will show, the number of operators with given charge $m \in T^3$ under the torus action is finite. It thus makes sense to write a partition function $Z_{D,N}$ for the BPS baryonic operators weighted by a T^3 charge $q = (q_1, q_2, q_3)$. $Z_{D,N}$ will be a polynomial in the q_i such that to every monomial $n\ q_1^{m_1} q_2^{m_2} q_3^{m_3}$ we associate n BPS D3 brane states with the R-charge and the two flavor charges parametrized by $\sum_{i=1}^{d}(< m, n_i > a_i + N c_i a_i)$.

The computation of the weighted partition function is done in two steps. We first compute a weighted partition function Z_D, or character, counting the sections of $\mathcal{O}(D)$; these correspond to the h_m which are the elementary constituents of the baryons. In a second time, we determine the total partition function $Z_{D,N}$ for the states $|h_{m_1}...h_{m_N} >$ in $\mathcal{H}_\mathcal{D}$.

6.1 The character Z_D

We want to resum the character, or weighted partition function,

$$Z_D = \mathrm{Tr}\{\mathrm{q}|\mathrm{H}^0(\mathrm{X}, \mathcal{O}(\mathrm{D})\} = \sum_{\mathrm{m} \in \mathrm{P_D} \cap \mathrm{M}} \mathrm{q}^{\mathrm{m}} \tag{6.1}$$

counting the integer points in the polytope P_D weighted with their charge under the T^3 torus action.

In the trivial case $\mathcal{O}(D) \sim \mathcal{O}$, Z_D is just the partition function for holomorphic functions discussed in [33, 35], which can be computed using the Atyah-Singer index theorem [35]. Here we show how to extend this method to the computation of Z_D for a generic divisor D.

Suppose that we have a smooth variety with a holomorphic action of T^k (with $k = 1, 2, 3$ and $k = 3$ is the toric case) and a line bundle $\mathcal{O}(D)$. Suppose also that the higher dimensional cohomology of the line bundle vanishes, $H^i(X, \mathcal{O}(D)) = 0$, for $i \geq 1$. The character (6.1) then coincides with the Leftschetz number

$$\chi(q, D) = \sum_{p=0}^{3} (-1)^p \mathrm{Tr}\{\mathrm{q}|\mathrm{H}^\mathrm{p}(\mathrm{X}, \mathcal{O}(\mathrm{D}))\} \tag{6.2}$$

which can be computed using the index theorem [50]: we can indeed write $\chi(q, D)$ as a sum of integrals of characteristic classes over the fixed locus of the T^k action. In this paper, we will only consider cases where T^k has isolated fixed points P_I. The general case can be handled in a similar way. In the case of isolated fixed points, the general cohomological formula[18] considerably simplifies and can be computed by linearizing the T^k action near the fixed points. The linearized action can be analysed as follows. Since P_I is a fixed point, the group T^k acts linearly on the normal (=tangent) space at P_I, $TX_{P_I} \sim \mathbb{C}^3$. The tangent space will split into three one dimensional representations $TX_{P_I} = \sum_{\lambda=1}^{3} L^\lambda$ of the abelian group T^k. We denote

[18] Equivariant Riemann-Roch, or the Lefschetz fixed point formula, reads

$$\chi(q, D) = \sum_{F_i} \int_{F_i} \frac{\mathrm{Todd}(F_i) Ch^q(D)}{\prod_\lambda (1 - q^{m^i_\lambda} e^{-x_\lambda})} \tag{6.3}$$

where F_i are the set of points, lines and surfaces which are fixed by the action of $q \in T^k$, Todd(F) is the Todd class Todd$(F) = 1 + c_1(F) + ...$ and, on a fixed locus, $Ch^q(D) = q^{m^0} e^{c_1(D)}$ where m^0 is the weight of the T^k action. The normal bundle N_i of each fixed submanifold F_i has been splitted in line bundles; x_λ are the basic characters and m^i_λ the weights of the q action on the line bundles.

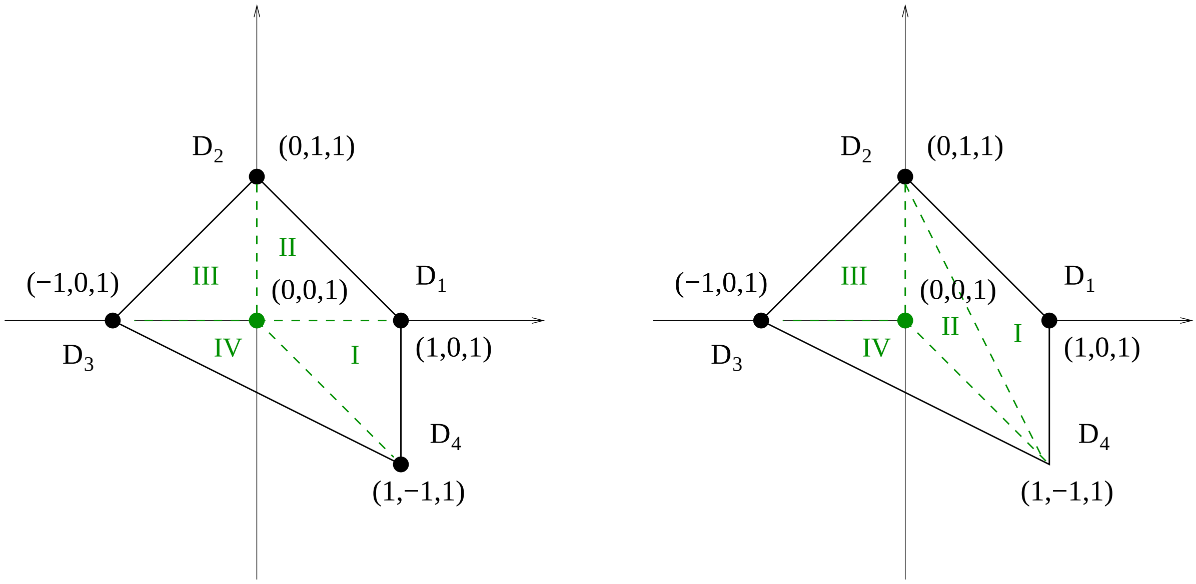


Figure 6: Two triangulation for the toric diagram of $Y^{2,1}$. The internal point $(0,0,1)$ has been blown up. One line and four planes have been added to the original fan. There are four maximal cones and corresponding fixed points, denoted I,II,III and IV.

the corresponding weights for the q action with $m_I^\lambda, \lambda = 1,2,3$. Denote also with m_I^0 the weight of the action of q on the $\mathbb{C}$ fiber of the line bundle $\mathcal{O}(D)$ over P_I. The equivariant Riemann-Roch formula expresses the Leftschetz number as a sum over the fixed points

$$\chi(q,D) = \sum_{P_I} \frac{q^{m_I^0}}{\prod_{\lambda=1}^3 (1 - q^{m_I^\lambda})} \tag{6.4}$$

We would like to apply the index theorem to our Calabi-Yau cone. Unfortunately, $X = C(H)$ is not smooth and a generic element of T^k has a fixed point at the apex of the cone, which is exactly the singular point. To use Riemann-Roch we need to resolve the cone X to a smooth variety $\tilde{X}$ and to find a line bundle $\mathcal{O}(\tilde{D})$ on it with the following two properties: i) it has the same space of sections, $H^0(\tilde{X}, \mathcal{O}(\tilde{D})) = H^0(X, \mathcal{O}(D))$, ii) it has vanishing higher cohomology $H^i(\tilde{X}, \mathcal{O}(\tilde{D})) = 0, i \geq 1$.

Notice that the previous discussion was general and apply to all Sasaki-Einstein manifolds H. It gives a possible prescription for computing Z_D even in the non toric case. In the following we will consider the case of toric cones where the resolution $\tilde{X}$ and the divisor $\tilde{D}$ can be explicitly found.

Toric Calabi-Yau cones have a pretty standard resolution by triangulation of the toric diagram, see Figure 6. The fan of the original variety X consists of a single maximal cone, with a set of edges, or one-dimensional cones, $\Sigma(1)$ whose generators n_i are not linearly independent in $\mathbb{Z}^3$. The resolutions of X consist of all the possible subdivisions of the fan in smaller three dimensional cones σ_I. The new variety $\tilde{X}$ is still a Calabi-Yau if all the minimal generators n_i of the one-dimensional cones lie on a plane. This process looks like a triangulation of the toric diagram. If each three-dimensional cone is generated by linearly independent primitive vectors, the variety is smooth. The smooth Calabi-Yau resolutions of X thus consist of all the triangulation of the toric diagram which cannot be further subdivided. Each three dimensional cone σ_I is now a copy of $\mathbb{C}^3$ and the smooth variety $\tilde{X}$ is obtained by gluing all these $\mathbb{C}^3$ according to the rules of the fan. T^3 acts on each σ_I in a simple way: the three weights of the T^3 action on a copy of $\mathbb{C}^3$ are just given by the primitive inward normal vectors m_I^λ to the three faces of σ_I. Notice that each σ_I contains exactly one fixed point of T^3 (the origin in the copy of $\mathbb{C}^3$) with weights given by the vectors m_I^λ.

The line bundles on $\tilde{X}$ are given by $\tilde{D} = \sum_i c_i D_i$ where the index i runs on the set of one-dimensional cones $\tilde{\Sigma}(1)$, which is typically bigger than the original $\Sigma(1)$. Indeed, each integer internal point of the toric diagram gives rise in the resolution $\tilde{X}$ to a new divisor. The space of sections of $\tilde{D}$ are still determined by the integral points of the polytope

$$\tilde{P}_D = \{u \in M_{\mathbb{R}} | < u, n_i > \geq -c_i \, , \ \forall i \in \tilde{\Sigma}(1)\} \tag{6.5}$$

It is important for our purposes that each maximal cone σ_I determines a integral point $m_I^0 \in M$ as the solution of this set of three equations:

$$< m_I^0, n_i >= -c_i, \ \ n_i \in \sigma_I, \tag{6.6}$$

In a smooth resolution $\tilde{X}$ this equation has always integer solution since the three generators n_i of σ_I are a basis for $\mathbb{Z}^3$. As shown in [43], m_I^0 is the charge of the local equation for the divisor $\tilde{D}$ in the local patch σ_I. It is therefore the weight of the T^3 action on the fiber of $\mathcal{O}(D)$ over the fixed point contained in σ_I.

The strategy for computing Z_D is therefore the following. We smoothly resolve X and find a divisor $\tilde{D} = \sum_i c_i D_i$ by assigning values c_i to the new one-dimensional cones in $\tilde{\Sigma}(1)$ that satisfies the two conditions

- It has the same space of sections, $H^0(\tilde{X}, \mathcal{O}(\tilde{D})) = H^0(X, \mathcal{O}(D))$. Equivalently, the polytope $\tilde{P}_D$ has the same integer points of P_D.
- It has vanishing higher cohomology $H^i(\tilde{X}, \mathcal{O}(\tilde{D})) = 0, i \geq 1$. As shown in [43] this is the case if there exist integer points $m_I^0 \in M$ that satisfy the convexity condition [19]

$$\begin{aligned} < m_I^0, n_i > &= -c_i, \ \ n_i \in \sigma_I \\ < m_I^0, n_i > &\geq -c_i, \ \ n_i \notin \sigma_I \end{aligned} \tag{6.7}$$

There are many different smooth resolution of X, corresponding to the possible complete triangulation of the toric diagram. It is shown in the Appendix B that we can always find a compatible resolution $\tilde{X}$ and a minimal choice of c_i that satisfy the two given conditions.

[19] The m_I^0s determine a continuous piecewise linear function ψ_D on the fan as follows: in each maximal cone σ_I the function ψ_D is given by $< m_I^0, v >$, $v \in \sigma_I$. As shown in [43], the higher dimensional cohomology vanishes, $H^i(\tilde{X}, \mathcal{O}(D)) = 0$, $i \geq 1$, whenever the function ψ_D is upper convex.

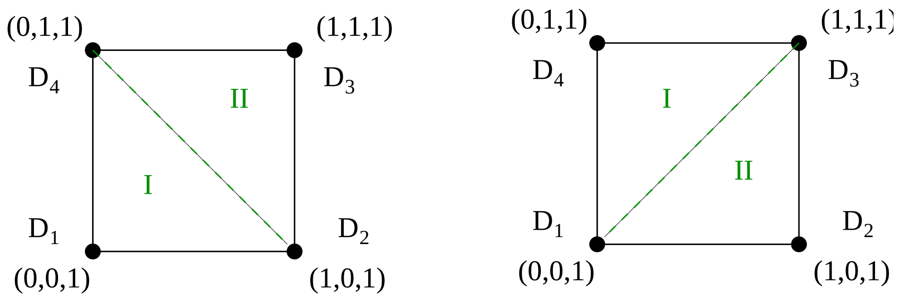


Figure 7: The two resolutions for the conifold. No internal points have been blown up. In each case, one line have been added to the original fan; there are two maximal cones and corresponding fixed points, denoted I,II.

The function Z_D is then given as

$$Z_D = \sum_{P_I} \frac{q^{m_I^0}}{\prod_{\lambda=1}^{3}(1 - q^{m_I^\lambda})} \tag{6.8}$$

where in the toric case for every fixed point P_I there is a maximal cone σ_I, m_I^λ are the three inward primitive normal vectors of σ_I and m_I^0 are determined by equation (6.7). This formula can be conveniently generalized to the case where the fixed points are not isolated but there are curves or surfaces fixed by the torus action.

6.2 Examples

6.2.1 The conifold

The four primitive generators for the one dimensional cones of the conifold are $\{(0,0,1),(1,0,1),(1,1,1),(0,1,1)\}$ and we call the associated divisors D_1, D_2, D_3 and D_4 respectively. They satisfy the equivalence relations $D_1 \sim D_3 \sim -D_2 \sim -D_4$. There is only one baryonic symmetry under which the four homogeneous coordinates transform as

$$(x_1, x_2, x_3, x_4) \sim (x_1\mu, x_2/\mu, x_3\mu, x_4/\mu) \tag{6.9}$$

The conifold case is extremely simple in that the chiral fields of the dual gauge theory are in one-to-one correspondence with the homogeneous coordinates: $(x_1, x_2, x_3, x_4) \sim (A_1, B_1, A_2, B_2)$. Recall that the gauge theory is $SU(N) \times SU(N)$ with chiral fields A_i and B_p transforming as $(N, \bar{N})$ and $(\bar{N}, N)$ and as $(2,1)$ and $(1,2)$ under the enhanced $SU(2)^2$ global flavor symmetry.

The two possible resolutions for the conifold are presented in Figure 7. We first compute the partition function for the divisor D_1 using the resolution on the left hand side of the figure. Regions I and II correspond to the two maximal cones in the resolution and, therefore, to the two fixed points of the T^3 action. Denote also $q = (q_1, q_2, q_3)$. Using the prescriptions given above, we compute the three primitive inward normals to each cone and the weight of the T^3 action on the fiber. It is manifest that the conditions required in equation (6.7) are satisfied.

Region I	$m_I^\lambda = \{((1,0,0),(0,1,0),(-1,-1,1)\}$	$m_I^0 = (1,1,-1)$
Region II	$m_{II}^\lambda = \{(0,-1,1),(-1,0,1),(1,1,-1)\}$	$m_{II}^0 = (0,0,0)$

$$Z_{D_1} = \frac{q_1(q_2-q_3)+q_3-q_2q_3}{(1-q_1)(1-q_2)(1-q_3/q_1)(1-q_3/q_2)q_3} \tag{6.10}$$

For simplicity, let us expand Z_{D_1} along the direction of the Reeb vector $(3/2, 3/2, 3)$ by putting $q_1 = q_2 = q, q_3 = q^2$. This corresponds to count mesonic excitations according to their R-charge, forgetting about the two $U(1)^2$ flavor indices.

$$Z_{D_1} = \frac{2}{(1-q)^3} = \sum_{n=0}^{\infty}(n+1)(n+2)q^n = 2+6q+12q^2+... \tag{6.11}$$

This counting perfectly matches the list of operators in the gauge theory. In the sector of Hilbert space with baryonic charge +1 we find the operators (3.2)

$$A_i\,, \qquad A_iB_pA_j\,, \qquad A_iB_pA_jB_qA_k\,, \qquad .... \tag{6.12}$$

The F-term equations $A_iB_pA_j = A_jB_pA_i$, $B_pA_iB_q = B_qA_iB_p$ guarantee that the $SU(2)\times SU(2)$ indices are totally symmetric. The generic operator is then of the form $A(BA)^n$ transforming in the $(n+2, n+1)$ representation of $SU(2)\times SU(2)$ thus exactly matching the q^n term in Z_{D_1}. The R-charge of the operators in Z_{D_1} is accounted by the exponent of q by adding the factor $q^{\sum c_iR_i} = q^{1/2}$ which is common to all the operators in this sector (cfr. equation (4.2)). The result perfectly matches with the operators $A(BA)^n$ since the exact R-charge of A_i and B_i is 1/2. We could easily include the $SU(2)^2$ charges in this counting.

Analogously, we obtain for Z_{D_3}

$$Z_{D_3} = \frac{q_1(q_2-q_3)+q_3-q_2q_3}{(1-q_1)(1-q_2)(q_1-q_3)(q_2-q_3)} = q_3Z_{D_1}/(q_2q_1) \tag{6.13}$$

Since $D_1 \sim D_3$ the polytope P_{D_3} is obtained by P_{D_1} by a translation and the the two partition functions Z_{D_1} and Z_{D_3} are proportional. Finally, the partition functions for D_2 and D_4 are obtained by choosing the resolution in the right hand side of figure 7, for which is possible to satisfy the convexity condition (6.7)

$$\begin{aligned} Z_{D_2} &= \frac{q_2(q_1+q_2-q_1q_2-q_3)}{(1-q_1)(1-q_2)(q_1-q_3)(q_2-q_3)} \\ Z_{D_4} &= \frac{q_1(q_1+q_2-q_1q_2-q_3)}{(1-q_1)(1-q_2)(q_1-q_3)(q_2-q_3)} = q_1Z_{D_2}/q_2 \end{aligned} \tag{6.14}$$

6.2.2 Other examples: $Y^{p,q}$, delPezzo and $L^{p,q,r}$

In this Section we give other examples of partition functions Z_D considering the $Y^{p,q}$, the delPezzo and $L^{p,q,r}$ manifolds.

The $Y^{p,q}$ toric diagram has four vertices and one baryonic charge. The dual gauge theory has an $SU(2)\times U(1)$ flavor symmetry. We consider the simplest example, $Y^{2,1}$. The fan for $Y^{2,1}$ has four primitive generators $\{(1,0,1),(0,1,1),(-1,0,1),(1,-1,1)\}$. The equivalence relations among divisors give $D_2 \sim D_4 \sim -2D_3$ and $D_1 = 3D_3$ and the corresponding homogeneous coordinates scale as

$$(x_1, x_2, x_3, x_4) \sim (x_1\mu^3, x_2/\mu^2, x_3\mu, x_4/\mu^2) \tag{6.15}$$

under the baryonic symmetry.

There are two different completely smooth resolutions that are presented in Figure 6. The toric diagram has one internal point; the corresponding four cycle is blown up in each smooth resolution of the cone and introduces a new divisor D_5. In each resolution there are four fixed points for the action of T^3.

To compute the partition functions we need to chose a resolution and the number c_5 that satisfy the convexity condition (6.7). The partition function for Z_{D_3} can be computed by using the resolution on the left hand side in the figure and the number $c_5 = 0$.

Region I	$m_I^\lambda = \{((-1,0,1),(0,-1,0),(1,1,0)\}$	$m_I^0 = (0,0,0)$
Region II	$m_{II}^\lambda = \{((-1,-1,1),(0,1,0),(1,0,0)\}$	$m_{II}^0 = (0,0,0)$
Region III	$m_{III}^\lambda = \{(1,-1,1),(0,1,0),(-1,0,0)\}$	$m_{III}^0 = (1,0,0)$
Region IV	$m_{IV}^\lambda = \{(1,2,1),(0,-1,0),(-1,-1,0)\}$	$m_{IV}^0 = (1,1,0)$

$$Z_{D_3} = \frac{-q_3^2 + q_2^2(q_1^2 q_3 - q_3^2 + q_1(1+q_3-q_3^2)) - q_2(-1+q_3+q_3^2-q_1-q_1q_3+q_1q_3^2)}{(1-q_3/(q_1q_2))(1-q_3/q_1)(q_2-q_1q_3)(1-q_1q_2^2q_3)}$$

Z_{D_3} can be expanded using the geometric series by setting $q_3 = qq_1q_2$. It is immediate to verify that the first terms in the expansion $Z_{D_3} = 1+q_1+q_1q_2+q(q_2+1+q_1+...)+...$ exactly match the list of field theory operators given in Section 5 (cfr Table 1).

The partition functions for the other three elementary divisors can be computed in a similar way. In order to satisfy the convexity condition we use the resolution on the left of figure 6 for D_2 and D_4 and the resolution on the right for D_1. In all cases we can safely put $c_5 = 0$.

$$\begin{aligned}
Z_{D_1} &= \frac{q_1^2q_2^2 + q_2(1+q_1^2(1+(1+q_1)(q_2+q_2^2)))q_3 - q_1(1+(1+q_1)(q_2+q_2^2+q_2^3))q_3^2}{(q_1q_2-q_3)(q_1-q_3)(q_2-q_1q_3)(1-q_1q_2^2q_3)} \\
Z_{D_2} &= \frac{q_1^2q_2^2q_3 - q_1^2q_2^2q_3^2 - q_3(q_2+(1+q_2+q_2^2)q_3) + q_1(1+q_2)(q_3+q_2+q_2^2q_3-q_2q_3^2)}{(q_1q_2-q_3)(1-q_3/q_1)(q_2-q_1q_3)(1-q_1q_2^2q_3)} \\
Z_{D_4} &= q_2 Z_{D_2}
\end{aligned}$$

The proportionality of Z_{D_4} and Z_{D_2} follows from the equivalence $D_2 \sim D_4$.

Similarly, one can compute the partition functions for the other $Y^{p,q}$ manifolds and, more generally, for the $L^{p,q,r}$ manifolds which correspond to the most general toric diagram with four external points. The flavor symmetry for $L^{p,q,r}$ is $U(1)^2$ and, for smooth manifolds, there is exactly one baryonic symmetry. The number of

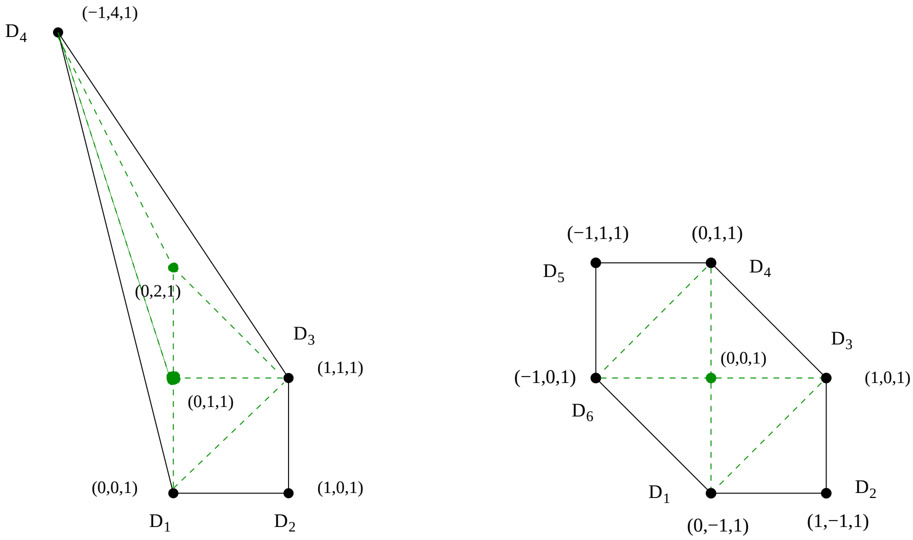


Figure 8: $L^{1,5,2}$ and dP_3. On the left: a compatible resolution for the D_4 partition function of $L^{1,5,2}$, with six fixed points. On the right: a compatible resolution for the D_2 partition function for dP_3, with six fixed points. In both cases, one can safely choose the c_i of the extra divisors equal to zero.

internal points increases with p, q, r thus making computations more involved. As an example, we present the partition function for the D_4 divisor in $L^{1,5,2}$. We refer to Figure 8 for notations and the choice of a compatible resolution.

$$\begin{aligned} Z_{D_4} &= \frac{P(q_1,q_2,q_3)}{(1-q_2)(1-q_1^4 q_2)(q_1-q_3)(q_1^3 q_2^2 - q_3^5)} \\ P(q_1,q_2,q_3) &= q_1 q_2 (q_1^6 q_2^2 - q_3^5 - q_1 q_3^5 - q_1^2 q_3^2(-1+q_2-q_3+q_2 q_3+q_3^3) \\ &+ q_1^5 q_2 (q_3(1+q_3) - q_2(-1+q_3+q_3^2)) + q_1^4(-q_2^2 q_3 + q_3^4 + q_2(1+q_3-q_3^4)) \\ &+ q_1^3(q_3^3+q_3^4-q_3^5-q_2(-1+q_3^3+q_3^4))) \end{aligned} \tag{6.16}$$

Finally, we give an example based on the dP_3 manifolds, whose toric diagram has six external points and thus three different baryonic symmetries. Once again we refer to Figure 8 for notations.

$$\begin{aligned} & Z_{D_2} = \frac{P(q_1,q_2,q_3)}{(q_1-q_3)(1-q_3/q_2)(q_3-q_1 q_2)(1-q_1 q_3)(1-q_2 q_3)(1-q_1 q_2 q_3)} \\ & P(q_1,q_2,q_3) = (q_2+q_1(1+2q_2+q_1(1+q_2)(1+q_2(1+q_1+q_2))))q_3^2 \\ & -(1+q_1((1+q_2)^2+q_1(1+q_2)(1+2q_2)+q_1^2 q_2(1+q_2+q_2^2)))q_3^3 \\ & +q_1(1+(1+q_1)q_2(1+q_1 q_2))q_3^4 - q_1^2 q_2 - q_1 q_2(q_1+q_2)q_3 \end{aligned}$$

Following this procedure one is able to compute the partition function Z_D for every divisor of a generic CY conical toric singularity.

6.3 The partition function for BPS baryonic operators

The BPS baryonic states in a sector of the Hilbert space associated with the divisor D are obtained from the h_m by considering the N-fold symmetrized combinations $|h_{m_1},....,h_{m_N}>$. The partition function $Z_{D,N}$ for BPS baryon is obtained from Z_D by solving a combinatorial problem [30, 33].

Given Z_D as sum of integer points in the polytope P_D

$$Z_D(q) = \sum_{m \in P_D \cap M} q^m \tag{6.17}$$

the generating function $G_D(p,q)$ for symmetrized products of elements in Z_D is given by

$$G_D(p,q) = \prod_{m \in P_D \cap M} \frac{1}{1-pq^m} = \sum_{N=0}^{\infty} p^N Z_{D,N}(q) \tag{6.18}$$

This formula is easy to understand: if we expand $(1-pq^m)^{-1}$ in geometric series, the coefficient of the term p^k is given by all possible products of k elements in P_D, and this is clearly a k-symmetric product.
It is easy to derive the following relation between $Z_D(q)$ and $G_D(p,q)$

$$G_D(p,q) = e^{\sum_{k=1}^{\infty} \frac{p^k}{k} Z_D(q^k)} \tag{6.19}$$

In the case we have computed $Z_D(q)$ in terms of the fixed point data of a compatible resolution as in equation (6.8)

$$Z_D(q) = \sum_I \frac{q^{m_I^0}}{\prod_{\lambda=1}^3 (1-q^{m_I^\lambda})} = \sum_I \sum_{s_I^1, s_I^2, s_I^3} q^{m_I^0} q^{\sum_{\lambda=1}^3 s_I^\lambda m_I^\lambda}$$

formula (6.19) allows, with few algebraic manipulation, to write the generating function as follows

$$G_D(p,q) = \prod_{P_I} \prod_{s_I^\lambda=0}^{\infty} \frac{1}{1-pq^{m_I^0} q^{\sum_{\lambda=1}^3 s_I^\lambda m_I^\lambda}} \tag{6.20}$$

We are eventually interested in the case of BPS baryonic operators associated with the symmetrized elements $|h_{m_1},....,h_{m_N}>$, and thus to the N-fold symmetric partition function:

$$Z_{D,N}(q) \equiv \frac{1}{N!} \frac{\partial^N G_D(p,q)}{\partial p^N}\Big|_{p=0} \tag{6.21}$$

Thanks to $G_D(0,q) = 1$ (see eq. (6.19)) we can easily write $Z_{D,N}$ in function of Z_D. For example we have:

$$Z_{D,1}(q) = Z_D(q)$$

$$Z_{D,2}(q) = \frac{1}{2}(Z_D(q^2) + Z_D^2(q))$$

$$Z_{D,3}(q) = \frac{1}{6}(2Z_D(q^3) + 3Z_D(q^2)Z_D(q) + Z_D^3(q))$$

Once we know Z_D for a particular baryonic sector of the BPS Hilbert space it is easy to write down the complete partition function $Z_{D,N}$.

7 Volumes of divisors

One of the predictions of the AdS/CFT correspondence for the background $AdS_5 \times H$ is that the volume of H is related to the central charge a of the CFT, and the volumes of the three cycles wrapped by the D3-branes are related to the R-charges of the corresponding baryonic operators [4, 37]. We already used this information in formula (4.6). To many purposes, it is useful to consider the volumes as functions of the Reeb Vector b. Recall that each Kähler metric on the cone, or equivalently a Sasakian structure on the base H, determines a Reeb vector $b = (b_1, b_2, b_3)$ and that the knowledge of b is sufficient to compute all volumes in H [17]. Denote with $\mathrm{Vol_H(b)}$ the volume of the base of a Kähler cone with Reeb vector b. The Calabi-Yau condition $c_1(X)$ requires $b_3 = 3$ [17]. As shown in [17, 35], the Reeb vector $\bar{b}$ associated with the Calabi-Yau metric can be obtained by minimizing the function $\mathrm{Vol_H(b)}$ with respect to $b = (b_1, b_2, 3)$. This volume minimization is the geometrical counterpart of a-maximization in field theory [51]; the equivalence of a-maximization and volume minimization has been explicitly proven for all toric cones in [19] and for a class a non toric cones in [28]. For each Reeb vector $b = (b_1, b_2, b_3)$ we can also define the volume of the three cycle obtained by restricting a divisor D to the base, $\mathrm{Vol_D(b)}$. We can related the value $\mathrm{Vol_D(\bar{b})}$ at the minimum to the exact R-charge of the lowest dimension baryonic operator associated with the divisor D [4, 37] as in formula (4.6).

All the geometrical information about volumes can be extracted from the partition functions. The relation between the character $Z_{\mathcal{O}}(q)$ for holomorphic functions on $C(H)$ and the volume of H was suggested in [52] and proved for all Kähler cones in [35]. If we define $q = (e^{-b_1 t}, e^{-b_2 t}, e^{-b_3 t})$, we have [35, 52]

$$\mathrm{Vol_H(b)} = \pi^3 \lim_{\mathrm{t} \to 0} \mathrm{t}^3 \mathrm{Z}_{\mathcal{O}}(\mathrm{e}^{-\mathrm{bt}}) \tag{7.1}$$

This formula can be interpreted as follows: the partition function $Z_{\mathcal{O}}(q)$ has a pole for $q \to 1$, and the order of the pole - three - reflects the complex dimension of $C(H)$ while the coefficient is related to the volume of H.

Here we propose that, similarly, the partition functions Z_D contain the information about the three-cycle volumes $\mathrm{Vol_D(b)}$. Indeed we will show that, for small t,

$$\frac{Z_D(e^{-bt})}{Z_{\mathcal{O}}(e^{-bt})} \sim 1 + t \frac{\pi \mathrm{Vol_D(b)}}{2\mathrm{Vol_H(b)}} + \dots \tag{7.2}$$

Notice that the leading behaviour for all partition functions Z_D is the same and proportional to the volume of H; for $q \to 1$ the main contribution comes from states with arbitrarily large dimension and it seems that states factorized in a minimal determinant times gravitons dominate the partition function. The proportionality to $\mathrm{Vol_H}$ is then expected by the analogy with giant gravitons probing the volume of H. The subleading term of order $1/t^2$ in Z_D then contains information about the dimension two complex divisors. Physically it is easy to understand that Z_D contains the information about the volumes of the divisors. We can think at Z_D as a semiclassical parametrization of the holomorphic non-trivial surfaces in X, with a particular set of charges related to D; while $Z_{\mathcal{O}}$ parametrizes the set of trivial surfaces in X. Thinking in this way it is clear that both know about the volume of the compact space, but only Z_D has information on the volumes of the non-trivial three cycles.

For divisors D associated with elementary fields we can rewrite equation (7.2) in a simple and suggestive way in terms of the R-charge, or dimension, of the elementary field (see equation (4.6))

$$\frac{Z_D(e^{-bt})}{Z_{\mathcal{O}}(e^{-bt})} \sim 1 + t\frac{3R_D(b)}{2} + ... = 1 + t\Delta(b) + ... \tag{7.3}$$

As a check of formula (7.2), we can expand the partition functions for the conifold computed in the previous Section

$$\frac{Z_{D_i}}{Z_{\mathcal{O}}} \sim (1+\frac{(b_1-b_3)(b_2-b_3)t}{b_3}, 1+\frac{(b_1b_3-b_1b_2)t}{b_3}, 1+\frac{b_1b_2t}{b_3}, 1+\frac{(b_2b_3-b_1b_2)t}{b_3}) \tag{7.4}$$

and compare it with the formulae in [17]

$$\mathrm{Vol_{D_i}(b)} = \frac{2\pi^2 \det\{\mathrm{n_{i-1}, n_i, n_{i+1}}\}}{\det\{\mathrm{b, n_{i-1}, n_i}\}\det\{\mathrm{b, n_i, n_{i+1}}\}}, \qquad \mathrm{Vol_H(b)} = \frac{\pi}{6}\sum_{\mathrm{i=1}}^{\mathrm{d}} \mathrm{Vol_{D_i}(b)} \tag{7.5}$$

One can perform similar checks for $Y^{2,1}$ and the other cases considered in the previous section, with perfect agreement. A sketch of a general proof for formula (7.2) is given in the Appendix A.

We would like to notice that, by expanding equation (6.8) for $q = e^{-bt} \to 1$ and comparing with formula (7.2), we are able to write a simple formula for the volumes of divisors in terms of the fixed point data of a compatible resolution

$$\mathrm{Vol_D(b)} = 2\pi^2 \sum_{\mathrm{P_I}} \frac{(-\mathrm{m_I^0, b})}{\prod_{\lambda=1}^3 (\mathrm{m_I^\lambda, b})} \tag{7.6}$$

This formula can be conveniently generalized to the case where the fixed points are not isolated but there are curves or surfaces fixed by the torus action.

The previous formula is not specific to toric varieties. It can be used whenever we are able to resolve the cone $C(H)$ and to reduce the computation of Z_D to a sum over isolated fixed points (and it can be generalized to the case where there are fixed submanifolds). As such, it applies also to non toric cones. The relation between volumes and characters may give a way for computing volumes of divisors in the general non toric case, where explicit formulae like (7.5) are not known.

8 Conclusion and Outlook

In this paper we proposed a general procedure to construct partition functions counting both baryonic and non baryonic BPS operators of a field theory dual to a toric geometry. We also explained how one can extract the volumes of the three cycles from the partition functions. It would be interesting to understand better the counting of multiplicity, and to find a way to write down a complete partition function for the BPS gauge invariant scalar operators [47].

Our computation is done on the supergravity side, and it is therefore valid at strong coupling. Similarly to the partition function for BPS mesonic operators [30, 32, 33], we expect to be able to extrapolate the result to finite value for the coupling.

It would be also interesting to understand better the non toric case. Most of the discussions in this paper apply to this case as well. The classical configurations of BPS D3 branes wrapping a divisor D are still parameterized by the generic section of $H^0(X, \mathcal{O}(D))$ and Beasley's prescription for constructing the BPS Hilbert space is unchanged. The partition function $Z_D(q)$ can be still defined, with the only difference that $q \in T^k$ with k strictly less than three. $Z_D(q)$ can be still computed by using the index theorem as explained in Section 6 and the relation between $Z_D(q)$ and the three cycles volumes should be still valid. In particular, when X has a completely smooth resolution with only isolated fixed points for the action of T^k, formulae (6.8) and (7.6) should allow to compute the partition functions and the volume. What is missing in the non-toric case is an analogous of the homogeneous coordinates, the polytopes and the existence of a canonical smooth resolution of the cone. But this seems to be just a technical problem.

Acknowledgments

We would like to thank Amihay Hanany and Alessandro Tomasiello for useful discussions. D.F. thanks Jose D. Edelstein, Giuseppe Milanesi and Marco Pirrone for useful discussions. This work is supported in part by INFN and MURST under contract 2005-024045-004 and 2005-023102 and by the European Community's Human Potential Program MRTN-CT-2004-005104.

A Localization formulae for the volumes of divisors

Relation (7.2) can be easily proven in the case where the Reeb action is regular, by adapting an argument in [52, 53], as refined in [35]. For a regular action, H is a $U(1)$ principal bundle over a Kähler manifold V and X can be written as a line bundle $L \to V$. We can blow up V and apply equivariant Riemann-Roch to the resulting manifold. Since the Reeb vector acts on the fibers of L, its fixed locus is the entire V with weight 1. We thus obtain

$$Z_D(q) = \int_V \frac{\text{Todd}(V)Ch(D)}{1 - qe^{-c_1(L)}} \tag{A.1}$$

Put $q = e^{-bt}$ in this formula. The denominator must be expanded in a formal power series of forms before taking the limit $t \to 0$

$$\frac{1}{1 - qe^{-c_1(L)}} = \frac{1}{1 - e^{-bt}} - \frac{e^{-bt}}{(1 - e^{-bt})^2} c_1(L) + \frac{e^{-bt} + e^{-2bt}}{2(1 - e^{-bt})^3} c_1(L)^2$$

Since the integral over V selects forms of degree four we obtain

$$Z_D(q) = \frac{1}{(bt)^3} \int_V c_1(L)^2 - \frac{1}{(bt)^2} \int_V c_1(L) \wedge \text{Todd}(V) \wedge Ch(D)\Big|_{\text{degree 4}} + ...$$

The only information we need about the Todd class is that $\text{Todd}(V) = 1 + ...$ We thus obtain

$$\frac{Z_D(q)}{Z_{\mathcal{O}}} = 1 - b\,t \frac{\int_V c_1(L) \wedge c_1(D)}{\int_V c_1(L) \wedge c_1(L)} + ...$$

The volumes of H and of the three cycle $D \cap H$, which are $U(1)$ fibrations over V and $D \cap V$, are proportional to

$$\begin{aligned} \text{Vol}_D(b) &\sim \int_D \omega_V = \int_V \omega_V \wedge c_1(D) \\ \text{Vol}_H(b) &\sim \int_V \frac{\omega_V^2}{2} \end{aligned} \quad \text{(A.2)}$$

Considering that the first Chern class of L is proportional to the Kähler form ω_V on V [20], we finally obtain

$$\frac{Z_D(q)}{Z_{\mathcal{O}}} = 1 + t \frac{\pi \mathrm{Vol_D(b)}}{2 \mathrm{Vol_H(b)}} + ...$$

Formula (7.2) could be proven for a generic Sasaki-Einstein by generalizing arguments in [35]. We only suggest a possible proof, leaving to experts the subtle task of filling mathematical details. As argued in [35], it is enough to prove (7.2) for quasi regular actions; since a rational $b \in T^k$ defines a Sasaki structure on H with quasi regular Reeb action, formula (7.2) would be true for all rational b and, therefore, for continuity, for all b. For a quasi regular action, $L \to V$ is an orbifold and we should use the Kawasaki-Riemann-Roch formula [54] which have extra contributions with respect to (A.1). However, for isolated orbifold singularities, the extra contributions are characteristic classes integrated over points; these contribute to $Z_D(q)$ only at order $1/t$ and should be irrelevant for our purposes.

It would be interesting to fully understand formula (7.2) in terms of localization. It seems that some localization theorem is at work here. Considering that the action of the Reeb field

$$\xi = \sum_{i=1}^{k} b_i \frac{\partial}{\partial \phi_i}$$

(we have chosen k angular coordinates for the torus T^k action, $k = 1, 2, 3$) is hamiltonian, we can define the equivariantly closed form $\omega^\xi = \omega - H$ starting from the

[20] We use the normalizations of [35]: $c_1(L) = -bc_1(V)/3$ and $\omega_V = \pi c_1(V)/3$. These formulae are valid also for a quasi regular action. The length of the $U(1)$ fiber is $2\pi/b$.

kähler class ω [21]. As shown in [35], the hamiltonian for the Reeb action is $H = r^2/2$. Analogously we can define the equivariant first Chern class $c_1^\xi(D)$ associated with the divisor D. The expression (7.6) for the volumes can be now reinterpreted as the integral of equivariantly characteristic classes,

$$\frac{1}{2}\int_X e^{\omega^\xi} \wedge c_1^\xi(D) = 2\pi^2 \sum_{\sigma_I} \frac{(-m_I^0, b)}{\prod_{\lambda=1}^3 (m_I^\lambda, b)} \tag{A.3}$$

This formula can be proven as follows. Suppose that we have found a smooth resolution $\tilde{X}$ of the cone X and a divisor $\tilde{D}$ that smoothly approach D in the singular limit. We may then compute the previous integral for $\tilde{X}$ and $\tilde{D}$ and take afterwards the limit $\tilde{X} \to X$. Integrals of equivariantly closed forms, like (A.3), can be computed by using localization theorems. Indeed given an equivariantly closed form α and an action along a direction in T^k ($k = 1, 2, 3$) with only isolated fixed points, it can be shown that

$$\int \alpha = (2\pi)^3 \sum_{P_I} \frac{\alpha|_{P_I}}{\prod_{\lambda=1}^3 (m_I^\lambda, b)} \tag{A.4}$$

where m_I^λ are, as usual, the weights of the action of ξ on the tangent space at P_I. The integral over a point P_I takes contribution only from the forms with zero degree in the equivariant forms,

$$\begin{aligned} \omega^\xi &\to -H(P_I) \\ c_1^\xi(D) &\to -\frac{(m_I^0, b)}{2\pi} \end{aligned} \tag{A.5}$$

where m_I^0 is the weight of the action on the line bundle fiber over P_I [22]. Taking into account that in the singular limit all the P_Is collapse to the apex of the cone where $H = 0$, we finally obtain formula (A.3).

Our general expression for the volumes (7.6), which, in case there exist smooth resolutions for X with isolated fixed points, is completely equivalent to the general formula (7.2), would be proved if we were able to show that

$$\int_D e^{\omega^\xi} = \int_X e^{\omega^\xi} \wedge c_1^\xi(D) \tag{A.6}$$

for a T invariant divisor D. Indeed the expression on left hand side is just the volume of the base $D \cap H$

$$\mathrm{Vol_D(b)} = \frac{1}{2}\int_{\mathrm{D}} \mathrm{e}^{-\mathrm{r}^2/2} \frac{\omega^2}{2} = \frac{1}{2}\int_{\mathrm{D}} \mathrm{e}^{\omega^\xi} \tag{A.7}$$

The relation between equivariant cohomology and homology seems to be not completely understood (to us at least), and we do not know under what condition the equation (A.6) is valid, even if this is probably well known to experts. A better understanding of this formula could give a simple alternative proof for (7.2).

[21] Given a vector field ξ the equivariant derivative d_ξ of a form α is $d_\xi \alpha = d\alpha + i_\xi \alpha$; ω^ξ is clearly equivariantly closed, because ω is closed and H is the Hamiltonian of the Reeb vector field ($i_\xi \omega = dH$).

[22] The second formula follows from the standard replacement $c_1 \to c_1 - w/(2\pi)$, with w the weight for the group action, for line bundles over fixed submanifolds.

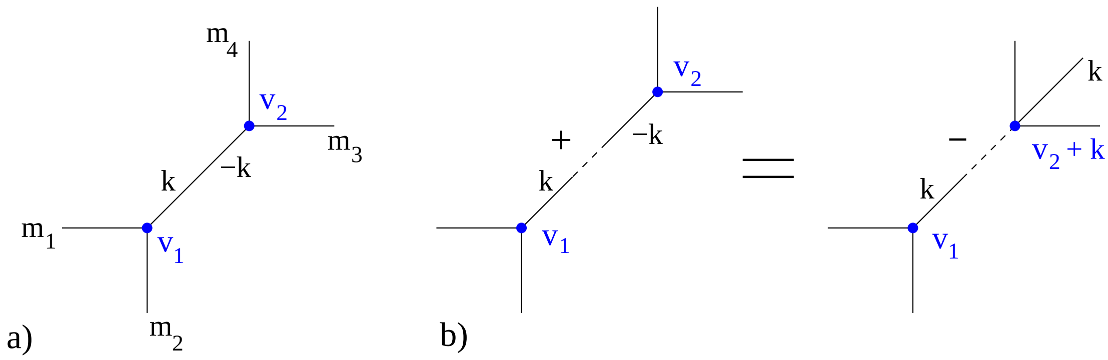


Figure 9: a) The (p,q) web for the conifold resolution corresponding to the first diagram in Figure 7. b) A pictorial description of equation (B.5).

B Convexity condition and integer counting

In this Appendix we give an alternative explanation of the formula (6.8) for computing the partition function Z_D defined in (6.1) and we explain why there always exists a suitable resolution of $X = C(H)$ and a suitable choice of weights $\tilde{c}_i$ for equation (6.8). In Section 6.1 we explained how to derive this formula from the equivariant Riemann-Roch theorem. However since Z_D counts the holomorphic sections of $\mathcal{O}(D)$ and since we know that, for toric singularities, these sections are in one to one correspondence with the integer points inside the polytope P_D, we can simply look at this problem as that of counting integer points inside a polytope, with the weights $q = (q_1, q_2, q_3)$ being associated with the three cartesian coordinates: $Z_D = \sum_{m \in P_D \cap M} q^m$. This simple point of view allows to have a direct understanding of the counting problem.

To be concrete consider for instance the case of the conifold; let us write the character that counts integer points inside the dual fan, which is equivalent to counting holomorphic functions, or sections of the trivial bundle $\mathcal{O}$. The dual fan is generated by the four vectors:

$$m_1 = (1,0,0) \quad m_2 = (0,1,0) \quad m_3 = (-1,0,1) \quad m_4 = (0,-1,1) \tag{B.1}$$

all attached at the origin $n = (0,0,0)$. To compute the character we will use for instance the first resolution in Figure 7, whose corresponding (p,q) web is drawn in Figure 9 a). Let us define the vector $k = (-1,-1,1)$. We split the (p,q) web into two subwebs corresponding to region I and II respectively of the resolution in Figure 7:

$$\begin{array}{lll} \text{Region I} & \{m_1, m_2, k\} & v_1 = (0,0,0) \\ \text{Region II} & \{m_3, m_4, -k\} & v_2 = (0,0,0) \end{array} \tag{B.2}$$

We denote with v_1 (v_2) the integer point to which the three vectors of region I (II) are attached. In this case $v_1 = v_2 = (0,0,0)$. Since we have completely resolved the conifold, according to [35], the character is:

$$Z_{\mathcal{O}} = \frac{1}{(1-q^{m_1})(1-q^{m_2})(1-q^{k})} + \frac{1}{(1-q^{m_3})(1-q^{m_4})(1-q^{-k})} \tag{B.3}$$

where as usual $q^h \equiv q_1^{h_1} q_2^{h_2} q_3^{h_3}$. It is simple to give an interpretation of this formula in terms of counting of integer points. Let us for instance expand the first term, associated with Region I, in equation (B.3). We get

$$\left(\sum_{i=0}^{\infty} q^{i\,m_1}\right)\left(\sum_{j=0}^{\infty} q^{j\,m_2}\right)\left(\sum_{h=0}^{\infty} q^{h\,k}\right) = \sum_{i,j,h\geq 0} q^{i\,m_1+j\,m_2+h\,k} \tag{B.4}$$

and this is just the partition function that counts integer points inside the cone generated by the vectors $\{m_1, m_2, k\}$ attached to the origin $v_1 = (0,0,0)$. In fact each integer point inside this cone can be written in a unique way as a linear combination of $\{m_1, m_2, k\}$ with positive integers due to the fact that $\det(m_1, m_2, k) = 1$. This is equivalent to the statement that region I is a triangle with integer points with minimal area (=1/2).

Note that the cone generated by $\{m_1, m_2, k\}$ centered in $(0,0,0)$ strictly includes the cone generated by the four vectors m_i; this is dual to the statement that region I is included in the fan of the conifold. The second term in (B.3) exactly cancels the integer points that belong to $\{m_1, m_2, k\}$ but not to the cone generated by the m_i.

The important observation is that expansion (B.4) is valid in the region $\{q^{m_1} < 1, q^{m_2} < 1, q^k < 1\}$. Since the second term in (B.3) contains the factor q^{-k} before expanding it in the usual way we can rearrange the expression (B.3) for $Z_{\mathcal{O}}$ as

$$Z_{\mathcal{O}} = \frac{1}{(1-q^{m_1})(1-q^{m_2})(1-q^{k})} - \frac{q^k}{(1-q^{m_3})(1-q^{m_4})(1-q^{k})} \tag{B.5}$$

Now we can expand both these terms in the region: $q^{m_i} < 1, \forall i = 1, \dots 4$ and $q^k < 1$. The factor of q^k in front of the second term simply translates the origin: we get the partition function that counts integer points inside the cone $\{m_1, m_2, k\}$ with origin $(0,0,0)$ minus the partition function that counts integer points inside the cone $\{m_3, m_4, k\}$ centered at the integer point k. Note that since k is a primitive vector, along the line with direction k and passing through $(0,0,0)$, the point k is the first integer point after the origin $(0,0,0)$. We have thus canceled all the integer points we didn't want to count and hence $Z_{\mathcal{O}}$ is the correct partition function. A pictorial description of equation (B.5) is given in Figure 9 b); we project the edges of the cones on the (p,q) web plane (first two coordinates). The reader should try to imagine the process in three dimensions.

Obviously one can repeat the same process exchanging $k \leftrightarrow -k$: we expand the second term of (B.3) for $\{q^{m_3} < 1, q^{m_4} < 1, q^{-k} < 1\}$ and rearrange the first term of (B.3); we see that the same expansion for $Z_{\mathcal{O}}$ is valid in the region $q^{m_i} < 1, \forall i = 1, \dots 4$ and $q^{-k} < 1$. Combining with the previous result, we obtain that $Z_{\mathcal{O}}$ can be expanded in the region $q^{m_i} < 1, \forall i = 1, \dots 4$, that is only the external vectors m_i matter. Obviously taking the second resolution for the conifold of Figure 7 one arrives at the same function $Z_{\mathcal{O}}$.

It is easy to see now how to write the partition function that counts integer points inside a polytope obtained by moving the origins v_1 and v_2 of the two cones at arbitrary, non coincident, points. We obtain:

$$\frac{q^{v_1}}{(1-q^{m_1})(1-q^{m_2})(1-q^k)} + \frac{q^{v_2}}{(1-q^{m_3})(1-q^{m_4})(1-q^{-k})} \tag{B.6}$$

since the factors q^{v_J} are simply translating the origins. For instance in the case of Z_{D_1} in Section 6.2.1 we had $v_1 = (1, 1, -1)$ and $v_2 = (0, 0, 0)$, there called m_I^0 and m_{II}^0. This is another explanation of formula (6.8).

It is not difficult to generalize this example to all cases we are interested in. Suppose you have a rational convex polytope in $\mathbb{R}^3$ with integer vertices; call m_J^0 its vertices and m_J^λ the edges attached to each vertex m_J^0. We normalize the m_J^λ to primitive integer vectors (all exiting from the vertex m_J^0). At each vertex suppose that the infinite cone generated by the m_J^λ is of Calabi-Yau type (meaning that the dual cone has generators lying on the plane $z = 1$); as we will see later this is our case. Then one can easily compute the partition function $Z_J(q)$ that counts integer points inside the infinite cone with vertex in $(0, 0, 0)$ generated by the vectors m_J^λ, with fixed J, for instance by taking any resolution of the associated fan. Repeating the trick above, it is easy to see that the partition function that counts integer points inside the original polytope is given by the sum $\sum_J q^{m_J^0} Z_J(q)$ over all vertices J of the polytope.

Now we go back to the original problem of Section 6.1. To fix the notation, let c_i be the generic integer weights assigned to each vertex n_i of a toric diagram that define the bundle: $\mathcal{O}(\sum_i c_i D_i)$, $i \in \Sigma(1)$; where $\Sigma(1)$ is the set of vertices $n_i = (y_i, z_i, 1)$ of the toric diagram. Let P_D the polytope in $\mathbb{R}^3$ defined by the equations:

$$P_D = \{m \in \mathbb{R}^3 | \langle m, n_i\rangle + c_i \geq 0, \, \forall i \in \Sigma(1)\} \tag{B.7}$$

One can compute the intersections of these planes and reconstruct the edges and the vertices of P_D. There is a plane for each vertex V_i. An example is reported in Figure 10 a): in this case the polytope has 7 planes and 4 vertices v_J. In general P_D is convex and has rational edges, however its vertices are only rational and may not be integer. Therefore we define another convex polytope $\tilde{P}_D$ as the convex hull of all integer points inside P_D. Therefore $\tilde{P}_D \subseteq P_D$ and all integer points in P_D belong also to $\tilde{P}_D$: the original problem is reduced to the problem of counting integer points inside $\tilde{P}_D$.

It is easy to see that $\tilde{P}_D$ can be alternatively described by adding equations to those defining P_D (B.7), since the infinite edges of P_D, being described by rational equations, pass through integer points. An example is reported in Figure 10 b), where we draw projected on the (p,q) web plane the edges of $\tilde{P}_D$ corresponding to the polytope P_D in Figure 10 a). Note that besides the 7 infinite pieces of planes delimiting P_D we have added two finite pieces of planes; in the dual description this corresponds to refine the resolution of the toric diagram by adding two internal points $\tilde{n}_i$, the perpendiculars to the two planes.

An important fact is that the resolution we need to describe $\tilde{P}_D$ is again Calabi-Yau, meaning that the only planes we need to add to equations (B.7) are those with

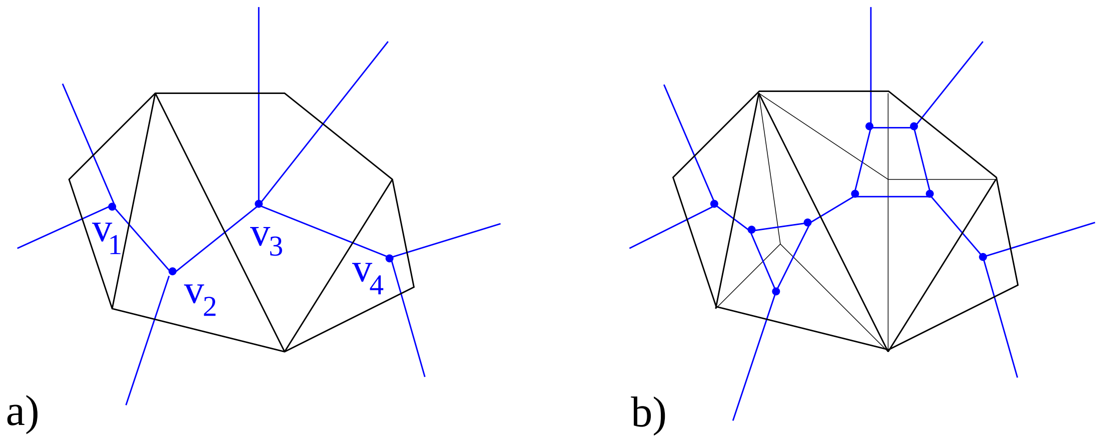


Figure 10: a) Edges of the polytope P_D projected on the (p,q) web plane (in blue) and dual resolved toric diagram (in black). b) The same picture for $\tilde{P}_D$.

perpendiculars $\tilde{n}_i \in \tilde{\Sigma}(1)$ where $\tilde{\Sigma}(1)$ is the set of integer vectors $\tilde{n}_i$ lying on the plane of the toric diagram and inside the toric diagram. Consider for any integer vector $\tilde{n}_i \in \tilde{\Sigma}(1)$ the plane $\langle m, \tilde{n}_i \rangle + a = 0$; by varying a it is easy to see that for large positive a the plane does not intersect $\tilde{P}_D$; hence there is a maximal value of a for which the plane has a non empty intersection with the closed polytope $\tilde{P}_D$. Define $\tilde{c}_i$ such value for a. Obviously if $\tilde{n}_i = n_i$ coincides with an external vertex of the toric diagram we obtain for $\tilde{c}_i$ the original value c_i. Note that all $\tilde{c}_i$ are integers since $\tilde{P}_D$ has integer vertices. Now define the polytope Q:

$$Q = \{m \in \mathbb{R}^3 | \langle m, \tilde{n}_i \rangle + \tilde{c}_i \geq 0, \forall i \in \tilde{\Sigma}(1)\} \tag{B.8}$$

with the $\tilde{c}_i$ defined as before. It is not difficult to prove that $Q = \tilde{P}_D$. In fact from the definitions we straightforwardly obtain that: $\tilde{P}_D \subseteq Q \subseteq P_D$. Now the convex polytope Q can be seen as the convex hull of its vertices and of the integer points along its infinite external edges. If we prove that Q has integer vertices then we would prove also $Q \subseteq \tilde{P}_D$ since $\tilde{P}_D$ is the convex hull of all integer points inside P_D; and hence $Q = \tilde{P}_D$.

By computing the intersections of the planes in the definition (B.8) we can obtain the corresponding resolution of the toric diagram and the vertices of the polytope Q. For example one could obtain the resolution in Figure 10 b), where the toric diagram has been divided into 9 regions ρ_J, $I = 1 \dots 9$, each corresponding to a vertex m_J^0 of the convex polytope Q. The vertex m_J^0 is the intersection of the planes $\langle m, \tilde{n}_i \rangle + \tilde{c}_i = 0$, for all the vertices $\tilde{n}_i$ of the region ρ_J. If the region ρ_J we are considering has no internal integer point, then by the Pick's theorem [43] it is a triangle with minimal area $1/2$; call its integer vertices $\tilde{n}_1$, $\tilde{n}_2$ and $\tilde{n}_3$. Since for this triangle $\det(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3) = 1$, m_J^0 is an integer point. Instead if the region ρ_J has integer points inside it is easy to see from the construction of Q that all the planes $\langle m, \tilde{n}_i \rangle + \tilde{c}_i = 0$, for any integer $\tilde{n}_i$ internal to ρ_J, pass through the vertex m_J^0. Hence we can take any complete resolution of the region ρ_J into minimal triangles and compute m_J^0 as the intersection of the planes associated to its three vertices. Hence again m_J^0 is integer. All minimal triangles belonging to the region ρ_J identify the same m_J^0.

We just proved that Q has integer vertices and hence $Q = \tilde{P}_D$. Since $\tilde{P}_D$ is Calabi-Yau, to compute the partition function Z_D that counts its integer points we can use the method derived above:

$$Z_D = \sum_{\rho_J} q^{m_J^0} Z_J(q) \tag{B.9}$$

where as before Z_J is the partition function counting integer points inside the cone with apex in $(0, 0, 0)$ generated by the edges m_J^λ attached to vertex m_J^0. Moreover the partition functions Z_J can be computed using any complete resolution of the regions ρ_J; we obtain therefore a refined resolution of the toric diagram in minimal triangles σ_I. The resulting partition function is just formula (6.8).

Note that there is some ambiguity in choosing the complete resolution of the toric diagram into triangles σ_I; however this resolution must be compatible with the starting ρ_J resolution. To summarize we have given an alternative proof of (6.8) and we have explicitly built the integers $\tilde{c}_i$ associated with the internal points $\tilde{n}_i \in \tilde{\Sigma}(1)$. Then the equations (B.8) define a resolution ρ_J of the toric diagram that can be further refined. Note that the two conditions given in Section 6.1 are naturally satisfied with this geometrical choice of $\tilde{c}_i$; in particular convexity (6.7) follows from the fact that m_J^0 are the vertices of $\tilde{P}_D$.

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