SISSA preprint 58/2018/MATE

Two-dimensional Schrödinger operators with point interactions: Threshold expansions, zero modes and L^p-boundedness of wave operators

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Received 4 April 2018
Accepted 23 October 2018
Published

We study the threshold behavior of two-dimensional Schrödinger operators with finitely many local point interactions. We show that the resolvent can either be continuously extended up to the threshold, in which case we say that the operator is of regular type, or it has singularities associated with s or p-wave resonances or even with an embedded eigenvalue at zero, for whose existence we give necessary and sufficient conditions. An embedded eigenvalue at zero may appear only if we have at least three centers.

When the operator is of regular type, we prove that the wave operators are bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. With a single center, we always are in the regular type case.

Keywords: Two-dimensional point interaction; threshold expansion; resonances at threshold; embedded eigenvalue at threshold; L^p-boundedness of wave operators.

Mathematics Subject Classification 2010: 35P15, 35J10, 47A40, 81Q10

1. Introduction and Main Results

Let $Y = \{y_1, \dots, y_N\}$ be N points in the plane $\mathbb{R}^2$ with $1 \leq N < \infty$. Let T_0 be the densely defined non-negative symmetric operator in the Hilbert space $L^2(\mathbb{R}^2)$ defined by

$$T_0 \colon = -\Delta|_{C_0^\infty(\mathbb{R}^2\setminus Y)}, \quad \Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

A Schrödinger operator on $\mathbb{R}^2$ with point interactions at Y is any self-adjoint extension of T_0. In this paper, we are concerned with local point interactions at Y which are parametrized by the interaction strengths $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$. The corresponding operators are denoted by $H_{\alpha,Y}$ and are defined via the resolvent equation:

$$(H_{\alpha,Y} - z^2)^{-1} - (H_0 - z^2)^{-1} = \sum_{j,k=1}^{N} \{\Gamma_{\alpha,Y}(z)\}_{jk}^{-1}\, \mathcal{G}_z(\cdot - y_j) \otimes \overline{\mathcal{G}_z(\cdot - y_k)}, \tag{1}$$

where $z \in \mathbb{C}^+ = \{z \in \mathbb{C} \colon \Im z > 0\}$; more details on the right-hand side of (1) are given next. $\mathcal{G}_z(x)$ is the convolution kernel of $(-\Delta - z^2)^{-1}$ in $L^2(\mathbb{R}^2)$:

$$\mathcal{G}_z(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix\xi} d\xi}{\xi^2 - z^2} \tag{2}$$

and, in terms of the Hankel function of the first kind (see, e.g., [3, (10.8.2)]),

$$\mathcal{G}_z(x) = \frac{i}{4} H_0^{(1)}(z|x|) \tag{3}$$

where

$$\begin{aligned}\frac{i}{4} H_0^{(1)}(z) = &\left(-\frac{1}{2\pi}\log\left(\frac{z}{2}\right) + \frac{i}{4} - \frac{\gamma}{2\pi}\right) J_0(z) \\ &- \frac{1}{2\pi}\left(\frac{\frac{1}{4}z^2}{(1!)^2} - \left(1 + \frac{1}{2}\right)\frac{(\frac{1}{4}z^2)^2}{(2!)^2} + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\frac{(\frac{1}{4}z^2)^3}{(3!)^2} - \cdots\right),\end{aligned} \tag{4}$$

and $J_0(z)$ is the 0th order Bessel function

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z^2}{4}\right)^k. \tag{5}$$

$\Gamma_{\alpha,Y}(z)$ is the $N \times N$ matrix whose (j,k)-entry is the function of $z \in \overline{\mathbb{C}}^+ \backslash \{0\}$ given by

$$\Gamma_{\alpha,Y}(z)_{jk} = \left(\alpha_j + \frac{1}{2\pi}\log\left(\frac{z}{2}\right) - \frac{i}{4} + \frac{\gamma}{2\pi}\right)\delta_{jk} - \mathcal{G}_z(y_j - y_k)\hat{\delta}_{jk}, \tag{6}$$

where δ_{jk} is the Kronecker delta, $\hat{\delta}_{jk} = 1 - \delta_{jk}$ and γ the Euler constant.

Throughout this work, we shall make the following genericity assumption on the set Y of singular points and on the coupling parameters α: *the matrix* $\Gamma_{\alpha,Y}(\lambda)$ *shall be non-singular for* $\lambda \in \mathbb{R}\backslash\{0\}$.

This is indeed reckoned to be the generic case, and the collection of parameters α for which $\Gamma_{\alpha,Y}(\lambda)$ becomes singular for some non-zero point on the real line is expected to be a small (measure-zero) set, although an explicit proof of this claim is still lacking. A first simple indication towards the validity of such claim can be found by observing that if $|\lambda|$ is large enough, then $\Gamma_{\alpha,Y}(\lambda)$ is non-singular because the Hankel function decays to zero as $|\lambda| \to \infty$; as the map $(\alpha, \lambda) \mapsto \det \Gamma_{\alpha,Y}(\lambda)$ is holomorphic, its real zeros must be finite in number.

The following facts are well known (see [2, pp. 163–165]).

(1) Equation (1) defines a unique self-adjoint operator $H_{\alpha,Y}$ in $L^2(\mathbb{R}^2)$ with domain

$$\left\{ u(x) = v(x) + \sum_{j,k=1}^{N} [\Gamma_{\alpha,Y}(z)^{-1}]_{jk} v(y_k) \mathcal{G}_z(x-y_j) \colon v \in H^2(\mathbb{R}^2) \right\} \tag{7}$$

which is independent of $z \in \mathbb{C}^+$ whenever $\Gamma_{\alpha,Y}(z)^{-1}$ exists.

(2) Given z, the function $v \in H^2(\mathbb{R}^2)$ of (7) is uniquely determined by $u \in D(H_{\alpha,Y})$ and

$$(H_{\alpha,Y} - z^2)u = (-\Delta - z^2)v.$$

(3) $H_{\alpha,Y}$ is a real local operator, viz. $H_{\alpha,Y}u$ is real if u is real, and if $u = 0$ in an open set U, then $H_{\alpha,Y}u = 0$ in U.

(4) The spectrum of $H_{\alpha,Y}$ consists of an absolutely continuous part $[0,\infty)$ denoted in short with AC, and at most N non-positive eigenvalues. Positive eigenvalues and singular continuous spectrum are absent.

(5) $H_{\alpha,Y}$ is a rank N perturbation of $-\Delta$ and, by virtue of Kato–Birman–Rosenblum theorem ([9]), the wave operators $W_\pm$ defined as the strong limits in $L^2(\mathbb{R}^2)$,

$$W_\pm = \lim_{t\to\pm\infty} e^{itH_{\alpha,Y}} e^{-itH_0} \tag{8}$$

exist and are complete in the sense that ran $W_\pm = L^2_{ac}(H_{\alpha,Y})$, the AC subspace of $L^2(\mathbb{R}^2)$ for $H_{\alpha,Y}$. Hence

$$W_\pm^* W_\pm = 1, \quad W_\pm W_\pm^* = P_{ac}(H_{\alpha,Y}),$$

where $P_{ac}(H_{\alpha,Y})$ is the orthogonal projection onto $L^2_{ac}(H_{\alpha,Y})$. The wave operators satisfy the intertwining property

$$f(H_{\alpha,Y})P_{ac}(H_{\alpha,Y}) = W_\pm f(H_0) W_\pm \tag{9}$$

for any Borel function f on $\mathbb{R}$.

The Hankel function has the following integral representation

$$\mathcal{G}_z(x) = (i/4)H_0^{(1)}(z|x|) = \frac{e^{iz|x|}}{2^{\frac{3}{2}}\pi} \int_0^\infty e^{-t} t^{-\frac{1}{2}} \left(\frac{t}{2} - iz|x|\right)^{-\frac{1}{2}} dt, \tag{10}$$

see [13, Formula (3), on p. 168]. From (10), we see that for any $c > 0$,

$$\mathcal{G}_z(x) = e^{iz|x|}\omega(z|x|), \quad |\partial_\lambda^\alpha \omega(\lambda)| \le C_\alpha \langle\lambda\rangle^{-\frac{1}{2}-|\alpha|}, \quad \lambda \in \mathbb{R},\ |\lambda| \ge c, \tag{11}$$

viz. $(1-\chi(\lambda))\omega \in S^{-\frac{1}{2}}(\mathbb{R})$ (i.e. the space of one-dimensional symbols of order $-1/2$) whenever $\chi \in C_0^\infty(\mathbb{R})$ is such that $\chi(\lambda) = 1$ near $\lambda = 0$. Also, for purely imaginary $z = ia \in \mathbb{C}^+$ with $a > 0$ we have that $\mathcal{G}_{ia}(x)$ is positive, hence $\Gamma_{\alpha,Y}(ia)$ is real and symmetric.

If $\sigma \in \mathbb{R}$, let $L^2_\sigma = L^2_\sigma(\mathbb{R}^2, \langle x\rangle^{2\sigma}dx)$ be the weighted L^2 space and $\mathbf{B}_\sigma = \mathbf{B}(L^2_\sigma, L^2_{-\sigma})$ the Banach space of bounded operators from L^2_σ to $L^2_{-\sigma}$.

Let $\mathcal{E} \subset \mathrm{i}(0,\infty)$ denote the finite set of square roots of negative eigenvalues of $H_{\alpha,Y}$. Let $\sigma > \frac{1}{2}$ and $z \in \mathbb{C}^+ \backslash \mathcal{E}$. The celebrated Agmon–Kuroda theory [1, 10] of limiting absorption principle for $(-\Delta - z^2)^{-1}$ and the properties of the Hankel function (4), (11), imply that the $\mathbf{B}_\sigma$-valued analytic function $(H_{\alpha,Y} - z^2)^{-1}$ admits a boundary value $(H_{\alpha,Y} - \lambda^2)^{-1}$ for $\lambda \in \mathbb{R}\backslash\{0\}$ which is locally Hölder continuous. However, it can be singular at $\lambda = 0$.

We shall show that $(H_{\alpha,Y} - \lambda^2)^{-1}$ can either be continuously extended to the whole $\overline{\mathbb{C}}^+ \backslash \mathcal{E}$, namely the closed half plane minus the non-zero singularity set, in which case we say $H_{\alpha,Y}$ is of *regular type*, or it has singularities of one of the three kinds associated with resonances of s-wave or p-wave types or zero energy eigenvalue.

In the regular case, we then show that the wave operators are bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$. We write λ instead of z when we want to emphasize that λ is in $\overline{\mathbb{C}}^+ \backslash \{0\}$, not only in $\mathbb{C}^+$.

For stating our main results, we need some more notation. We introduce the vectors

$$\widehat{\mathcal{G}}_{\lambda,Y}(x) = \begin{pmatrix} \mathcal{G}_\lambda(x-y_1) \\ \vdots \\ \mathcal{G}_\lambda(x-y_N) \end{pmatrix}, \quad \widehat{G}_{0,Y}(x) = \begin{pmatrix} G_0(x-y_1) \\ \vdots \\ G_0(x-y_N) \end{pmatrix},$$

where $G_0(x)$ is the Green function of the 2-dimensional $-\Delta$:

$$G_0(x) = -\frac{1}{2\pi}\log|x|, \quad (-\Delta)^{-1}u(x) = \int_{\mathbb{R}^2} G_0(x-y)u(y)dy,$$

so that the right-hand side of (1) may be expressed as

$$D(\lambda, x, y) = \langle \widehat{\mathcal{G}}_{\lambda,Y}(x), \Gamma_{\alpha,Y}(\lambda)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle, \tag{12}$$

where $\langle \mathbf{a}, \mathbf{b}\rangle = \sum a_j b_j$ (without complex conjugation). Also:

$$\mathbf{e} = \frac{1}{\sqrt{N}}\hat{\mathbf{1}}, \quad \hat{\mathbf{1}} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad P = \mathbf{e} \otimes \mathbf{e}, \quad S = 1 - P. \tag{13}$$

Moreover, $\tilde{\mathcal{D}} = \mathcal{D}(\alpha, Y)$ and $\mathcal{G}_1(Y)$ are $N \times N$ real symmetric matrices given by

$$\tilde{\mathcal{D}} = \left(\delta_{jk}\alpha_j + \frac{\hat{\delta}_{jk}}{2\pi}\log|y_j - y_k|\right), \quad \mathcal{G}_1(Y) = -\left(\frac{\hat{\delta}_{jk}}{4N}|y_j - y_k|^2\right). \tag{14}$$

For an integral operator K, we denote by $K(x,y)$ its integral kernel and we often identify K with $K(x,y)$. We will use the function

$$g(\lambda) = -\frac{1}{2\pi}\log\left(\frac{\lambda}{2}\right) + \frac{i}{4} - \frac{\gamma}{2\pi} \tag{15}$$

which appears in front of $J_0(z)$ in (4) as one of the scales for the asymptotic expansions as $\lambda \to 0$, the other being λ. We have for small $|\lambda||x|$ that

$$\mathcal{G}_\lambda(x) = g(\lambda) + G_0(x) + O(\lambda^2|x|^2 g(\lambda|x|)). \tag{16}$$

The representation of any point $x \in \mathbb{R}^2$ in polar coordinates will be $x = r\omega$, where $r \equiv |x| \geq 0$ and $\omega \in \mathbb{S}^1$. For $u, v \in L^2(\mathbb{R}^2)$, $u \otimes v$ denotes the rank-1 operator $f \mapsto u\langle v, f\rangle$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^2)$, anti-linear in the first entry and linear in the second. The notation $\langle f, g\rangle$ will also be used whenever the dual product is meaningful, say for $f \in \mathcal{S}$ and $g \in \mathcal{S}'$. For the Fourier transform in $\mathbb{R}^d$, we use the convention

$$(\mathcal{F}f)(\xi) \equiv \widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}x\xi} f(x)\,\mathrm{d}x.$$

We often write $f \leq_{|\,\cdot\,|} g$ when $|f| \leq |g|$. When not specified otherwise, C denotes a universal positive constant and 1 is the identity operator on the space that is clear from the context. Since the centers Y and the strengths α will be fixed throughout the paper, we shall often omit them from the notation whenever we think no confusion can occur.

Here is our first main result.

Theorem 1.1. *Let $\sigma > 1$. Then, as a $\mathbf{B}_\sigma$-valued function of $\lambda \in \overline{\mathbb{C}}^+ \setminus (\mathcal{E} \cup \{0\})$, the resolvent $(H_{Y,\alpha} - \lambda^2)^{-1}$ satisfies the following properties:*

(1) *The linear map $S\tilde{\mathcal{D}}S$ in $S\mathbb{C}^N$ is non-singular if and only if $(H_{Y,\alpha} - \lambda^2)^{-1}$ can be extended to a continuous function on $\overline{\mathbb{C}}^+$ and*

$$\begin{aligned}(H_{Y,\alpha} - \lambda^2)^{-1}(x,y) &= G_0(x-y) - N^{-1}(\langle \hat{G}_{0,Y}(x), \hat{\mathbf{1}}\rangle + \langle \hat{\mathbf{1}}, \hat{G}_{0,Y}(y)\rangle) - N^{-2}\langle \hat{\mathbf{1}}, \tilde{\mathcal{D}}\hat{\mathbf{1}}\rangle \\ &\quad + \langle [S\tilde{\mathcal{D}}S]^{-1} S(\hat{G}_{0,Y}(x) - N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}}), S(\hat{G}_{0,Y}(y) - N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}})\rangle + O(g(\lambda)^{-1}),\end{aligned}$$

where $O(g(\lambda)^{-1})$ satisfies $\|O(g(\lambda)^{-1})\|_{\mathbf{B}_\sigma} \leq C|g(\lambda)^{-1}|$ as $\lambda \to 0$.

(2) *Suppose that $\mathrm{Ker}_{S\mathbb{C}^N} S\tilde{\mathcal{D}}S \neq 0$ and let T be the orthogonal projection in $S\mathbb{C}^N$ onto $\mathrm{Ker}_{S\mathbb{C}^N} S\tilde{\mathcal{D}}S$. Then $\mathrm{rank}\, T\tilde{\mathcal{D}}^2T \leq 1$. Moreover,*

   (a) *If $T\tilde{\mathcal{D}}^2T$ is non-singular in $T\mathbb{C}^N$, then $\mathrm{rank}\, T = 1$ and, if we write $T = \mathbf{f} \otimes \mathbf{f}$, $\mathbf{f} = {}^t(f_1, \dots, f_N)$ and $\langle \mathbf{f}, \tilde{\mathcal{D}}^2\mathbf{f}\rangle = \gamma_0^{-2}$, $\gamma_0 > 0$, then*

$$(H_{Y,\alpha} - \lambda^2)^{-1} = \gamma_0^{-2} g(\lambda)\varphi \otimes \varphi + O(1) \quad (\lambda \to 0), \tag{17}$$

where $\varphi(x) \in \mathbb{R}$ satisfies $-\Delta\varphi(x) = \sum_{j=1}^N f_j \delta(x - y_j)$ and as $|x| \to \infty$

$$\varphi(x) = \left\langle \mathbf{f}, \hat{G}_{0,Y}(x) - \frac{\tilde{\mathcal{D}}\hat{\mathbf{1}}}{N} \right\rangle = b + \frac{a_1x_1 + a_2x_2}{|x|^2} + O\left(\frac{1}{|x|^2}\right) \tag{18}$$

where a_1, a_2 are real constants and

$$b = -N^{-1}\langle \mathbf{f}, \tilde{\mathcal{D}}\hat{\mathbf{1}}\rangle \neq 0. \tag{19}$$

(b) *If* $T\tilde{\mathcal{D}}^2T$ *is singular in* $T\mathbb{C}^N$, *then* $T\tilde{\mathcal{D}}^2T = 0$ *and* $T\tilde{\mathcal{D}} = \tilde{\mathcal{D}}T = 0$. *If* $T\mathcal{G}_1(Y)T$ *is non-singular in* $T\mathbb{C}^N$, *then*

$$\begin{aligned}(H_{Y,\alpha} - \lambda^2)^{-1}(x,y) \\ &= -(Ng\lambda^2)^{-1}\langle T\hat{G}_{0,Y}(x), [T\mathcal{G}_1(Y)T]^{-1}T\hat{G}_{0,Y}(y)\rangle + O(\lambda^{-2}). \\ &= -(Ng\lambda^2)^{-1}\sum_{j=1}^{n}\varphi_j(x)\varphi_j(y) + O(\lambda^{-2}) \quad (\lambda \to 0),\end{aligned} \tag{20}$$

where $n = \operatorname{rank} T$, $(a_{j1}, a_{j,2}) \neq 0$, $j = 1, \dots, n$ *are real constants and, as* $|x| \to \infty$

$$\varphi_j(x) = \frac{a_{j1}x_1 + a_{j2}x_2}{|x|^2} + O(|x|^{-2}). \tag{21}$$

(c) *Both above operators* $T\tilde{\mathcal{D}}^2T$ *and* $T\mathcal{G}_1(Y)T$ *are singular in* $T\mathbb{C}^N$ *if and only if* $H_{\alpha,Y}$ *has an eigenvalue at zero. More precisely, let* T_1 *denote the orthogonal projection onto* $\operatorname{Ker} T\mathcal{G}_1(Y)T$ *and denote by*:

$$\mathcal{G}_2(Y) = -\left(\frac{\hat{\delta}_{jk}}{8\pi N}|y_j - y_k|^2 \log\left(\frac{e}{|y_j - y_k|}\right)\right). \tag{22}$$

Then $T_1\mathcal{G}_2(Y)T_1$ *is non-singular in* $T_1\mathbb{C}^N$ *and*

$$\begin{aligned}(H_{Y,\alpha} - \lambda^2)^{-1}(x,y) \\ &= -(N\lambda^2)^{-1}\langle T_1\hat{G}_{0,Y}(x), [T_1\mathcal{G}_2(Y)T_1]^{-1}T_1\hat{G}_{0,Y}(y)\rangle \\ &\quad + O(\lambda^{-2}g^{-1}) \qquad (23) \\ &= -(N\lambda^2)^{-1}\sum_{j=1}^{m}\psi_j(x)\psi_j(y) + O(\lambda^{-2}g^{-1}), \quad \lambda \to 0, \qquad (24)\end{aligned}$$

where $m = \operatorname{rank} T_1\mathcal{G}_2(Y)T_1$ *and* $\psi_1, \dots, \psi_m$ *are zero energy eigenfunctions of* $H_{\alpha,Y}$.

We say that $H_{\alpha,Y}$ is of regular type in the case (1) and that $H_{\alpha,Y}$ has zero energy resonance of s-wave type in the case (2.a) and p-wave type in the case (2.b). Note that in all cases the leading term as $\lambda \to 0$ of $(H_{\alpha,Y} - \lambda^2)^{-1}$ is an operator of finite rank. The behaviors of $\varphi(x)$ in the s-wave resonance or $\varphi_1(x), \dots, \varphi_n(x)$ in the p-wave resonance case are similar to the corresponding resonance functions of Schrödinger operators with regular very short range potentials (cf. [8, 5]).

Remarks. (1) If $N = 1$, $H_{\alpha,Y}$ is always of regular type. This directly follows from (1) where the $\log\lambda$ singularity cancels identically.

(2) Let $N = 2$. Then:

(i) $H_{\alpha,Y}$ is of regular type if and only if $\alpha_1 + \alpha_2 \neq \pi^{-1}\log|y_1 - y_2|$.

(ii) $H_{\alpha,Y}$ has a resonance of s-wave type if and only if

$$\alpha_1 + \alpha_2 = \pi^{-1} \log|y_1 - y_2| \quad \text{and}$$
$$(\alpha_1 - (2\pi)^{-1} \log|y_1 - y_2|)^2 + (\alpha_2 - (2\pi)^{-1} \log|y_1 - y_2|)^2 > 0.$$

(iii) $H_{\alpha,Y}$ has a resonance of p-wave type if and only if

$$\alpha_1 = \alpha_2 = (2\pi)^{-1} \log|y_1 - y_2|.$$

(iv) $H_{\alpha,Y}$ cannot have a zero energy eigenvalue.

(3) If $N \geq 3$, we shall prove that both $T\tilde{\mathcal{D}}^2T$ and $T\mathcal{G}_1(Y)T$ can be singular and a zero eigenvalue can exist. A similar argument also applies to the three dimensional case, thus the statement on the absence of zero eigenvalue for point interactions in [2] is incorrect.

More precisely, we have the following result:

Proposition 1.2. *Let $N \geq 3$. Assume that $\mathbf{a} = {}^t(a_1, \ldots, a_N) \in \mathbb{R}^N \backslash \{0\}$ satisfies*

$$\sum_{j=1}^{N} a_j = 0, \quad \sum_{j=1}^{N} a_j y_j = 0, \quad \text{and} \quad \tilde{\mathcal{D}}\mathbf{a} = 0. \tag{25}$$

Then the function

$$\psi(x) = -\sum_{j=1}^{N} \frac{a_j}{2\pi} \log|x - y_j| \tag{26}$$

belongs to the domain of $H_{\alpha,Y}$ and $H_{\alpha,Y}\psi = 0$. Moreover, the converse is also true: any eigenfunction which obeys $H_{\alpha,Y}\psi = 0$ must be of the form (26) where $\mathbf{a} \in \mathbb{R}^N \backslash \{0\}$ obeys (25).

For $N = 3$ and $y_1, y_2, y_3 \in \mathbb{R}^2$ which are collinear or for $N \geq 4$ and arbitrary $y_1, \ldots, y_N \in \mathbb{R}^2$, there exists $\mathbf{a} \in \mathbb{R}^N \backslash \{0\}$ which satisfies the first two equations of (25). Then we can always find α such that $\tilde{D}\mathbf{a} = 0$ and, hence, $H_{\alpha,Y}$ has an eigenvalue at zero. We will also prove in Lemma 3.1 that a zero mode (if it exists) is always non-degenerate when $N \leq 4$ and we conjecture that this is always true.

The third main result of our paper is the following theorem:

Theorem 1.3. *Suppose that $H_{\alpha,Y}$ is of regular type. Then the wave operators $W_\pm$ are bounded in $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.*

It has been long known (see [7]) that the wave operators for one-dimensional Schrödinger operators with point interactions are bounded in $L^p(\mathbb{R}^1)$ for all $1 < p < \infty$ and, in three dimensions it was recently shown ([4]) that they are bounded in $L^p(\mathbb{R}^3)$ if and only if $3/2 < p < 3$. Thus, there is a sharp contrast between the results in dimensions one or two compared to dimension three. We also note that for Schrödinger operators with multiplicative short-range potentials it has been recently proved [6] that the wave operators remain bounded in $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$ even when there is an s-wave resonance or an eigenvalue at threshold.

The intertwining property (9) reduces the mapping properties of the AC part of the functions $f(H_{\alpha,Y})$ of $H_{\alpha,y}$ to that of $f(H_0)$ and there is a large body of literature on the L^p mapping properties of the wave operators (for this we refer to the reference of [4, 14]). The same intertwining property (9) and the well-known L^p-L^q estimates for the free propagator imply the corresponding property of $e^{-itH_{\alpha,Y}}$. We write $\|u\|_p = \|u\|_{L^p(\mathbb{R}^2)}$ for $1 \le p \le \infty$ and p' is the dual exponent of p defined by $1/p + 1/p' = 1$.

Corollary 1.4. *For any $2 \le p < \infty$, there exists a constant C_p such that*

$$\|e^{-itH_{\alpha,Y}} P_{ac}(H_{\alpha,Y})u\|_p \le C_p |t|^{1/p-1/2} \|u\|_{p'}, \quad u \in L^p(\mathbb{R}^2) \cap L^2(\mathbb{R}^2). \tag{27}$$

An immediate corollary of the L^p-L^q estimates (27) are the Strichartz estimates in two dimensions: We say (p, r) is a two-dimensional Strichartz exponent if it satisfies

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 < q \le \infty.$$

Corollary 1.5. *Suppose that $H_{\alpha,Y}$ is of regular type. Let (p, q) and (s, r) be 2-dimensional Strichartz exponents. Then there exists a constant $C > 0$ such that:*

$$\left(\int_{\mathbb{R}} \|e^{-itH_{\alpha,Y}} u\|^q_{L^p(\mathbb{R}^2)} dt \right)^{1/q} \le C\|u\|_2,$$

$$\left\| \int_0^t e^{-i(t-s)H_{\alpha,Y}} P_{ac}(H_{\alpha,Y})\, u(s)\, \mathrm{d}s \right\|_{L^q(\mathbb{R}_t, L^p(\mathbb{R}^2_x))} \le C\|u\|_{L^{s'}(\mathbb{R}_t, L^{r'}(\mathbb{R}^3_x))}.$$

For more about Schrödinger operators with point interactions we refer to the monograph [2], while for L^p boundedness of wave operators we refer to our previous papers [4, 14] and references therein.

The structure of the remaining text is as follows:

- In Sec. 2, we give a detailed analysis of the behavior of $\Gamma(\lambda)^{-1}$ near $\lambda = 0$ and we classify its possible singularities.
- In Sec. 3, we prove both Theorem 1.1 and Proposition 1.2, results which completely characterize the threshold behavior of the class of zero point interactions we consider here.
- Finally, in Sec. 4, we give the proof of Theorem 1.3 concerning the L^p boundedness of wave operators.

2. The Small λ Behavior of $\Gamma(\lambda)^{-1}$

We begin with the study of the small λ behavior of $\Gamma(\lambda)^{-1}$. In this section, the notation $O(g(\lambda)^j \lambda^k)$, $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, represents a scalar or a matrix-valued function which has an asymptotic expansion when $\lambda \to 0$ as

$$O(g(\lambda)^j \lambda^k) = C_j g(\lambda)^j \lambda^k + C_{j-1} g(\lambda)^{j-1} \lambda^k + \cdots \quad \mathrm{mod}\ O(\lambda^{k+1}) \tag{28}$$

which may be differentiated term by term. For simplicity, *we will often omit the variable λ from various functions and write, e.g., g for $g(\lambda)$, F for $F(\lambda)$* and so on. The expression (a_{jk}) will denote an $N \times N$ matrix with entries a_{jk}.

We shall repeatedly use the following lemma due to Jensen and Nenciu ([8]) in the case when $\mathcal{H}$ is finite dimensional.

Lemma 2.1. *Let A be a closed operator in a Hilbert space $\mathcal{H}$ and S a projection. Suppose $A+S$ has a bounded inverse. Then, A has a bounded inverse if and only if*

$$B = S - S(A+S)^{-1}S$$

has a bounded inverse in $S\mathcal{H}$ and, in this case,

$$A^{-1} = (A+S)^{-1} + (A+S)^{-1}SB^{-1}S(A+S)^{-1}. \tag{29}$$

From (4) and (5) and the definition (15) of $g(\lambda)$, we have as $\lambda \to 0$ that

$$\frac{i}{4}H_0^{(1)}(\lambda) = g(\lambda) - \frac{1}{4}g(\lambda)\lambda^2 - \frac{1}{8\pi}\lambda^2 + O(\lambda^4 g(\lambda)) \tag{30}$$

and, for $j \neq k$,

$$\begin{aligned} -\mathcal{G}_\lambda(y_j - y_k) = -g(\lambda) + \frac{1}{2\pi}\log|y_j - y_k| + \frac{\lambda^2}{4}g(\lambda)|y_j - y_k|^2 \\ + \frac{\lambda^2}{8\pi}|y_j - y_k|^2 \log\left(\frac{e}{|y_j - y_k|}\right) + O(\lambda^4 g). \end{aligned} \tag{31}$$

It follows from (6) and (31) that as $\lambda \to 0$

$$\begin{aligned} \Gamma(\lambda) &= -g(\lambda)\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \tilde{\mathcal{D}} + g(\lambda)\lambda^2\left(\frac{\hat{\delta}_{jk}}{4}|y_j - y_k|^2\right) \\ &\quad + \frac{\lambda^2}{8\pi}\left(\hat{\delta}_{jk}|y_j - y_k|^2\log\left(\frac{e}{|y_j - y_k|}\right)\right)_{jk} + O(\lambda^4 g) \\ &= -Ng(P - N^{-1}g(\lambda)^{-1}\tilde{\mathcal{D}} + \lambda^2\mathcal{G}_1(Y) + \lambda^2 g(\lambda)^{-1}\mathcal{G}_2(Y) + O(\lambda^4)) \\ &=: -Ng(\lambda)A(\lambda), \end{aligned} \tag{32}$$

where $\tilde{\mathcal{D}}$ and $\mathcal{G}_1(Y)$ are defined in (14) and $\mathcal{G}_2(Y)$ in (22).

We apply Lemma 2.1 to the pair consisting of the operator A appearing in (32) and $S = 1 - P$ in the space $\mathcal{H} = \mathbb{C}^N$. For simplicity we write

$$F(\lambda) = -N^{-1}g(\lambda)^{-1}\tilde{\mathcal{D}} \tag{33}$$

so that as $\lambda \to 0$

$$\begin{aligned} A(\lambda) + S &= 1 + F(\lambda) + R_1 + O(\lambda^4), \\ R_1 = R_1(\lambda, Y) &= \lambda^2\mathcal{G}_1(Y) + \lambda^2 g(\lambda)^{-1}\mathcal{G}_2(Y) \; (= O(\lambda^2)). \end{aligned}$$

For small $0 < \lambda < \lambda_0$, the inverse $(1+F)^{-1}$ exists and

$$(1+F)^{-1} = 1 - F + \cdots + (-F)^{n-1} + O(g^{-n}), \quad \lambda \to 0. \tag{34}$$

Moreover, $A(\lambda) + S$ is invertible and

$$\begin{aligned}(A+S)^{-1} &= (1+F)^{-1}(1 + (R_1 + O(\lambda^4))(1+F)^{-1})^{-1} \\ &= (1+F)^{-1} - (1+F)^{-1}R_1(1+F)^{-1} + O(\lambda^4) \\ &= (1+F)^{-1} - R_2 + O(\lambda^2 g(\lambda)^{-2}),\end{aligned} \tag{35}$$

where

$$R_2 = \lambda^2\mathcal{G}_1(Y) + \lambda^2 g^{-1}(N^{-1}\mathcal{G}_1(Y)\tilde{\mathcal{D}} + N^{-1}\tilde{\mathcal{D}}\mathcal{G}_1(Y) + \mathcal{G}_2(Y)) \ (= O(\lambda^2)). \tag{36}$$

From (35), we have

$$B = S(1 - (A+S)^{-1})S = S(F(1+F)^{-1} + R_2 + O(\lambda^2 g^{-2}))S. \tag{37}$$

2.1. *The case when $S\tilde{\mathcal{D}}S$ is invertible in $S\mathbb{C}^N$*

Suppose that $S\tilde{\mathcal{D}}S$ is invertible in $S\mathbb{C}^N$. Since

$$F(1+F)^{-1} = F - F^2 + F^3 - \cdots = -N^{-1}g^{-1}\tilde{\mathcal{D}} + O(g^{-2}),$$

by absorbing $R_2 + O(\lambda^2 g^{-2})$ of (37) into $O(g^{-2})$ we have:

$$B = S(-N^{-1}g^{-1}[S\tilde{\mathcal{D}}S] + O(g^{-2}))S. \tag{38}$$

Thus, B is invertible in $S\mathbb{C}^N$ for small $|\lambda| > 0$ and

$$B^{-1} = -Ng(\lambda)S[S\tilde{\mathcal{D}}S]^{-1}S + SO(1)S. \tag{39}$$

Combining (29), (35) and (39), we see that

$$\begin{aligned}A^{-1} = (1+F)^{-1} &- Ng(1+F)^{-1}S([S\tilde{\mathcal{D}}S]^{-1} + O(g^{-1}))S(1+F)^{-1} \\ &+ O(g\lambda^2).\end{aligned} \tag{40}$$

Lemma 2.2. *$\Gamma(\lambda)^{-1}$ is bounded as $\lambda \to 0$ if and only if $S\tilde{\mathcal{D}}S$ is non-singular in $S\mathbb{C}^N$. In this case,*

$$\begin{aligned}\Gamma(\lambda)^{-1} &= -N^{-1}g^{-1}(1+F)^{-1} \\ &\quad + (1+F)^{-1}S([S\tilde{\mathcal{D}}S]^{-1} + O(g^{-1}))S(1+F)^{-1} + O(\lambda^2) \qquad (41) \\ &= [S\tilde{\mathcal{D}}S]^{-1} + O(g^{-1}). \qquad (42)\end{aligned}$$

There exists a constant $\lambda_0 > 0$ such that all entries of $\Gamma^{-1}(\lambda)$ satisfy

$$\partial_\lambda^\ell[\Gamma^{-1}(\lambda)]_{jk} \leq_{|\cdot|} C\lambda^{-\ell}, \quad \ell = 0, 1, \ldots \quad 0 < |\lambda| < \lambda_0. \tag{43}$$

Proof of the "if" part. We substitute (40) for A in $\Gamma^{-1}(\lambda) = -N^{-1}g^{-1}A^{-1}$ and pick up the leading (bounded) term. Recall that the asymptotic expansion for (28) may be differentiated term by term and

$$(d/d\lambda)^j O(g^{-1}) \le_{|\cdot|} C\lambda^{-j}, \quad j = 0, 1, \dots.$$

This proves (43) by virtue of (42).

The proof of the "only if" part of Lemma 2.2 can be completed only when we finish proving all other lemmas in this section.

As we shall see in Sec. 4, (43) is sufficient for studying the L^p-boundedness of the wave operators, however, we need the more detailed structure (41) and (42) in Sec. 3 for studying the behavior of $(H_{\alpha,Y} - \lambda^2)^{-1}$ as $\lambda \to 0$.

2.2. *The case when $S\tilde{\mathcal{D}}S$ is singular in $S\mathbb{C}^N$*

Suppose that the leading term $S\tilde{\mathcal{D}}S$ in (38) is singular in $S\mathbb{C}^N$. We expand $(1+F)^{-1}$ in (37) up to the order $O(g^{-3})$ as

$$F(1+F)^{-1} = -N^{-1}g^{-1}\tilde{\mathcal{D}} + N^{-2}g^{-2}\tilde{\mathcal{D}}^2 - N^{-3}g^{-3}R_3, \quad R_3 = O(1), \tag{44}$$

where the definition of $R_3 = R_3(\lambda)$ should be obvious (see (34)). We introduce the operator

$$B = -N^{-1}g^{-1}(S\tilde{\mathcal{D}}S - N^{-1}g^{-1}S\tilde{\mathcal{D}}^2S + S(N^{-2}g^{-2}R_3 + NgR_2)S) \tag{45}$$

(recall from (36) that $R_2 = O(\lambda^2)$). In order to find the small λ behavior of B^{-1} in $S\mathbb{C}^N$ we introduce the pair (A_1, T) where

$$A_1 = S\tilde{\mathcal{D}}S + N^{-1}g^{-1}S\tilde{\mathcal{D}}^2S - S(N^{-2}g^{-2}R_3 + NgR_2)S \tag{46}$$

and T is the orthogonal projection in $S\mathbb{C}^N$ onto $\operatorname{Ker} S\tilde{\mathcal{D}}S$. Then we again apply Lemma 2.1 to the pair (A_1, T). Since $(S\tilde{\mathcal{D}}S + T)^{-1}$ is invertible in $S\mathbb{C}^N$ and $A_1 = S\tilde{\mathcal{D}}S + O(g(\lambda)^{-1})$, then $A_1 + T$ is also invertible for small $0 < |\lambda|$ and

$$\begin{aligned}
&(A_1+T)^{-1} \\
&\quad = (S\tilde{\mathcal{D}}S + T)^{-1}(1 + (N^{-1}g^{-1}S\tilde{\mathcal{D}}^2S + SO(g^{-2})S)(S\tilde{\mathcal{D}}S + T)^{-1})^{-1} \\
&\quad = (S\tilde{\mathcal{D}}S + T)^{-1} - N^{-1}g^{-1}(S\tilde{\mathcal{D}}S + T)^{-1}S\tilde{\mathcal{D}}^2S(S\tilde{\mathcal{D}}S + T)^{-1} + SO(g^{-2})S.
\end{aligned}$$

Since $TS = ST = T$ and $T(S\tilde{\mathcal{D}}S + T)^{-1} = (S\tilde{\mathcal{D}}S + T)^{-1}T = T$, the operator B_1 which corresponds to B when $A_1 = A$ satisfies

$$B_1 := T - T(A_1+T)^{-1}T = N^{-1}g^{-1}T\tilde{\mathcal{D}}^2T + TO(g^{-2})T. \tag{47}$$

Here we state the following lemma:

Lemma 2.3. *The matrix $T\tilde{\mathcal{D}}^2T$ has rank 1. It is singular in $T\mathbb{C}^N$ if and only if $\tilde{\mathcal{D}}T = T\tilde{\mathcal{D}} = 0$ and, if it is non-singular, then $\dim T\mathbb{C}^N = 1$.*

Proof. Choose an orthonormal basis $\{\mathbf{e}_2, \dots, \mathbf{e}_N\}$ of $S\mathbb{C}^N$ so that $\{\mathbf{e}, \mathbf{e}_2, \dots, \mathbf{e}_N\}$ is an orthonormal basis of $\mathbb{C}^N = \langle \mathbf{e} \rangle \oplus S\mathbb{C}^N$, where $\langle \mathbf{e} \rangle$ denotes the linear span of $\mathbf{e}$. Let K denote the matrix of $S\tilde{\mathcal{D}}S$. Let

$$\mathcal{M} = \begin{pmatrix} a & {}^t\mathbf{a} \\ \mathbf{a} & K \end{pmatrix} \tag{48}$$

be the block matrix representation of $\tilde{\mathcal{D}}$ in this basis with respect to the decomposition $\mathbb{C}^N = \langle \mathbf{e} \rangle \oplus S\mathbb{C}^N$. Then

$$\mathcal{M}^2 = \begin{pmatrix} a^2 + |\mathbf{a}|^2 & a\,{}^t\mathbf{a} + {}^t\mathbf{a}K \\ a\mathbf{a} + K\mathbf{a} & \mathbf{a} \otimes \mathbf{a} + K^2 \end{pmatrix}.$$

We identify T with its matrix. Since T projects onto the kernel of $S\tilde{\mathcal{D}}S$ in $S\mathbb{C}^N$, $T\tilde{\mathcal{D}}^2T$ has the matrix representation $T\mathbf{a} \otimes T\mathbf{a}$ with respect to the basis $\{\mathbf{e}_2, \dots, \mathbf{e}_N\}$ and has rank 1. It follows that $T\tilde{\mathcal{D}}^2T$ is singular in $T\mathbb{C}^N$ if and only if $T\tilde{\mathcal{D}}^2T = (\tilde{\mathcal{D}}T)^*(\tilde{\mathcal{D}}T) = 0$ or $\tilde{\mathcal{D}}T = T\tilde{\mathcal{D}} = 0$ and if it is non-singular, it must be that $\dim T\mathbb{C}^N = 1$. □

(a) The case when $T\tilde{\mathcal{D}}^2T$ is non-singular in $T\mathbb{C}^N = \mathrm{Ker}\, S\tilde{\mathcal{D}}S$. If $T\tilde{\mathcal{D}}^2T$ is non-singular in $T\mathbb{C}^N$, then $\dim T\mathbb{C}^N = 1$ by virtue of Lemma 2.3; Eq. (47) implies that B_1 is invertible in $T\mathbb{C}^N$ and

$$B_1^{-1} = NgT[T\tilde{\mathcal{D}}^2T]^{-1}T + O(1). \tag{49}$$

Then, by virtue of Lemma 2.1, A_1 is invertible in $S\mathbb{C}^N$ and

$$A_1^{-1} = (A_1 + T)^{-1} + (A_1 + T)^{-1}TB_1^{-1}T(A_1 + T)^{-1}. \tag{50}$$

Since $(A_1 + T)^{-1}$ is bounded as $\lambda \to 0$ and $(A_1 + T)^{-1}T = T(A_1 + T)^{-1} = T + O(g^{-1})$, we conclude from (49) and (50) that in the space $S\mathbb{C}^N$

$$A_1^{-1} = NgT[T\tilde{\mathcal{D}}^2T]^{-1}T + O(1). \tag{51}$$

Thus, $B = -N^{-1}g^{-1}A_1$ is invertible in $S\mathbb{C}^N$ and

$$B^{-1} = -NgA_1^{-1} = -N^2g^2T[T\tilde{\mathcal{D}}^2T]^{-1}T + O(g). \tag{52}$$

Lemma 2.4. *Suppose $S\tilde{\mathcal{D}}S$ is singular in the space $S\mathbb{C}^N$ and let T be the orthogonal projection onto the kernel of $S\tilde{\mathcal{D}}S$ in $S\mathbb{C}^N$. Suppose that $T\tilde{\mathcal{D}}^2T$ is non-singular in $T\mathbb{C}^N$. Then, $T\tilde{\mathcal{D}}^2T$ has rank* 1 *and $\Gamma(\lambda)^{-1}$ has* log *singularities as $\lambda \to 0$. More precisely,*

$$\begin{aligned} \Gamma(\lambda)^{-1} &= -N^{-1}g^{-1}(1+F)^{-1} \\ &\quad + Ng(1+F)^{-1}(T[T\tilde{\mathcal{D}}^2T]^{-1}T + TO(g^{-1})T)(1+F)^{-1} + O(\lambda^2 g) \qquad (53) \\ &= Ng[T\tilde{\mathcal{D}}^2T]^{-1} + O(1). \qquad (54) \end{aligned}$$

Proof. We combine (52) with (32), (29) and (35). Recalling that $(A + S) = (1 + F)^{-1} + O(\lambda^2)$ and $(1 + F)^{-1} = 1 + O(g^{-1})$, we obtain the above result. □

(b) The case when $T\tilde{\mathcal{D}}^2T$ is singular in $T\mathbb{C}^N$ but $T\mathcal{G}_1(Y)T$ is non-singular. If $T\tilde{\mathcal{D}}^2T$ is singular in $T\mathbb{C}^N$, then, by virtue of Lemma 2.3, $T\tilde{\mathcal{D}} = \tilde{\mathcal{D}}T = 0$ and $R_3T = TR_3 = 0$, see (44) for the definition of R_3. Define

$$L = S\tilde{\mathcal{D}}S + N^{-1}g^{-1}S\tilde{\mathcal{D}}^2S - SN^{-2}g^{-2}R_3S,$$

so that $TL = LT = 0$ and, by virtue of (46),

$$A_1 = L + SNgR_2S.$$

Recall that $R_2 = \lambda^2\mathcal{G}_1(Y) + \lambda^2 g^{-1}(N^{-1}\mathcal{G}_1(Y)\tilde{\mathcal{D}} + N^{-1}\tilde{\mathcal{D}}\mathcal{G}_1(Y) + \mathcal{G}_2(Y))$ (see (36)). It follows that in the direct sum decomposition of $S\mathbb{C}^N = (1-T)\mathbb{C}^N \oplus T\mathbb{C}^N$,

$$L + T = \begin{pmatrix} L^{\perp} & 0 \\ 0 & T \end{pmatrix} = L^{\perp} \oplus T$$

where $L^{\perp}$ is the part of L in $(1-T)\mathbb{C}^N$ and

$$(L^{\perp})^{-1} = \{(1-T)S\tilde{\mathcal{D}}S(1-T)\}^{-1} + O(g^{-1}).$$

It follows that

$$\begin{aligned}(A_1 + T)^{-1} &= (L + T + SNgR_2S)^{-1} \\ &= (L^{\perp} \oplus T)^{-1} - (L^{\perp} \oplus T)^{-1}(SNgR_2S)(L^{\perp} \oplus T)^{-1} \\ &\quad + O(\lambda^4 g^2). \end{aligned} \tag{55}$$

Define

$$R_4(Y) := N^{-1}\mathcal{G}_1(Y)\tilde{\mathcal{D}} + N^{-1}\tilde{\mathcal{D}}\mathcal{G}_1(Y) + \mathcal{G}_2(Y).$$

Then the operator $B_1 = T - T(A_1 + T)^{-1}T$ of (47) is given as

$$\begin{aligned} B_1 &= -TNg\lambda^2(\mathcal{G}_1(Y) + g^{-1}R_4(Y) + O(\lambda^2 g^2))T \\ &= -TNg\lambda^2(\mathcal{G}_1(Y) + g^{-1}\mathcal{G}_2(Y) + O(\lambda^2 g^2))T \end{aligned} \tag{56}$$

where we used (36), the identity $T(L^{\perp} \oplus T)^{-1} = \mathbf{0} \oplus 1$ and that $\tilde{\mathcal{D}}T = T\tilde{\mathcal{D}} = 0$ in the final step.

If $T\mathcal{G}_1(Y)T$ is non-singular in $T\mathbb{C}^N$, we see from (56) that B_1 is invertible in $T\mathbb{C}^N$ for small $0 < |\lambda|$ and

$$B_1^{-1} = -N^{-1}g^{-1}\lambda^{-2}T[T\mathcal{G}_1(Y)T]^{-1}T + O(\lambda^{-2}g^{-1}).$$

We again apply Lemma 2.1 to A_1, B_1 and T and use (55) for $(A_1+T)^{-1}$. Then, since $(A_1 + T)^{-1} = O(g(\lambda))$ as $\lambda \to 0$,

$$\begin{aligned} A_1^{-1} &= (A_1 + T)^{-1} - (A_1 + T)^{-1}TB_1^{-1}T(A_1 + T)^{-1} \\ &= N^{-1}g^{-1}\lambda^{-2}T[T\mathcal{G}_1(Y)T]^{-1}T + O(\lambda^{-2}g^{-1}) \end{aligned}$$

and

$$B^{-1} = -Ng(\lambda)A_1^{-1} = -\lambda^{-2}T[T\mathcal{G}_1(Y)T]^{-1}T + SO(\lambda^{-2}g^{-1})S.$$

By using (35), we have proved the following lemma:

Lemma 2.5. *Suppose that $S\tilde{\mathcal{D}}S$ is singular in the space $S\mathbb{C}^N$ and $T\tilde{\mathcal{D}}^2T = 0$. Suppose further that $T\mathcal{G}_1(Y)T$ is non-singular in $T\mathbb{C}^N$. Then, as $\lambda \to 0$ we have*:

$$\begin{aligned}\Gamma(\lambda)^{-1} &= -N^{-1}g^{-1}(1+F)^{-1} + N^{-1}g^{-1}\lambda^{-2} \\ &\quad \times (1+F)^{-1}S(T[T\mathcal{G}_1(Y)T]^{-1}T + O(g^{-1}))S(1+F)^{-1} + O(g^{-1}) \qquad (57) \\ &= N^{-1}g^{-1}\lambda^{-2}T[T\mathcal{G}_1(Y)T]^{-1}T + O(\lambda^{-2}), \qquad (58)\end{aligned}$$

where we wrote $ST = T$ for simplicity in (58).

(c) The case when both $T\tilde{\mathcal{D}}^2T$ and $T\mathcal{G}_1(Y)T$ are singular in $T\mathbb{C}^N$. We note the identity

$$\mathcal{G}_2(Y) = \frac{1}{2\pi}\mathcal{G}_1(Y) + \tilde{\mathcal{G}}_2(Y), \quad \text{where } \tilde{\mathcal{G}}_2(Y) = \left(\frac{\hat{\delta}_{jk}}{8\pi N}|y_j - y_k|^2 \log|y_j - y_k|\right). \qquad (59)$$

Lemma 2.6. *Suppose that both $T\tilde{\mathcal{D}}^2T$ and $T\mathcal{G}_1(Y)T$ are singular. Let T_1 be the orthogonal projection onto $\mathrm{Ker}\, T\mathcal{G}_1(Y)T$ in $T\mathbb{C}^N$. Then both $T_1\mathcal{G}_2(Y)T_1$ and $T_1\tilde{\mathcal{G}}_2(Y)T_1$ are non-singular in $T_1\mathbb{C}^N$.*

Proof. Define the $N \times N$ matrix M by

$$M = (\hat{\delta}_{jk}|y_j - y_k|^2 \log(|y_j - y_k|^2)).$$

Due to the presence of T_1, it suffices to show that T_1MT_1 is non-singular in $T_1\mathbb{C}^N$. Because all the matrices we worked with until now were real and symmetric, we may choose their eigenvectors to be real. Thus the matrices of T_1 and of T_1MT_1 are also real and self-adjoint. Hence we can choose the eigenvectors of T_1MT_1 to be real. Let $f = T_1 f$ be a normalized real eigenvector of T_1MT_1 associated with the smallest eigenvalue. We show that necessarily $\langle f, Mf\rangle > 0$, hence T_1MT_1 is positive definite and non-singular on the range of T_1.

For $\mathbf{f} = {}^t(f_1, \dots, f_N) \in T_1\mathbb{C}^N$, define the function

$$F(\lambda) = \sum_{1\le j\neq k\le N} f_j f_k |y_j - y_k|^2 \log(|y_j - y_k|^2 + \lambda), \quad \lambda \ge 0.$$

We want to show that $F(0) > 0$. We observe that $F(\lambda)$ is smooth for $\lambda \ge 0$ and that $\lim_{\lambda\to\infty} F(\lambda) = 0$ because $\sum_{1\le j\neq k\le N} f_j f_k|y_j - y_k|^2 = \langle \mathbf{f}, T_1\mathcal{G}_1(Y)T_1\mathbf{f}\rangle = 0$ for $\mathbf{f} \in T_1\mathbb{C}^n$ and

$$F(\lambda) = \sum_{1\le j\neq k\le N} f_j f_k |y_j - y_k|^2(\log(|y_j - y_k|^2 + \lambda) - \log\lambda).$$

We will prove that $F'(\lambda) < 0$ for all $\lambda > 0$, which implies $F(0) > \lim_{\lambda\to\infty} F(\lambda) = 0$ and finishes the proof. In order to do that, we compute

$$F'(\lambda) = \sum_{j,k=1}^{N} f_j f_k \frac{|y_j - y_k|^2}{|y_j - y_k|^2 + \lambda} = -\sum_{j,k=1}^{N} f_j f_k \frac{\lambda}{|y_j - y_k|^2 + \lambda}.$$

For $t > 0$, we have the identity:

$$e^{-t(|y_j-y_k|^2+\lambda)} = \frac{e^{-\lambda t}}{4\pi t} \int_{\mathbb{R}^2} e^{ip(y_j-y_k)} e^{-p^2/(4t)} dp,$$

and also:

$$\frac{1}{|y_j - y_k|^2 + \lambda} = \lim_{n\to\infty} \int_{n^{-1}}^{n} e^{-t(|y_j-y_k|^2+\lambda)} dt.$$

Thus,

$$F'(\lambda) = -\lambda \lim_{n\to\infty} \int_{n^{-1}}^{n} dt\, \frac{e^{-\lambda t}}{4\pi t} \int_{\mathbb{R}^2} dp\, e^{-p^2/(4t)} \left| \sum_{j=1}^{N} e^{ip\cdot y_j} f_j \right|^2 \leq 0.$$

The above inequality is in fact strict for every $\lambda > 0$, since $F'(\lambda) = 0$ for some $\lambda > 0$ would imply $\sum_{j=1}^{N} e^{ip\cdot y_j} f_j = 0$ for all $p \in \mathbb{R}^2$, which is equivalent with $f_1 = \cdots = f_N = 0$. □

For studying B_1^{-1} of (56), we let A_2 be the linear operator in $T\mathbb{C}^N$ inside the parenthesis of (56):

$$A_2 = T(\mathcal{G}_1(Y) + g^{-1}\mathcal{G}_2(Y) + O(\lambda^2 g^2))T, \tag{60}$$

and apply Lemma 2.1 once again to the pair (A_2, T_1).

The inverse $(A_2 + T_1)^{-1}$ exists in $T\mathbb{C}^N$ for small $0 < |\lambda|$ and, omitting the variable Y,

$$\begin{aligned}(A_2 + T_1)^{-1} &= (T\mathcal{G}_1 T + T_1)^{-1} - g^{-1}(T\mathcal{G}_1 T + T_1)^{-1} T\mathcal{G}_2(Y)T(T\mathcal{G}_1 T + T_1)^{-1} \\ &\quad + O(g^{-2}).\end{aligned}$$

We need to consider the invertibility of

$$B_2 = T_1 - T_1(A_2 + T_1)^{-1}T_1.$$

Since $T_1 T\mathcal{G}_1 T = T\mathcal{G}_1 T T_1 = 0$ and $T_1(T\mathcal{G}_1 T + T_1)^{-1} = (T\mathcal{G}_1 T + T_1)^{-1}T_1 = T_1$, we have $B_2 = -g^{-1}T_1\tilde{\mathcal{G}}_2(Y)T_1 + O(g^{-2})$. Because $T_1\tilde{\mathcal{G}}_2(Y)T_1$ is non-singular in $T_1\mathbb{C}^N$ by virtue of Lemma 2.6, B_2^{-1} exists for small $|\lambda| > 0$ and

$$B_2^{-1} = -gT_1[T_1\tilde{\mathcal{G}}_2(Y)T_1]^{-1}T_1 + O(1).$$

Then, by virtue of Lemma 2.1, A_2^{-1} also exists for small $|\lambda| > 0$ and

$$\begin{aligned}A_2^{-1} &= (A_2 + T_1)^{-1} - (A_2 + T_1)^{-1}T_1 B_2^{-1} T_1 (A_2 + T_1)^{-1} \\ &= gT_1[T_1\tilde{\mathcal{G}}_2(Y)T_1]^{-1}T_1 + O(1)\end{aligned}$$

where we used $(A_2+T_1)^{-1}=O(1)$ and $(A_2+T_1)^{-1}T_1=T_1(A_2+T_1)^{-1}=T_1+O(g^{-1})$ in the second step. Thus, we have

$$\begin{aligned} B_1^{-1} &= -T^{-1}N^{-1}g(\lambda)^{-1}A_2^{-1} \\ &= -T^{-1}N^{-1}\lambda^{-2}T_1[T_1\tilde{\mathcal{G}}_2(Y)T_1]^{-1}T_1T + O(\lambda^{-2}g(\lambda)^{-1}). \end{aligned} \tag{61}$$

Then, exactly as in the case (b), we have

$$A_1^{-1} = T^{-1}N^{-1}\lambda^{-2}T_1[T_1\tilde{\mathcal{G}}_2(Y)T_1]^{-1}T_1T + O(\lambda^{-2}g(\lambda)^{-1})$$

and

$$B^{-1} = -Ng(\lambda)A_1^{-1} = T^{-1}N^{-1}\lambda^{-2}T_1[T_1\tilde{\mathcal{G}}_2(Y)T_1]^{-1}T_1T + O(\lambda^{-2}g(\lambda)^{-1}).$$

Repeating the argument in the last part of the proof of Lemma 2.2, we prove the following lemma:

Lemma 2.7. *Suppose that $T\tilde{\mathcal{D}}^2T=0$ and $T\mathcal{G}_1(Y)T$ are both singular in $T\mathbb{C}^N$. Let T_1 be the projection to $\operatorname{Ker} T\mathcal{G}_1(Y)T$ in $T\mathbb{C}^N$. Then $T_1\tilde{\mathcal{G}}_2(Y)T_1$ is non-singular in $T_1\mathbb{C}^N$ and, as $\lambda\to 0$ we have:*

$$\begin{aligned} \Gamma(\lambda)^{-1} &= -N^{-1}g^{-1}(1+F)^{-1} - N^{-1}\lambda^{-2}(1+F)^{-1} \\ &\qquad \times S(T_1[T_1\tilde{\mathcal{G}}_2(Y)T_1]^{-1}T_1 + O(g^{-1}))S(1+F)^{-1} + O(1) \qquad (62) \\ &= -N\lambda^{-2}T_1[T_1\tilde{\mathcal{G}}_2(Y)T_1]^{-1}T_1 + O(g^{-1}\lambda^{-2}) \qquad (63) \end{aligned}$$

where in (63) the first T_1 should be considered as a linear map from $T_1\mathbb{C}^N$ into $\mathbb{C}^N$ and the last one from $\mathbb{C}^N$ into $T_1\mathbb{C}^N$.

3. Proof of Theorem 1.1 and Proposition 1.2

We start with Proposition 1.2 because it is strongly related to the part (2)(c) of Theorem 1.1 and it completely characterises the zero modes of $H_{\alpha,Y}$.

3.1. *Proof of Proposition* 1.2

We first show that ψ in (26) belongs to the domain of $H_{\alpha,Y}$ and satisfies $H_{\alpha,Y}\psi=0$. It is obvious that $\psi\in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. For large $|x|$ we have

$$\log|x-y_j| = \log|x| - \frac{x\cdot y_j}{|x|^2} + \mathcal{O}(|x|^{-2})$$

and (25) implies that $\psi(x)$ behaves like $|x|^{-2}$ at infinity hence it is square-integrable. We next show $\psi\in D(H_{\alpha,Y})$. Let $\mu\in\mathbb{C}^+$ be such that $\Gamma_{\alpha,Y}(\mu)$ is invertible and define the vector

$$v_\mu(x) := \psi(x) - \sum_{j=1}^N a_j\mathcal{G}_\mu(x-y_j) = \sum_{j=1}^N a_j\left(-\frac{1}{2\pi}\log|x-y_j| - \mathcal{G}_\mu(x-y_j)\right).$$

Clearly, $v_\mu \in H^2(\mathbb{R}^2)$ because the logarithmic singularities of ψ are removed. Moreover, we have:

$$v_\mu(y_k) = \sum_{j=1}^N a_j \left(\hat{\delta}_{jk}(-(2\pi)^{-1} \log|y_k - y_j| - \mathcal{G}_\mu(y_k - y_j)) + \frac{\delta_{jk}}{2\pi}(\log(\mu/2i) + \gamma) \right)$$

$$= \sum_{j=1}^N [\Gamma_{\alpha,Y}(\mu)]_{kj} a_j,$$

where we used the fact that $\tilde{D}\mathbf{a} = 0$. Thus we have:

$$\psi(x) = v_\mu(x) + \sum_{j,k=1}^N [\Gamma_{\alpha,Y}(\mu)]^{-1}_{jk} v_\mu(y_k) \mathcal{G}_\mu(x - y_j)$$

which (see (7)) shows that ψ belongs to the domain of $H_{\alpha,Y}$. By computing the distributional Laplacian of v_μ we obtain:

$$(-\Delta - \mu^2)v_\mu = \mu^2 \sum_{j=1}^N \frac{a_j}{2\pi} \log|x - y_j| = -\mu^2\psi = (H_{\alpha,Y} - \mu^2)\psi,$$

which confirms that $H_{\alpha,Y}\psi = 0$.

We now prove the converse. Assume that ψ is in the domain of $H_{\alpha,Y}$ and $H_{\alpha,Y}\psi = 0$. Let μ be such that $\Gamma_{\alpha,Y}(\mu)$ is invertible, viz. $\mu \in \mathbb{C}^+ \backslash \mathcal{E}$, $\mathcal{E} \subset i[0,\infty)$ being the square roots of negative eigenvalues of $H_{\alpha,Y}$. Then there must exist a function $v_\mu \in H^2(\mathbb{R}^2)$ such that

$$\psi(x) = v_\mu(x) + \sum_{j,k=1}^N [\Gamma_{\alpha,Y}(\mu)]^{-1}_{jk} v_\mu(y_k) \mathcal{G}_\mu(x - y_j). \tag{64}$$

Define $\mathbf{a} = {}^t(a_1, \ldots, a_N)$ by

$$a_j = \sum_{k=1}^N [\Gamma_{\alpha,Y}(\mu)]^{-1}_{jk} v_\mu(y_k), \quad j = 1, \ldots, N.$$

The vector $\mathbf{a}$ must be independent of μ because all its components can be directly expressed in terms of ψ by using (16) in (64):

$$a_j = -2\pi \lim_{x \to y_j} \psi(x)(\log|x - y_j|)^{-1}, \quad j = 1, \ldots, N.$$

Since $(H_{\alpha,Y} - \mu^2)\psi = (-\Delta - \mu^2)v_\mu$, we have the equation

$$-\Delta v_\mu(x) = -\mu^2 \sum_j^N a_j \mathcal{G}_\mu(x - y_j).$$

In momentum coordinates we have:

$$\hat{v}_\mu(p) = -\frac{\mu^2}{2\pi} \sum_{j=1}^N a_j \frac{e^{-ip\cdot y_j}}{p^2(p^2 - \mu^2)}.$$

Since $v_\mu \in L^2(\mathbb{R}^2)\backslash\{0\}$, $\mathbf{a}$ must obey the first two equations of (25).

We now show that if we take $\mu = i\lambda$ with sufficiently small $\lambda > 0$ and $\mu \to 0$, then $\|v_\mu\|_{H^2} \to 0$, hence $v_\mu(x) \to 0$ uniformly. Since $\mathbf{a}$ satisfies the first two equations of (25) this would imply

$$\lim_{\mu\to 0} \Gamma_{\alpha,Y}(\mu)\mathbf{a} = \tilde{\mathcal{D}}\mathbf{a} = 0$$

and the desired identity

$$\psi(x) = -\frac{1}{2\pi}\sum_{j=1}^{N} a_j \log|x - y_j|.$$

To show that $\|v_\mu\|_{H^2} \to 0$ we first observe the trivial estimate

$$\|\Delta v_{i\lambda}\| = \|\lambda^2\Delta(-\Delta + \lambda^2)^{-1}\psi\| \le \lambda^2\|\psi\|.$$

In momentum coordinates we have that

$$\hat{v}_{i\lambda}(p) = \lambda^2(p^2 + \lambda^2)^{-1}\hat{\psi}(p) \le_{|\,\cdot\,|} \hat{\psi}(p) \quad \text{and} \quad \lim_{\lambda\to 0}\hat{v}_{i\lambda}(p) = 0 \quad \text{if } p \neq 0.$$

It follows by the dominated convergence theorem,

$$\|v_{i\lambda}\|_{L^2(\mathbb{R}^2)} \to 0 \quad (\lambda \to 0).$$

Hence $\|v_{i\lambda}\|_{H^2} \to 0$ as $\lambda \to 0$. This finishes the proof of Proposition 1.2.

Lemma 3.1. *Let $N \le 4$ and assume that $H_{\alpha,Y}$ has an embedded eigenvalue at zero. Then the eigenvalue is non-degenerate.*

Proof. We know that we need at least $N = 3$ in order to have a zero mode. Without any loss of generality, up to a translation, a scaling and a relabeling, we may always assume that $y_1 = 0$, $|y_2| = 1$, and $1 \le |y_j|$ for all $j \ge 3$.

If $N = 3$ then y_2 and y_3 must be linearly dependent, otherwise the first two constraints of (25) impose $\mathbf{a} = 0$ and no zero mode can exist. If y_2 and y_3 are collinear then up to a translation, a scaling and a relabeling we may assume that y_2 and y_3 have the same direction and $|y_1| = 0 < |y_2| = 1 < |y_3|$. We write $a_2 = -|y_3|a_3$ and $a_1 = -a_2 - a_3 = (|y_3| - 1)a_3$. Thus all the compatible $\mathbf{a}$'s belong to a one-dimensional subspace generated by the vector with components $\mathbf{a}_1 = {}^t(|y_3| - 1, -|y_3|, 1)$. Now from the equation $\tilde{D}\mathbf{a}_1 = 0$ we can find the right combination of α's (uniquely determined by $|y_3|$) for which a zero mode can exist. Thus if a zero mode exists, it must be non-degenerate.

Now let $N = 4$. We know that y_2, y_3 and y_4 are linearly dependent. There are two possibilities: either these three vectors are all collinear or they are not.

If they are not collinear, for example y_2 and y_3 are linearly independent, then given any $a_4 \in \mathbb{R}$ we may uniquely determine a_2 and a_3 from the equation $a_2y_2 + a_3y_3 = -a_4y_4$ and also $a_1 = -a_2 - a_3 - a_4$. Thus we are again in a situation in which the compatible $\mathbf{a}$'s form an one-dimensional family. As in the $N = 3$ case, if a zero mode exists, it must be unique.

Let us now assume that all four points are collinear. We may also assume without loss of generality that y_2, y_3 and y_4 have the same direction and

$$|y_1| = 0 < |y_2| = 1 < |y_3| < |y_4|.$$

Then we have $a_2 = -a_3|y_3| - a_4|y_4|$ and $a_1 = (|y_3| - 1)a_3 + (|y_4| - 1)a_4$. This time, the family of compatible $\mathbf{a}$'s is two-dimensional, generated by the following two linearly independent vectors:

$$\mathbf{a}_1 = {}^t(|y_3| - 1, -|y_3|, 1, 0) \quad \text{and} \quad \mathbf{a}_2 = {}^t(|y_4| - 1, -|y_4|, 0, 1).$$

To each generator we can separately find some α's for which a zero mode would exist, but we want to see if we can find one joint α for which both equations $\tilde{D}\mathbf{a}_1 = 0$ and $\tilde{D}\mathbf{a}_2 = 0$ are simultaneously satisfied. By solving for α in both equations we obtain four compatibility relations involving $|y_3|$ and $|y_4|$. The one involving α_1 imposes the condition:

$$\frac{\log|y_3|}{|y_3| - 1} = \frac{\log|y_4|}{|y_4| - 1}.$$

But the function $(\log t)/(t-1)$ is strictly decreasing if $t \in (1, \infty)$, hence the above equality cannot hold true. Thus the zero mode is unique if it exists. □

Remark. If $N \geq 5$, the family of $\mathbf{a}$'s which are compatible with the first two equations in (25) is always at least two dimensional. The compatibility relations (only involving the y's) which are obtained from the condition that the α's must be the same, are much more complicated. Nevertheless, they can always be written as an equation of the type $F(y_3, \dots, y_N) = 0$ where $F : \mathbb{R}^{N-2} \mapsto \mathbb{R}^N$ is a rather complicated function; here $y_1 = 0$ and $y_2 = {}^t(1, 0)$ are fixed and no two y's can coincide. We conjecture that no degeneracy is possible when $N \geq 5$.

3.2. *Proof of Theorem* 1.1: *Preliminaries*

We will study the operator

$$D(\lambda) = (H_{\alpha,Y} - \lambda^2)^{-1} - (H_0 - \lambda^2)^{-1} = \langle \widehat{\mathcal{G}}_{\lambda,Y}(x), \Gamma_{\alpha,Y}(\lambda)^{-1} \widehat{\mathcal{G}}_{\lambda,Y}(y) \rangle \qquad (65)$$

when $\lambda \in \overline{\mathbb{C}}^{+} \setminus (\mathcal{E} \cup \{0\})$ converges to zero, by using the results of Sec. 2. Our results will be stated for $\lambda > 0$, however, they hold for $\lambda \in \overline{\mathbb{C}}^{+} \setminus (\mathcal{E} \cup \{0\})$ with the same proof. As before, we identify operators with their integral kernels.

We define

$$R_0(\lambda, x) = \mathcal{G}_\lambda(x) - g(\lambda|x|) = \mathcal{G}_\lambda(x) - g(\lambda) - G_0(x)$$

and use the vector notation

$$\widehat{R}_{0,Y}(\lambda, x) = \begin{pmatrix} R_0(\lambda, x - y_1) \\ \vdots \\ R_0(\lambda, x - y_N) \end{pmatrix}, \quad \widehat{g}_\lambda = g(\lambda)\hat{\mathbf{1}}$$

so that

$$\widehat{\mathcal{G}}_{\lambda,Y}(x) = \hat{g}(\lambda) + \widehat{G}_{0,Y}(x) + \widehat{R}_{0,Y}(\lambda, x). \tag{66}$$

By virtue of (30) for the Hankel function for small λ, we have for any constant $C_1 > 0$ and for an arbitrary small $0 < \delta$ that

$$R_0(\lambda, x) \le_{|\,\cdot\,|} C_\delta |\lambda|x||^\delta, \quad |\lambda|x|| < C_1, \tag{67}$$

and from (11) for large λ that

$$\mathcal{G}_\lambda(x) \le_{|\,\cdot\,|} C|\lambda|x||^{-1/2}, \quad |\lambda|x|| \ge C_1. \tag{68}$$

We take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$ such that

$$\chi(x) = 1, \quad \text{for } |x| \le 1 \quad \text{and} \quad \chi(x) = 0, \quad \text{for } |x| \ge 2$$

and define for $\lambda > 0$

$$\chi_\lambda(x) = \chi(\lambda x), \quad \widehat{\mathcal{G}}^{\le}_{\lambda,Y}(x) = \chi_\lambda(x)\widehat{\mathcal{G}}_{\lambda,Y}, \quad \widehat{\mathcal{G}}^{\ge}_{\lambda,Y}(x) = (1-\chi_\lambda(x))\widehat{\mathcal{G}}_{\lambda,Y},$$

and likewise for other functions. To shorten the formulas, we often omit the variables from various functions.

Lemma 3.2. *For any $\lambda_0 > 0$ and $\sigma > 1$, there exists $C > 0$ such that the following estimates are satisfied for $0 < \lambda < \lambda_0$:*

$$\|\widehat{\mathcal{G}}^{\ge}_{\lambda,Y}\|_{L^2_{-\sigma}} \le C\lambda^{\sigma-1}, \quad \|\widehat{G}^{\ge}_{0,Y}\|_{L^2_{-\sigma}} \le C\lambda^{\sigma-1}\langle g(\lambda)\rangle. \tag{69}$$

$$\|\widehat{\mathcal{G}}^{\le}_{\lambda,Y}\|_{L^2_{-\sigma}} \le C\langle g(\lambda)\rangle, \quad \|\widehat{G}^{\le}_{0,Y}\|_{L^2_{-\sigma}} \le C. \tag{70}$$

For any $0 < \delta < \sigma - 1$, there exists $C > 0$ such that for $0 < \lambda < \lambda_0$

$$\|\widehat{R}_{0,Y}(\lambda, x)\|_{L^2_{-\sigma}} \le C\lambda^\delta. \tag{71}$$

Proof. By virtue of (68), we have for $\sigma > 1$ and for small $0 < \lambda < \lambda_0$ that

$$\|\widehat{\mathcal{G}}^{\ge}_{\lambda,Y}\|_{L^2_{-\sigma}} \le C\lambda^{-\frac{1}{2}} \left(\int_{|x| \ge C\lambda^{-1}} |x|^{-1-2\sigma} dx \right)^{1/2} = C\lambda^{\sigma-1}.$$

$$\|\widehat{G}^{\ge}_{0,Y}\|_{L^2_{-\sigma}} \le C \left(\int_{|x| \ge C\lambda^{-1}} \frac{(\log|x|)^2}{|x|^{2\sigma}} dx \right)^{1/2} = C\lambda^{\sigma-1}\log\lambda.$$

This proves (69). The first of the following estimates is obvious and the second follows from (67):

$$\left(\int_{|x| \le \lambda^{-1}} |\widehat{G}_{0,Y}(x)|^2 \langle x\rangle^{-2\sigma} dx \right)^{1/2} \le C < \infty,$$

$$\left(\int_{|x| \le \lambda^{-1}} |\widehat{\mathcal{G}}_{\lambda,Y}(x)|^2 \langle x\rangle^{-2\sigma} dx \right)^{1/2} \le C\langle g(\lambda)\rangle.$$

This yields (70). By virtue of (67), we have for any $0<\delta<\sigma-1$ that

$$\left(\int_{\mathbb{R}^2} |\widehat{R}^{\le}_{0,Y}(\lambda,x)|^2 \langle x\rangle^{-2\sigma}dx\right)^{1/2} \le C\lambda^\delta \left(\int_{|x|\le\lambda^{-1}} |x|^{2\delta}\langle x\rangle^{-2\sigma}dx\right)^{1/2}$$
$$\le C\lambda^\delta. \tag{72}$$

Estimate (69) implies $\|\widehat{R}^{\ge}_{0,Y}(\lambda)\|_{L^2_{-\sigma}} \le C\lambda^\delta$ for any $0<\delta<\sigma-1$, which completes the proof of the lemma. □

We will now study $D(\lambda)$ of (65) in the space $\mathbf{B}_\sigma = \mathbf{B}(L^2_\sigma(\mathbb{R}^2), L^2_{-\sigma}(\mathbb{R}^2))$, $\sigma>1$. Note that $L^2_\sigma(\mathbb{R}^2)\subset L^1(\mathbb{R}^2)$ when $\sigma>1$.

We begin with studying the contribution to $D(\lambda)$ of $-N^{-1}g^{-1}(1+F)^{-1}$ which is the common first term in the right-hand sides of the first formulas for $\Gamma(\lambda)$ in Lemmas 2.2, 2.4, 2.5 and 2.7.

Lemma 3.3. *Let $\sigma>1$. Then, as a $\mathbf{B}_\sigma$-valued function of $\lambda>0$ we have*

$$-N^{-1}g^{-1}\langle\widehat{\mathcal{G}}_{\lambda,Y}(x),(1+F)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle \tag{73}$$
$$= -g - N^{-1}(\langle\hat{G}_{0,Y}(x),\hat{\mathbf{1}}\rangle + \langle\hat{\mathbf{1}},\hat{G}_{0,Y}(y)\rangle + N^{-1}\langle\hat{\mathbf{1}},\tilde{\mathcal{D}}\hat{\mathbf{1}}\rangle) + O(g^{-1}), \tag{74}$$

where $O(g^{-1})$ is such that $\|O(g^{-1})\|_{\mathbf{B}_\sigma}\le C|g(\lambda)|^{-1}$ as $\lambda\to0$.

Proof. We substitute $\widehat{\mathcal{G}}_{\lambda,Y}(x) = \widehat{\mathcal{G}}^{\ge}_{\lambda,Y}(x)+\widehat{\mathcal{G}}^{\le}_{\lambda,Y}(x)$ and likewise for $\widehat{\mathcal{G}}_{\lambda,Y}(y)$ in (73). Then, (69) and (70) imply that as $\lambda\to0$

$$(73) = -N^{-1}g^{-1}\langle\widehat{\mathcal{G}}^{\le}_{\lambda,Y}(x),(1+F)^{-1}\widehat{\mathcal{G}}^{\le}_{\lambda,Y}(y)\rangle + O(\lambda^\delta) \tag{75}$$

for any $0<\delta<\sigma-1$. Multiplying (67) by $\chi_\lambda(x)$ we have

$$\widehat{\mathcal{G}}^{\le}_{\lambda,Y}(x) = \chi_\lambda(x)\hat{g}(\lambda) + \widehat{G}^{\le}_{0,Y}(x) + \widehat{R}^{\le}_{0,Y}(\lambda,x),$$

and likewise for $\widehat{\mathcal{G}}^{\le}_{\lambda,Y}(y)$ which we insert in the right of (75). This produces nine terms out of which five contain $\widehat{R}^{\le}_{0,Y}(\lambda,x)$ or $\widehat{R}^{\le}_{0,Y}(\lambda,y)$ and, by virtue of (70) and (71), they are bounded by $C\lambda^\delta$, $\delta<\sigma-1$ in $\mathbf{B}_\sigma$. We collect them into $O(g^{-1})$ of (74). Moreover, we trivially have

$$\|N^{-1}g(\lambda)^{-1}\langle\widehat{G}^{\le}_{0,Y}(x),(1+F(\lambda))^{-1}\widehat{G}^{\le}_{0,Y}(y)\rangle\|_{\mathbf{B}_\sigma} \le C\langle g(\lambda)\rangle^{-1}$$

and we include this too into $O(g^{-1})$. Thus, we only have the following three terms Z_1, Z_2 and Z_3 to deal with.

$$Z_1 = -N^{-1}g^{-1}\langle\chi_\lambda(x)\hat{g},(1+F)^{-1}\chi_\lambda(y)\hat{g}\rangle, \tag{76}$$

$$Z_2 = -N^{-1}g^{-1}\langle\widehat{G}^{\le}_{0,Y}(x),(1+F)^{-1}\chi_\lambda(y)\hat{g}\rangle, \tag{77}$$

$$Z_3 = -N^{-1}g^{-1}\langle\chi_\lambda(x)\hat{g},(1+F)^{-1}\widehat{G}^{\le}_{0,Y}(y)\rangle. \tag{78}$$

We have by using that $\|1-\chi_\lambda\|_{L^2_{-\sigma}} \le C\lambda^{\sigma-1}$ for $0<\lambda<C_1$

$$\begin{aligned} Z_1 &= -N^{-1}g^{-1}\langle \hat g, (1+F)^{-1}\hat g\rangle \chi_\lambda \otimes \chi_\lambda \\ &= -N^{-1}g^{-1}(\langle \hat g, \hat g\rangle - \langle \hat g, F\hat g\rangle + \langle \hat g, F^2(1+F)^{-1}\hat g\rangle)\chi_\lambda \otimes \chi_\lambda \\ &= (-g - N^{-2}\langle \hat{\mathbf{1}}, \tilde{\mathcal{D}}\hat{\mathbf{1}}\rangle)(1\otimes 1) + O(g^{-1}). \end{aligned} \tag{79}$$

In a similar fashion we have

$$Z_2 = -N^{-1}\langle \widehat{G}_{0,Y}(x), \hat{\mathbf{1}}\rangle + O(g^{-1}), \tag{80}$$

$$Z_3 = -N^{-1}\langle \hat{\mathbf{1}}, \widehat{G}_{0,Y}(y)\rangle + O(g^{-1}). \tag{81}$$

The combination of (79)–(81) concludes the proof of Lemma 3.3. □

The following corollary shows that the sum of the first term and the contribution by the common first term $-N^{-1}g^{-1}(1+F)^{-1}$ of $\Gamma(\lambda)$ in the Lemmas 2.2, 2.4, 2.5, and 2.7 to the second term on the right of

$$(H_{\alpha,Y}-\lambda^2)^{-1} = \mathcal{G}_\lambda(x-y) + \langle \widehat{\mathcal{G}}_{\lambda,Y}(x), \Gamma(\lambda)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle \tag{82}$$

is bounded in $\mathbf{B}_\sigma$:

Corollary 3.4. *Let $\sigma>1$. Then as $\lambda\to 0$,*

$$\begin{aligned} &\mathcal{G}_\lambda(x-y) - N^{-1}g^{-1}\langle \widehat{\mathcal{G}}_{\lambda,Y}(x), (1+F)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle \\ &\quad = G_0(x-y) - N^{-1}(\langle \hat G_{0,Y}(x), \hat{\mathbf{1}}\rangle + \langle \hat{\mathbf{1}}, \hat G_{0,Y}(y)\rangle + N^{-1}\langle \hat{\mathbf{1}}, \tilde{\mathcal{D}}\hat{\mathbf{1}}\rangle) + O(g^{-1}) \end{aligned} \tag{83}$$

which is bounded in $\mathbf{B}_\sigma$ as $\lambda\to 0$.

Proof. Substitute $\mathcal{G}_\lambda(x-y) = g(\lambda) + G_0(x-y) + O(\lambda^2 g(\lambda)|x-y|^2)$. Then (83) immediately follows Lemma 3.3. □

The second terms in the first formulas for $\Gamma(\lambda)$ in Lemmas 2.2, 2.4, 2.5, and 2.7 are all sandwiched by $(1+F)^{-1}S$ and $S(1+F)^{-1}$ and, for studying their contributions to $D(\lambda,x,y)$, we use the following lemma. Recall that T is a linear map defined in $S\mathbb{C}^N$ and if we identify T with $\mathbf{0}_{P\mathbb{C}^N}\oplus T$, then $TS=ST=T$.

Lemma 3.5. *Let $\sigma>1$ and $0<\delta<\sigma-1$. Then, there exists $\lambda_0>0$ such that for $0<\lambda<\lambda_0$ the following estimates are satisfied for some constant $C>0$:*

$$\|S(\widehat{\mathcal{G}}_{\lambda,Y} - \widehat{G}_{0,Y})\|_{L^2_{-\sigma}(\mathbb{R}^2)} \le C\lambda^\delta, \tag{84}$$

$$\|S(1+F)^{-1}\widehat{\mathcal{G}}_{\lambda,Y} - S(\widehat{G}_{0,Y} + N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}})\|_{L^2_{-\sigma}} \le C\langle g\rangle^{-1} \tag{85}$$

In particular, the $L^2_{-\sigma}(\mathbb{R}^2)$-valued analytic functions $S\widehat{\mathcal{G}}_{\lambda,Y}$ and $S(1+F)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}$ of $\lambda\in\mathbb{C}^+$ have continuous extensions to the closure $\overline{\mathbb{C}}^+$.

Proof. Since $S\hat{g} = 0$, we have $S(\widehat{\mathcal{G}}_{\lambda,Y} - \widehat{G}_{0,Y}) = S\widehat{R}_{0,Y}(\lambda, x)$ and (84) follows from (71). We write

$$S(1+F)^{-1}\widehat{\mathcal{G}}_{\lambda,Y} = S(\widehat{\mathcal{G}}_{\lambda,Y} - F\widehat{\mathcal{G}}_{\lambda,Y} + F^2(1+F)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}) \tag{86}$$

and, on the right-hand side, we substitute (66) for first two $\widehat{\mathcal{G}}_{\lambda,Y}$, $F = -N^{-1}g^{-1}\tilde{\mathcal{D}}$ in the second term, use $S\hat{g} = 0$ and arrange so that in the formula below the terms in the first line are independent of λ, while those in the second line are bounded by $C\langle g(\lambda)^{-1}\rangle$ in $L^2_{-\sigma}$ as $\lambda \to 0$:

$$\begin{aligned}(86) = S(&\widehat{G}_{0,Y} + N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}} + N^{-1}g^{-1}\tilde{\mathcal{D}}\widehat{G}_{0,Y} + \widehat{R}_{0,Y}(\lambda)\\ &+ F\widehat{R}_{0,Y}(\lambda) + F^2(1+F)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}).\end{aligned}$$

This proves (85). □

We now start proving each statement of Theorem 1.1 separately. By virtue of Corollary 3.4, we only have to study $\langle\widehat{\mathcal{G}}_{\lambda,Y}(x), \Gamma(\lambda)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle$ when $\Gamma(\lambda)^{-1}$ is replaced by the second terms in the first formulas for $\Gamma(\lambda)$ in Lemmas 2.2, 2.4, 2.5, and 2.7 for the corresponding cases.

3.3. *Proof of Theorem* 1.1(1)

We use Lemma 2.2. When $\Gamma(\lambda)^{-1}$ is replaced by (41), $\langle\widehat{\mathcal{G}}_{\lambda,Y}(x), \Gamma(\lambda)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle$ becomes

$$\langle S(1+F)^{-1}S\widehat{\mathcal{G}}_{\lambda,Y}(x), ([S\tilde{\mathcal{D}}S]^{-1} + O(g^{-1}))S(1+F)^{-1}S\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle + O(\lambda^2 g)$$

where we used that $\|\widehat{\mathcal{G}}_{\lambda,Y}\|_{L^2_{-\sigma}} \le C\langle g\rangle$ to obtain the term $O(\lambda^2 g)$. Statement (1) immediately follows by applying Lemma 3.5.

3.4. *Proof of Theorem* 1.1(2)

Proof of Statement (2-a). We apply Lemma 2.4 and Corollary 3.4. Replacing $\Gamma(\lambda)^{-1}$ in $\langle\widehat{\mathcal{G}}_{\lambda,Y}(x), \Gamma(\lambda)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle$ by (53) produces

$$\langle(1+F)^{-1}S\widehat{\mathcal{G}}_{\lambda,Y}(x), T(g[T\tilde{\mathcal{D}}^2T]^{-1} + O(1))T(1+F)^{-1}S\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle + O(\lambda^2 g^3). \tag{87}$$

Thus, if $T = \mathbf{f}\otimes\mathbf{f}$ with normalised $\mathbf{f} \in \mathbb{C}^N$ and $\langle\mathbf{f}, \tilde{\mathcal{D}}^2\mathbf{f}\rangle = \gamma_0^{-2}$, $\gamma_0 > 0$ then Lemma 3.5 implies that

$$(87) = \gamma_0^2 g\varphi(x)\varphi(y) + O(1), \quad \varphi(x) = \langle\widehat{G}_{0,Y}(x) + N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}}, \mathbf{f}\rangle.$$

Here $f_1 + \cdots + f_N = 0$ as $\mathbf{f} = (f_1, \ldots, f_N) \in S\mathbb{C}^N$ and

$$\begin{aligned}\langle\mathbf{f}, \hat{G}_{0,Y}(x)\rangle &= -\frac{1}{2\pi}\sum f_j \log(|x-y_j|) = -\frac{1}{2\pi}\sum f_j(\log(|x-y_j|) - \log|x|)\\ &= \sum f_j \left(\int_0^1 \frac{(x-\theta y_j)y_j}{2\pi|x-\theta y_j|^2}d\theta\right) = \sum f_j\frac{y_j\cdot x}{|x|^2} + O(|x|^{-2}).\end{aligned} \tag{88}$$

In the matrix representation of $\tilde{\mathcal{D}}$ in Lemma 2.3, $\mathbf{e}$ is represented by $\binom{1}{\mathbf{0}}$ and $\hat{\mathbf{1}}$ by $\binom{\sqrt{N}}{\mathbf{0}}$. It follows that $\tilde{\mathcal{D}}\hat{\mathbf{1}}$ is given by the vector $\binom{a}{\mathbf{a}}$. Then, $T\tilde{\mathcal{D}}\hat{\mathbf{1}} = T\mathbf{a}$ which does not vanish as $T\tilde{\mathcal{D}}^2T$ is non-singular in $T\mathbb{C}^N$ (see the proof of Lemma 2.3). Hence $\langle \tilde{\mathcal{D}}\hat{\mathbf{1}}, \mathbf{f}\rangle \neq 0$ and this proves Statement (2-a).

Proof of Statement (2-b). We use Lemma 2.5 and Corollary 3.4. As before, we only have to study

$$N^{-1}g^{-1}\lambda^{-2}\langle (1+F)^{-1}S\widehat{\mathcal{G}}_{\lambda,Y}(x), T[T\mathcal{G}_1(Y)T]^{-1}TS(1+F)^{-1}\widehat{\mathcal{G}}_{\lambda,Y}(y)\rangle \tag{89}$$

and the remainder is of order $O(\lambda^{-2}g^{-2})$. We diagonalize the symmetric matrix as

$$T[T\mathcal{G}_1(Y)T]^{-1}T = \sum_{j=1}^{n} a_j \mathbf{f}_j \otimes \mathbf{f}_j, \quad n = \operatorname{rank} T,$$

where $a_j \in \mathbb{R}\backslash\{0\}$ and $\mathbf{f}_j \in T\mathbb{C}^N$, $j = 1, \dots, N$ can be chosen to be real. Then, Lemma 3.5 implies

$$(89) = N^{-1}g^{-1}\lambda^{-2}\sum_{j=1}^{n} a_j\varphi_j(x)\varphi_j(y), \quad \varphi_j(x) = \langle \mathbf{f}_j, \hat{G}_{0,Y}(x) + N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}}\rangle.$$

Here $\langle \mathbf{f}_j, N^{-1}\tilde{\mathcal{D}}\hat{\mathbf{1}}\rangle = 0$ since $\tilde{\mathcal{D}}\mathbf{f}_j = 0$,

$$\varphi_j(x) = \langle \mathbf{f}_j, \hat{G}_{0,Y}(x)\rangle = \sum_{k=1}^{N} f_{jk}\log(|x-y_k|) = -2\sum_{k=1}^{N} f_{jk}\frac{y_k\cdot x}{|x|^2} + O(|x|^{-2}).$$

We must have that $\sum f_{jk}y_k \neq 0$ for $j = 1, \dots, n$ because for every real vector $\mathbf{f} \in T\mathbb{C}^N$ we have:

$$\langle T\mathcal{G}_1(Y)T\mathbf{f}, \mathbf{f}\rangle = -\frac{1}{4\pi}\sum_{j,k=1}^{N}|y_j - y_k|^2 f_j f_k = \frac{1}{2\pi}\left(\sum_{j=1}^{N} f_j y_j\right)^2 > 0 \tag{90}$$

where we use the assumption that $T\mathcal{G}_1(Y)T$ is non-singular.

Proof of Statement (2-c). We use Lemma 2.7. As in the proof of statement (2-b), we only need to study

$$-N^{-1}\lambda^{-2}\langle \hat{G}_{0,Y}(x), T_1[T_1\mathcal{G}_2(Y)T_1]^{-1}T_1\hat{G}_{0,Y}(y)\rangle \tag{91}$$

and the remainder is $O(\lambda^{-2}g^{-1})$. If we diagonalize

$$T_1\tilde{\mathcal{G}}_2(Y)T_1 = \sum_{j=1}^{m} a_j\mathbf{a}_j \otimes \mathbf{a}_j, \quad m = \operatorname{rank} T_1,$$

then we have

$$(91) = -N^{-1}\lambda^{-2}\sum_{j=1}^{n} a_j\psi_j(x)\otimes\psi_j(y), \quad \psi_j(x) = \langle \mathbf{a}_j, \hat{G}_{0,Y}(x)\rangle.$$

Here we have $\mathbf{a}_j = {}^t(a_{j1}, \ldots, a_{jN}) \in T_1\mathbb{C}^N \subset T\mathbb{C}^N \subset S\mathbb{C}^n$, hence,

$$a_{j1} + \cdots + a_{jN} = 0, \quad \tilde{\mathcal{D}}\mathbf{a}_j = 0, \quad a_{j1}y_1 + \cdots + a_{jN}y_N = 0$$

where the last equation is the result of (90) and $\langle T\mathcal{G}_1(Y)T\mathbf{f}, \mathbf{f}\rangle = 0$ for $\mathbf{f} \in T_1\mathbb{C}^N$. It follows from Proposition 1.2 that $\psi_j(x)$, $j = 1, \ldots, m$ are all eigenfunctions of $H_{\alpha,Y}$ with eigenvalue zero. This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

We only prove the theorem for W_+. The complex conjugation $u \mapsto \mathcal{C}u = \overline{u}$ then gives the proof for $W_- = \mathcal{C}^* W_+ \mathcal{C}$. In what follows we assume $H_{\alpha,Y}$ is of regular type and the results of Lemma 2.2 are satisfied.

4.1. *Stationary representation of the wave operators*

We use the *stationary representation* of W_+ as in the three dimensional case (see [4]). We need some preparation. We set

$$\mathcal{D}_* = \{u \in \mathcal{S}(\mathbb{R}^2) \,|\, \hat{u} \in C_0^\infty(\mathbb{R}^2 \backslash \{0\})\}. \tag{92}$$

Lemma 4.1. *For every $n \in \mathbb{N}$, $\mathcal{D}_*$ is a dense subspace of $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.*

Proof. It suffices to show that having fixed $f \in \mathcal{S}(\mathbb{R}^n)$, then for every $\varepsilon > 0$ there exists a $u \in \mathcal{D}_*$ such that $\|f - u\|_p < \varepsilon$. Take a $\chi \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \chi \le 1$ such that $\chi(\xi) = 1$ for $|\xi| < 1$ and $\chi(\xi) = 0$ for $|\xi| \ge 2$ and set $\chi_\rho(\xi) = \chi(\xi/\rho)$. If we define $u = (1 - \chi_\rho(D))\chi_N(D)f \in \mathcal{D}_*$ then

$$\|f - u\|_p \le \|f - \chi_N(D)f\|_p + \|\chi_\rho(D)f\|_p$$

and it suffices to show that $\|(\chi_N(D) - 1)f\|_p \to 0$ as $N \to \infty$ and $\|\chi_\rho(D)f\|_p \to 0$ as $\rho \to 0$. To see $\|(\chi_N(D) - 1)f\|_p \to 0$ as $N \to \infty$, we write

$$\begin{aligned}
(\chi_N(D) - 1)f(x) &= (2\pi)^{-1} \int_{\mathbb{R}^n} (N^n(\mathcal{F}^{-1}\chi)(N(x-y))f(y) - f(x))dy \\
&= (2\pi)^{-1} \int_{\mathbb{R}^n} (\mathcal{F}^{-1}\chi)(y)(f(x + N^{-1}y) - f(x))dy \\
&= (2\pi)^{-1} \int_{\mathbb{R}^n} N^{-1}y(\mathcal{F}^{-1}\chi)(y) \cdot \left(\int_0^1 \nabla f(x + \theta N^{-1}y)d\theta\right) dy
\end{aligned}$$

and apply Minkowski's inequality to obtain

$$\|(\chi_N(D) - 1)f(x)\|_p \le (2\pi)^{-1}N^{-1}\||y|(\mathcal{F}^{-1}\chi)\|_1 \|\nabla f\|_p. \tag{93}$$

For the second limit, we apply Young's inequality and obtain

$$\|\chi_\rho(D)f\|_p = (2\pi)^{-1}\|(\mathcal{F}^{-1}\chi_\rho) * f\|_p \le (2\pi)^{-1}\rho^{n(1-1/p)}\|(\mathcal{F}^{-1}\chi)\|_p\|f\|_1.$$ □

We define the operator Ω_{jk}, $j,k=1,\dots,N$ such that $(\Omega_{jk}u)(x)$ for $u\in\mathcal{D}_*$ is given by

$$\frac{1}{\pi \mathrm{i}}\lim_{\delta\downarrow 0}\int_0^{+\infty}\lambda\, e^{-\delta\lambda}\overline{(\Gamma_{\alpha,Y}(\lambda)^{-1})_{jk}}\,\mathcal{G}_{-\lambda}(x)\left(\int_{\mathbb{R}^2}(\mathcal{G}_\lambda(y)-\mathcal{G}_{-\lambda}(y))u(y)\mathrm{d}y\right)\mathrm{d}\lambda.$$

Then the following lemma may be proved by repeating line by line the proof of Proposition 3.2 of [4] for the corresponding formula in three dimensions.

Lemma 4.2. *Let $(T_{x_0}f)(x):=f(x-x_0)$ be the translation operator by x_0. Then, for $u,v\in\mathcal{D}_*$,*

$$(W^+_{\alpha,Y}u,v)=(u,v)+\sum_{j,k=1}^N(T_{y_j}\Omega_{jk}T^*_{y_k}u,v). \tag{94}$$

In order to prove our theorem it suffices to show that

$$\|\Omega_{jk}u\|_p\le C\|u\|_p,\quad u\in\mathcal{D}_*,\quad j,k=1,\dots,N \tag{95}$$

for any $1<p<\infty$ and for a constant C independent of u. We first remark here that the damping factor $e^{-\delta\lambda}$ in the definition of $\Omega_{jk}u$ is unnecessary. To see this we first note that $(\xi^2-z^2)^{-1}$ has a limit in $\mathcal{S}'(\mathbb{R}^2)$ as $z\to-\lambda+i0$, $\lambda>0$ and, for $v\in\mathcal{D}_*$:

$$\begin{aligned}\int_{\mathbb{R}^2}\overline{v(x)}\mathcal{G}_{-\lambda}(x)dx&=\lim_{\varepsilon\downarrow 0}\langle v,\mathcal{G}_{-\lambda+i\varepsilon}\rangle=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}(v,\mathcal{F}^*(\xi^2-(-\lambda+i\varepsilon)^2)^{-1})\\&=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}(\mathcal{F}v,(\xi^2-(\lambda-i\varepsilon)^2)^{-1})\\&=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\overline{\hat{v}(\xi)}}{\xi^2-\lambda^2+i\varepsilon}d\xi.\end{aligned} \tag{96}$$

Then, as a function of λ

$$\begin{aligned}&\int_{\mathbb{R}^2}(\mathcal{G}_\lambda(y)-\mathcal{G}_{-\lambda}(y))u(y)\mathrm{d}y\\&\quad=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\frac{1}{\eta^2-\lambda^2-i\varepsilon}-\frac{1}{\eta^2-\lambda^2+i\varepsilon}\right)\hat{u}(\eta)\mathrm{d}\eta\\&\quad=\frac{i}{2}\int_{\mathbb{S}^1}\hat{u}(\lambda\omega)d\omega\end{aligned} \tag{97}$$

is of class $C_0^\infty((0,\infty))$. Then we have:

$$(\Omega_{jk}u)(x)=\frac{1}{\pi\mathrm{i}}\int_0^{+\infty}\lambda\overline{(\Gamma_{\alpha,Y}(\lambda)^{-1})_{jk}}\,\mathcal{G}_{-\lambda}(x)\left(\int_{\mathbb{S}^1}\hat{u}(\lambda\omega)d\omega\right)\mathrm{d}\lambda. \tag{98}$$

4.2. *Decomposition of the operator Ω_{jk}*

For simplicity we define for $j,k=1,\dots,N$

$$\widetilde{\Gamma}_{jk}(\lambda)=[\overline{\Gamma_{\alpha,Y}(|\lambda|)}^{-1}]_{jk}.$$

We let $\widetilde{\Gamma}_{jk}(|D|)$ and K be the operators defined for $u \in \mathcal{D}_*$ respectively by

$$\tilde{\Gamma}_{jk}(|D|)u(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix\xi}\,\widetilde{\Gamma}_{jk}(|\xi|)(\mathcal{F}u)(\xi)d\xi, \tag{99}$$

$$Ku(x) = \frac{1}{\pi \mathrm{i}}\int_0^{+\infty} \mathcal{G}_{-\lambda}(x)\lambda\left(\int_{\mathbb{S}^1}(\mathcal{F}u)(\lambda\omega)d\omega\right)\mathrm{d}\lambda.$$

Lemma 4.3. *For every $j,k = 1,\ldots,N$ the operator Ω_{jk} is the product of $\tilde{\Gamma}_{jk}(|D|)$ and K:*

$$(\Omega_{jk}u)(x) = (K \circ \tilde{\Gamma}_{jk}(|D|))u(x), \quad u \in \mathcal{D}_*. \tag{100}$$

Proof. We may write the right-hand side of (98) in the form

$$\frac{1}{\pi \mathrm{i}}\int_0^{+\infty}\lambda\mathcal{G}_{-\lambda}(x)\left(\int_{\mathbb{S}^1}[\overline{\Gamma_{\alpha,Y}(\lambda)}^{-1}]_{jk}(\mathcal{F}u)(\lambda\omega)d\omega\right)\mathrm{d}\lambda.$$

Here $[\overline{\Gamma_{\alpha,Y}(\lambda)}^{-1}]_{jk}(\mathcal{F}u)(\lambda\omega) = \mathcal{F}(\tilde{\Gamma}_{jk}(D)u)(\lambda\omega)$ by the definition of $\tilde{\Gamma}_{jk}(|D|)$. The lemma follows. □

4.3. *Estimate of Ku*

In what follows, we shall prove that both K and $\widetilde{\Gamma}_{jk}(|D|)$, $j,k = 1,\ldots,N$ are bounded operators from $L^p(\mathbb{R}^2)$ to itself for $1 < p < \infty$. We deal with K first.

Lemma 4.4. *For any $1 < p < \infty$, there exists a constant $C > 0$ such that*

$$|\langle v, Ku\rangle| \le C\|u\|_p\|v\|_{p'}, \quad u, v \in \mathcal{D}_*$$

and K extends to a bounded operator from L^p to itself.

Proof. Let $u, v \in \mathcal{D}_*$. Define a signed measure μ_u on $(0,\infty)$ by

$$\mu_u(E) = \int_{\lambda\in E}\left(\int_{\mathbb{S}^1}\hat{u}(\lambda\omega)d\omega\right)\lambda d\lambda$$

for Borel sets E of $(0,\infty)$. The measure μ_u is supported on a compact subset of $(0,\infty)$ and

$$(v, Ku) = \frac{1}{\pi i}\int_{\mathbb{R}^2}\overline{v(x)}\left(\int_0^\infty \mathcal{G}_{-\lambda}(x)\mu_u(d\lambda)\right)dx.$$

Changing the order of integration by using Fubini theorem, we have

$$(v, Ku) = \frac{1}{\pi i}\int_0^\infty\left(\int_{\mathbb{R}^2}\overline{v(x)}\mathcal{G}_{-\lambda}(x)dx\right)\mu_u(d\lambda). \tag{101}$$

Since the limit as $\varepsilon \to 0$ converges uniformly on compact sets of λ in (96), we may change of order of the limit and the integral in (101) and, applying Fubini's theorem again we have

$$(v, Ku) = \lim_{\varepsilon\downarrow 0} \int_0^\infty \left(\frac{1}{2\pi^2 i} \int_{\mathbb{R}^2} \frac{\overline{\hat{v}(\xi)}d\xi}{\xi^2 - \lambda^2 + i\varepsilon} \right) \mu_u(d\lambda)$$

$$= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi^2 i} \int_{\mathbb{R}^2} \overline{\hat{v}(\xi)} \left(\int_0^\infty \frac{\mu_u(d\lambda)}{\xi^2 - \lambda^2 + i\varepsilon} \right) d\xi. \tag{102}$$

Here the inner integral in (102) is equal to

$$\int_0^\infty \left(\int_{\mathbb{S}^1} \frac{\hat{u}(\lambda\omega)}{\xi^2 - \lambda^2 + i\varepsilon} d\omega \right) \lambda d\lambda = \int_{\mathbb{R}^2} \frac{\hat{u}(\eta)}{\xi^2 - \eta^2 + i\varepsilon} d\eta$$

and, Fubini's theorem and the change of variables (ξ, η) to $(\xi + \eta, \eta)$ imply

$$(v, Ku) = \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi^2 i} \int_{\mathbb{R}^4} \frac{\overline{\hat{v}(\xi)}\hat{u}(\eta)}{\xi^2 - \eta^2 + i\varepsilon} d\xi d\eta = \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi^2 i} \int_{\mathbb{R}^4} \frac{\overline{\hat{v}(\xi+\eta)}\hat{u}(\eta)}{\xi^2 + 2\xi\eta + i\varepsilon} d\eta d\xi.$$

Substituting

$$\frac{1}{\xi^2 + 2\xi\eta + i\varepsilon} = -i \int_0^\infty e^{it(\xi^2 + 2\xi\eta + i\varepsilon)} dt$$

and using Fubini's theorem once more yield

$$(v, Ku) = \lim_{\varepsilon\downarrow 0} \frac{-1}{2\pi^2} \int_0^\infty e^{-\varepsilon t} \left\{ \int_{\mathbb{R}^2} e^{it\xi^2} \left(\int_{\mathbb{R}^2} e^{i2t\xi\eta} \overline{\hat{v}(\xi+\eta)}\hat{u}(\eta) d\eta \right) d\xi \right\} dt. \tag{103}$$

Apply Parseval identity to the inner most integral and change variables $(x, \xi) \to (x, (y - x)/2t)$. Then the function inside $\{\cdots\}$ becomes

$$\int_{\mathbb{R}^2} e^{it\xi^2} \left(\int_{\mathbb{R}^2} e^{ix\xi} \overline{v(x)} u(x + 2t\xi) dx \right) d\xi = \int_{\mathbb{R}^4} e^{i(y^2 - x^2)/4t} \overline{v(x)} u(y) t^{-2} dx dy.$$

Introduce this identity in (103) and change $t \to 1/4t$:

$$(v, Ku) = \lim_{\varepsilon\downarrow 0} \frac{-2}{\pi^2} \int_0^\infty e^{-\varepsilon/4t} \left(\int_{\mathbb{R}^2} e^{-itx^2} \overline{v(x)} dx \right) \left(\int_{\mathbb{R}^2} e^{ity^2} u(y) dy \right) dt. \tag{104}$$

Now we introduce the spherical mean:

$$M_u(r) = \frac{1}{2\pi} \int_{\mathbb{S}^1} u(r\omega) d\omega, \quad r > 0, \tag{105}$$

define $N_u(r) = M_u(\sqrt{r})$ for $r > 0$ and $N_u(r) = 0$ for $r \le 0$ and let $\mathcal{R}$ be the restriction operator to the positive half line:

$$(\mathcal{R}f)(r) = \begin{cases} f(r), & r > 0, \\ 0, & r \le 0. \end{cases}$$

Using polar coordinates, we then have

$$\int_{\mathbb{R}^2} e^{ity^2} u(y) dy = 2\pi \int_0^\infty e^{itr^2} M_u(r) r dr$$

$$= \pi \int_{\mathbb{R}} e^{itr} N_u(r) dr = \sqrt{2\pi}\pi (\mathcal{F}^* N_u)(t),$$

where $\mathcal{F}$ is the one dimensional Fourier transform and, likewise

$$\int_{\mathbb{R}^2} e^{itx^2} v(x)dx = \sqrt{2\pi}\pi(\mathcal{F}^* N_v)(t).$$

Since $(\mathcal{F}^* N_u)(t), (\mathcal{F}^* N_v)(t) \in L^2(\mathbb{R})$, the limit as $\varepsilon \to 0$ in (104) can be trivially taken and Parseval's identity implies

$$\begin{aligned}(v, Ku) &= -4\pi \int_0^\infty \overline{(\mathcal{F}^* N_v)(t)}(\mathcal{F}^* N_u)(t)dt \\ &= -4\pi(\mathcal{F}^* N_v, \mathcal{R}\mathcal{F}^* N_u) = -4\pi(N_v, \mathcal{F}\mathcal{R}\mathcal{F}^* N_u).\end{aligned}$$

As is well known, the operator

$$u(x) \mapsto (\mathcal{F}\mathcal{R}\mathcal{F}^* u)(x) = \frac{i}{2\pi}\int_{\mathbb{R}} \frac{u(y)}{x-y-i0}dy$$

is bounded in $L^p(\mathbb{R})$ for any $1 < p < \infty$. Thus Hölder's inequality implies

$$\begin{aligned}(v, Ku) \leq_{|\,\cdot\,|} C &\left(\int_0^\infty |M_v(\sqrt{r})|^{p'} dr\right)^{1/p'} \left(\int_0^\infty |M_u(\sqrt{r})|^p dr\right)^{1/p} \\ &\leq C\|v\|_{L^{p'}(\mathbb{R}^2)}\|u\|_{L^p(\mathbb{R}^2)}.\end{aligned}$$

This completes the proof. □

Remark. Equation (104) and the argument following it imply that

$$Ku(x) = \lim_{\varepsilon \downarrow 0} \frac{2i}{\pi^2}\int_{\mathbb{R}^2} \frac{u(y)dy}{x^2 - y^2 - i\varepsilon}.$$

4.4. *Proof of* Theorem 1.3, *the case* $N = 1$

Thanks to Lemmas 4.3 and 4.4, it suffices to prove that $\widetilde{\Gamma}_{jk}(|D|)$, $j, k = 1, \ldots, N$ are bounded from $L^p(\mathbb{R}^2)$ to itself for $1 < p < \infty$. We recall Mikhlin's multiplier theorem ([11]):

Lemma 4.5. *Let $k > n/2$ be an integer. Suppose $m \in C^k(\mathbb{R}^n\backslash\{0\})$ and*

$$\partial_\xi^\alpha m(\xi) \leq_{|\,\cdot\,|} C_\alpha |\xi|^{-|\alpha|}, \quad |\alpha| \leq k. \tag{106}$$

Then, the Fourier multiplier $m(D)$ by $m(\xi)$ defined by

$$m(D)u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\xi} m(\xi)\mathcal{F}u(\xi)d\xi$$

is bounded from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$.

When $N = 1$ we have for $\lambda > 0$ that

$$\widetilde{\Gamma}(\lambda) = \overline{\Gamma(\lambda)}^{-1} = \left(\alpha + \frac{1}{2\pi}\log\left(\frac{\lambda}{2}\right) - \frac{i}{4} + \frac{\gamma_0}{2\pi}\right)^{-1}.$$

It is obvious that $\widetilde{\Gamma}(\lambda) \in C^\infty((0,\infty))$ and

$$\widetilde{\Gamma}^{(\ell)}(\lambda) \le_{|\cdot|} C_\ell \lambda^{-\ell}, \quad \ell = 0, 1, \dots, \tag{107}$$

which implies

$$\partial_\xi^\alpha \widetilde{\Gamma}(|\xi|) \le_{|\cdot|} C_\ell |\xi|^{-|\alpha|}, \quad |\alpha| \ge 0.$$

Hence, Lemma 4.5 implies $\widetilde{\Gamma}(|D|)$ is bounded from $L^p(\mathbb{R}^2)$ to itself for any $1 < p < \infty$. This completes the proof of Theorem 1.3 for the case $N = 1$.

4.5. *Proof of* Theorem 1.3, *the case* $N \ge 2$

For $N \ge 2$, $\Gamma(\lambda)$ has the form

$$\Gamma(\lambda) = \{\alpha\} - g(\lambda) - J(\lambda), \quad J(\lambda) = (\mathcal{G}_\lambda(y_j - y_k)\hat{\delta}_{jk}).$$

where $\{\alpha\}$ is the diagonal matrix with entries $\alpha_1, \dots, \alpha_N$ and $g(\lambda)$ is the scalar matrix $g(\lambda)\mathbf{1}$. Thus, for $N \ge 2$, $\Gamma(\lambda)$ contains the term $\mathcal{G}_\lambda(y_j - y_k)$ which is oscillatory for large λ. This prevents to directly apply Lemma 4.5 to $\widetilde{\Gamma}_{jk}(|D|)$ and we need to split it into the low and high energy parts and treat them differently. Recall (11) that

$$\mathcal{G}_\lambda(x) = e^{i\lambda|x|}\omega(\lambda|x|) \tag{108}$$

where $\omega(\lambda)$ satisfies for $\lambda > c > 0$, c being any positive number,

$$\partial_\lambda^\ell \omega(\lambda) \le_{|\cdot|} C_\ell \langle\lambda\rangle^{-\frac{1}{2}-\ell}, \quad \ell = 0, 1, 2 \dots. \tag{109}$$

Low Energy Estimate of $\widetilde{\Gamma}(|D|)$. If $H_{\alpha,Y}$ is of regular type, then Lemma 2.2 shows that $((\Gamma(\lambda)^{-1})_{jk}$ satisfies (43) hence, so does $\widetilde{\Gamma}_{jk}(\lambda)$, $j, k = 1, \dots, N$. It follows that if $\chi \in C_0^\infty(\mathbb{R}^1)$ is such that $\chi(\lambda) = 1$ for $|\lambda| \le \lambda_0/2$ and $\chi(\lambda) = 0$ for $|\lambda| \ge \lambda_0$, λ_0 being as in (43), then $\chi(|\xi|)\widetilde{\Gamma}_{jk}(|\xi|)$ satisfies the condition of Lemma 4.5 and $\chi(|D|)\widetilde{\Gamma}_{jk}(|D|)$ is bounded from $L^p(\mathbb{R}^2)$ to itself for all $1 < p < \infty$.

High Energy Estimate of $\widetilde{\Gamma}(|D|)$. Thus, the proof of Theorem 1.3 will be completed if we prove $(1 - \chi(|D|))\widetilde{\Gamma}_{jk}(|D|)$ is also bounded from $L^p(\mathbb{R}^2)$ to itself for all $1 < p < \infty$. We use the following result due to Peral (p. 139, [12]).

Lemma 4.6 (Peral). *The translation invariant Fourier integral operator*

$$(Tf)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi + i|\xi|} \frac{\psi(\xi)}{|\xi|^b} \hat{f}(\xi) d\xi, \tag{110}$$

where $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ is such that $\psi(\xi) = 0$ in a neighbourhood of $\xi = 0$ and $\psi(\xi) = 1$ for $|\xi| \ge 2$, is bounded in $L^p(\mathbb{R}^n)$ if and only if

$$\left|\frac{1}{p} - \frac{1}{2}\right| \le \frac{b}{n-1}. \tag{111}$$

Lemma 4.7. *Let $\chi \in C_0^\infty(\mathbb{R})$ be as above. Then, we may write*

$$(1-\chi(\lambda))\widetilde{\Gamma}(\lambda) = (\Phi_{jk}(\lambda) + L_{jk}(\lambda))_{jk}, \tag{112}$$

where $\Phi = (\Phi_{jk})$ and $L = (L_{jk})$ satisfy the following properties:

(1) *For $j,k = 1,\dots,N$, $\Phi_{jk}(\lambda)$ is of the form*

$$\Phi_{jk}(\lambda) = \sum_{\ell=1}^{M} e^{ia_\ell\lambda} b_\ell(\lambda), \tag{113}$$

where $a_1,\dots,a_M > 0$ are constants and $b_1,\dots,b_M$ are symbols of order $-1/2$ on $\mathbb{R}$ (which, of course, depend on j,k but we suppress such dependence as the argument will be the same for all j,k).

(2) *For $j,k = 1,\dots,N$, $L_{jk}(\lambda)$ satisfy*

$$\partial_\lambda^\ell L_{jk}(\lambda) \le_{|\,\cdot\,|} C_\ell \langle\lambda\rangle^{-2}, \quad \ell = 0,1,\dots. \tag{114}$$

Proof. Since $(1-\chi(\lambda))\widetilde{\Gamma}(\lambda)_{jk}$ is smooth, it suffices to prove that the decomposition (112) is possible for $\lambda \gg 1$. As $(\{\alpha\} - g(\lambda))^{-1} \to 0$ as $\lambda \to \infty$ and

$$\partial_\lambda^\ell J_{jk}(\lambda) \le C_\ell \langle\lambda\rangle^{-\frac{1}{2}}, \quad \ell = 0,1,\dots \tag{115}$$

for large λ by virtue of (108), we may write

$$(1-\chi(\lambda))\widetilde{\Gamma}(\lambda) = (1-\chi(\lambda))(\{\alpha\} - \overline{g(\lambda)})^{-1}(1 - (\{\alpha\} - \overline{g(\lambda)})^{-1}\overline{J(\lambda)})^{-1},$$

which implies

$$\partial_\lambda^\ell (1-\chi(\lambda))\widetilde{\Gamma}(\lambda)_{jk} \le C_\ell, \quad \ell = 0,1,\dots. \tag{116}$$

We expand $(1 - (\{\alpha\} - \overline{g})^{-1}\overline{J})^{-1}$ and define $\Phi(\lambda)$ and $L(\lambda)$ as

$$\Phi(\lambda) = (1-\chi(\lambda)) \sum_{k=0}^{3} (\{\alpha\} - \overline{g(\lambda)})^{-1} (\overline{J(\lambda)}(\{\alpha\} - \overline{g(\lambda)})^{-1})^k,$$

$$L(\lambda) = (1-\chi(\lambda))\Gamma(\lambda)^{-1}(\{\alpha\} - \overline{g}(\lambda))^{-4}\overline{J(\lambda)}^4.$$

Then we have $(1-\chi(\lambda))\widetilde{\Gamma}(\lambda)^{-1} = \Phi(\lambda) + L(\lambda)$ and the entries of $\Phi(\lambda)$ satisfy the property (113) for large $\lambda > 0$. The entries of $L(\lambda)$ satisfy (114) by virtue of (116) and (115) which implies $J(\lambda)^4$ and its derivatives are bounded by $\langle\lambda\rangle^{-2}$. The lemma follows. □

We have

$$(1-\chi(|D|))\widetilde{\Gamma}_{jk}(|D|) = \Phi_{jk}(|D|) + L_{jk}(|D|).$$

Then by virtue of (113) Lemma 4.6 with $b = 1/2$ and $n = 2$ implies $\Phi_{jk}(|D|)$ is bounded from $L^p(\mathbb{R}^2)$ to itself; Mikhlin's Lemma 4.5 shows that the operator $L_{jk}(|D|)$ is bounded from $L^p(\mathbb{R}^2)$ to itself by virtue of (114). This concludes the proof of Theorem 1.3 also for $N \ge 2$.

Acknowledgments

K.Y. acknowledges the support by JSPS grant in aid for scientific research No. 16K05242. H.C. acknowledges financial support from the Danish Council of Independent Research–Natural Sciences, Grant DFF4181-00042.

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